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H_∞–CONTROL PROBLEMS UNBOUNDED OBSERVATION: THE HYPERBOLIC CASE

BY

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Abstract. In this paper we prove necessary and sufficient conditions for the existence of a solution in a H_∞–control problem for infinite dimensional systems with unbounded observation. These conditions are derived in terms of an algebraic Riccati equation.

1. Introduction. Let A be an infinitesimal generator of a C_0–semigroup e^{At} on a real Hilbert space $X, B_1 \in L(W,X), B_2 \in L(U,X), C \in L(D(A)),Z)$ and $D \in L(U,Z)$, where U, W, Z are also real Hilbert spaces.

We consider the input–output system given by

$$\begin{cases} x'(t) = Ax(t) + B_1 w(t) + B_2 u(t),\ t \in \boldsymbol{R}^+ = [0,\infty) \\ x(0) = x_0 \end{cases} \tag{1.1}$$

$$z(t) = Cx(t) + Du(t),\ t \in \boldsymbol{R}^+ \tag{1.2}$$

where $x(t) \in X$ is the state of the system, $u(t) \in U$ is the control input, $w(t) \in W$ is the disturbance input and $z(t) \in Z$ is the to–be–controlled output.

We shall denote by $|.|_X, |.|_Z, |.|_U, |.|_W$ the norms of X, Z, U, W and by $(.,), (.,.)_Z, (.,.)_U, (.,.)_W$, the corresponding scalar products. We denote by A^* the adjoint of A, and by $D(A)'$ the dual of $D(A)$. We shall denote by $(.,.)$ the pairing between $D(A)$ and $D(A)'$.

The following assumptions will be in effect throughout the paper:

i) For every $T > 0$ the operator $e^{A^*t}C^*$ admits a continuous extension form Z to $L^2(0,T;X)$ i.e., there exists $C_T > 0$ such that

$$\int_0^T |e^{A^*t}C^*z|_X^2 dt \leq C_T|z|_Z^2, \ \forall z \in Z. \tag{1.3}$$

ii) There exists $K \in L(Z, D(A)')$ such that $A_K = A+KC$ generates an exponenetially stable semigroup on X and for all $T > 0$ there exists $\widetilde{C}_T > 0$ such that

$$\int_0^T |K^*e^{A_K^*t}x|_Z^2 dt \leq \widetilde{C}_T|x|_X^2, \quad \forall x \in X.$$

iii) $D^*[C \quad D] = [0 \quad I]$.

Remark 1.1. Optimal control problems of LQ type with finite horizon and unbounded observation (with C and D satisfying (i)–(iii)) have been cosidered in [7] and [12].

Without any loss of generality for the problem considered, we take $0 \in \rho(A)$ where $\rho(A)$ is the resolvent of A.

Denote by $\mathcal{F}$ the set of feedback operators $F \in L(X,U)$ such that $A_F = A + B_2F$ generates an exponentially stable semigroup on X.

This means (see [8],[14]) that for every $x_0 \in X$ and $f \in L^2(\boldsymbol{R}^+;X)$ the Cauchy problem

$$\begin{aligned} &x' = Ax + B_2Fx + f \ \text{ in } \ \boldsymbol{R}^+ \\ &x(0) = x_0 \end{aligned}$$

has a solution $x \in C(\boldsymbol{R}^+;X) \cap L^2(\boldsymbol{R}^+;X)$.

Our aim is to find such a state–feedback that exponentially stabilizes the system (1.1)–(1.2) and ensures that the influence of $w(.)$ on $z(.)$ is "small" in a sense specified below.

For every $T > 0, x_0 \in X$ and inputs $u \in L^2(0,T;U), w \in L^2(0,T;W)$ the system (1.1) has a mild solution $x \in C([0,T];X)$ given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}(B_1w(s) + B_2u(s))ds, \ t \in [0,T]$$

Let us denote by $S_F(x_0,w)$ the solution of (1.2) where $u = Fx$ (hence $S_F(x_0,w) = Cx + DFx$) and x is the mild solution of the problem

$$\begin{aligned} &x' = A_Fx + B_1w \text{ in } \boldsymbol{R}^+ \\ &x(0) = x_0. \end{aligned}$$

Now we can formulate the H_∞–control problem as follows: given $\gamma > 0, \alpha > 0$ find $F \in \mathcal{F}$ such that

$$||S_F(x_0, w)||^2_{L^2(\boldsymbol{R}^+;Z)} \leq \rho^2 ||w||^2_{L^2(\boldsymbol{R}^+;W)} + \alpha |x_0|^2_X \tag{1.4}$$

for a $0 < \rho < \gamma$, and $\forall x_0 \in X, w \in L^2(\boldsymbol{R}^+; W)$.

2. The main result.

Theorem 2.1. *Suppose $\gamma, \alpha > 0$ and assumptions (i), (ii), (iii) hold. If there exists an operator $F \in L(X, U)$ such that (1.4) is fulfilled then $F \in \mathcal{F}$ and there exists $P \in L(X), P = P^* \geq 0, ||P|| \leq \alpha$ satisfying*

$$\begin{aligned}(Ax, Py) + (Px, Ay) - (P(B_2B_2^* - \gamma^{-2}B_1B_1^*)Px, y) + \\ +(Cx, Cy)_Z = 0, \ \forall x, y \in D(A)\end{aligned} \tag{2.1}$$

and such that $A - (B_2B_2^* - \gamma^{-2}B_1B_1^*)P$ gerarates an exponentially stable semigroup on X.

Conversely, if there exists P as above and $A - (B_2B_2^* - \gamma^{-2}B_1B_1^*)P$ generates an exponentially stable semigroup on X then $F = -B_2^*P$ stabilizes the system (1.1)–(1.2) and guarantees inequality (1.4) with $\rho = \gamma$.

Remark 2.1. For related results concerning the H_∞– control problem with bounded and unbounded operators we refer to [1],[2],[5],[6],[10],[11],[13].

3. Proof of the sufficiency. Suppose that there exists $P \in L(X)$, $P = P^* \geq 0, ||P|| \leq \alpha$ satisfying (2.1) and such that $A_1 := A - (B_2B_2^* - -\gamma^{-2}B_1B_1^*)P$ generates an exponentially stable semigroup. We mention that the proof of sufficiency is similar to that in [11] but, for reader convenienc we shall sketch it.

Lemma 3.1. *The state feedback $u(.) = -B_2^*Px(.)$, with P as in Theorem 2.1, exponentially stabilizes the system and satisfies (1.4) with $\rho = \gamma$.*

Proof. Firstly we show that $A_2 := A - B_2B_2^*P$ generates an exponentially stable semigroup e^{A_2t} on X. For this, following [6] it is sufficient to show that $e^{A_2(.)}x_0 \in L^2(\boldsymbol{R}^+; X)$ for all $x_0 \in X$.

Since e^{A_1t} is an exponentially stable semigroup on X and for $x_0 \in X$

$$e^{A_2t}x_0 = e^{A_1t}x_0 - \int_0^t e^{A_1(t-s)}\gamma^{-2}B_1B_1^*Pe^{A_2s}x_0 ds$$

to conclude it suffices to show that $B_1^*e^{A_2(.)}x_0 \in L^2(\boldsymbol{R}^+; X)$. For this, having in mind (2.1), for $x_0 \in D(A) = D(A_2)$ we have

$$\frac{d}{dt}(e^{A_2t}x_0, Pe^{A_2t}x_0) \leq -\gamma^{-2}|B_1^*Pe^{A_2t}x_0|^2_W.$$

Integrating the last inequality on $[0,T], T>0$ and tending with T to infinity one obtains

$$||B_1^* P e^{A_2(.)} x_0||^2_{L^2(\boldsymbol{R}^+;W)} \leq \gamma^2 (x_0.Px_0). \tag{3.1}$$

Now since $D(A)$ is dense in X, from (3.1) follows our assertion. Taking $u(.) = -B_2^* P x(.)$ the system (1.1)–(1.2) becomes

$$x(t) = e^{A_2 t} x_0 + \int_0^t e^{A_2(t-s)} B_1 w(s) ds,\ t \in \boldsymbol{R}^+ \tag{3.2}$$

$$z(t) = (C - DB_2^* P)x(t),\ t \in \boldsymbol{R}^+. \tag{3.3}$$

Let $w(.)$ be continuously differentiable and $x_0 \in D(A)$. Taking into accout (2.1), (3.2), (3.3) after differentiation we obtain

$$\frac{d}{dt}(x(t), Px(t)) = -|z(t)|^2_Z + \gamma^2 |w(t)|^2_W - \gamma^2 |w - \gamma^{-2} B_1^* P x(t)|^2_W. \tag{3.4}$$

But since $e^{A_2 t}$ is exponentially stable on X, it follows (see [11], pp.31) that $x(t) \to 0$ as $t \to \infty$.

Integrating (3.4) on $(0, +\infty)$ one gets

$$||S_{-B_2^* P}(x_0, w)||^2_{L^2(\boldsymbol{R}^+;Z)} \leq \gamma^2 ||w||^2_{L^2(\boldsymbol{R}^+;W)} + \alpha |x_0|^2_X. \tag{3.5}$$

Finally (3.5) can be extended to the general case $w \in L^2(\boldsymbol{R}^+; W)$ and $x_0 \in X$ using the density of $CCD(\boldsymbol{R}^+; W)$ in $L^2(\boldsymbol{R}^+; W)$ and of $D(A)$ in X. This concludes the lemma.

4. Deriving necessary conditions. Suppose that there exists $F \in L(X,U)$ such that (1.4) holds. We shall prove that $A + B_2 F$ generates an exponentially stable semigroup on X, in other words (cf.[8]) the solution of the equation

$$\begin{cases} x' = (A + B_2 F)x \text{ in } \boldsymbol{R}^+ \\ x(0) = x_0 \end{cases} \tag{4.1}$$

is in $L^2(\boldsymbol{R}^+; X)$.

For this we use the hypothesis (ii) and rewrite (4.1) as

$$\begin{cases} x' = (A+KC)x - KCx + B_2Fx,\ t \in \boldsymbol{R}^+ \\ x(0) = x_0 \end{cases}$$

or equivalently

(4.2)

$$x(t) = e^{A_K t}x_0 + \int_0^t e^{A_K(t-s)}B_2Fx(s)ds - \int_0^t e^{A_K(t-s)}KCx(s)ds, \quad t \in \boldsymbol{R}^+.$$

Now if we write (1.4) as

$$\int_0^\infty (|C\widetilde{x}|_Z^2 + |F\widetilde{x}|_U^2)dt \le \rho^2 \int_0^\infty |w|_W^2 dt + \alpha|x_0|_X^2$$

one obtains that

$$C\widetilde{x} \in L^2(\boldsymbol{R}^+; Z) \tag{4.3}$$

$$F\widetilde{x} \in L^2(\boldsymbol{R}^+; U) \tag{4.4}$$

where $\widetilde{x}$ is the solution of the equation

$$\begin{cases} \widetilde{x}' = (A+B_2F)\widetilde{x} + B_1 w \text{ in } \boldsymbol{R}^+ \\ \widetilde{x}(0) = x_0 \end{cases}$$

for $w \in L^2(\boldsymbol{R}^+; W)$.

As A_K generates an exponentially stable semigroup it implies that the first term in the right side in (4.2) is in $L^2(\boldsymbol{R}^+; X)$.

Also since $B_2 \in L(U, X)$ and from (4.4) it follows that (see also [11],[14]) the second term in (4.2) is in $L^2(\boldsymbol{R}^+; X)$.

To finish our assertion, that F is an exponentially stabilizing feedback, it remains to prove that the last term in (4.2) is in $L^2(\boldsymbol{R}^+; X)$. The difficulty consists in that the operator K is unbounded.

Lemma 4.1. *If $u \in L^2(\boldsymbol{R}^+; Z)$ then*

$$h(t) = \int_0^t e^{A_K(t-s)}Ku(s)ds \ \ \textit{is in} \ \ L^2(\boldsymbol{R}^+; X).$$

Proof. It is known (see [12],pp.657) that condition (ii) implies that there exists a positive constant C such that

$$|h(t)|_X \leq C||u||_{L^2(0,1;Z)}, \ \forall t \in [0,1]. \tag{4.5}$$

Also, $e^{A_K t}$ being an exponenetially stable semigroup on X there exist (see [8],[14]) positive constants M and ω such that

$$|e^{A_K t}x|_X \leq Me^{-\omega t}|x|_X, \text{ for all } t \in \boldsymbol{R}^+ \text{ and } x \in X. \tag{4.6}$$

For our purpose besides (4.5) and (4.6) we shall use systematically the following simple inequality

$$(a+b)^2 \leq (1+\beta)a^2 + \left(1+\frac{1}{\beta}\right)b^2, \quad \forall \beta, a, b \in \boldsymbol{R}, \beta > 0.$$

From (4.5) it follows

$$\int_0^1 |h(t)|_X^2 dt \leq C^2||u||^2_{L^2(0,1;Z)}.$$

Now using (4.6) for $1 \leq t < 2$ we have

$$\begin{aligned}
|h(t)|_X^2 &= |\int_0^1 e^{A_K(t-s)}Ku(s)ds + \int_1^t e^{A_K(t-s)}Ku(s)ds|_X^2 \\
&\leq (1+\beta)|\int_0^1 e^{A_K(t-1)}e^{A_K(1-s)}Ku(s)ds|_X^2 \\
&+ (1+\frac{1}{\beta})|\int_1^t e^{A_K(t-s)}Ku(s)ds|_X^2 \\
&\leq (1+\beta)M^2C^2e^{-2\omega(t-1)}||u||^2_{L^2(0,1;Z)} + (1+\frac{1}{\beta})C^2||u||^2_{L^2(1,2;Z)}.
\end{aligned}$$

Generally, for $n \leq t < n+1$

$$\begin{aligned}
|h(t)|_X^2 &= |\int_0^n e^{A_K(t-s)}Ku(s)ds + \int_n^t e^{A_K(t-s)}Ku(s)ds|_X^2 \\
&\leq (1+\beta)|\int_0^n e^{A_K(t-s)}Ku(s)ds|_X^2 + (1+\frac{1}{\beta})C^2||u||^2_{L^2(n,n+1;Z)}
\end{aligned}$$

Next

$$|\int_0^n e^{A_K(t-s)}Ku(s)ds|_X^2 =$$

$$= |\int_0^{n-1} e^{A_K(t-s)}Ku(s)ds + \int_{n-1}^n e^{A_K(t-n)}e^{A_K(n-s)}Ku(s)ds|_X^2$$

$$\leq (1+\beta)|\int_0^{n-1} e^{A_K(t-s)}Ku(s)ds|_X^2 +$$

$$+(1+\frac{1}{\beta})M^2C^2e^{-2\omega(t-n)}||u||^2_{L^2(n-1,n;Z)}.$$

So, going throught an introductive process one obtains (for $n \leq t < n+1$)

$$|h(t)|_X^2 \leq M^2C^2(1+\beta)^n e^{-2\omega(t-1)}||u||^2_{L^2(0,1;Z)} +$$

$$+M^2C^2\left(1+\frac{1}{\beta}\right)\sum_{k=1}^{n-1}(1+\beta)^{n-k}e^{-2\omega(t-k-1)}||u||^2_{L^2(k,k+1;Z)} +$$

$$+C^2\left(1+\frac{1}{\beta}\right)||u||^2_{L^2(n,n+1;Z)}.$$

Integrating this it follows

$$\int_n^{n+1}|h(t)|_X^2dt \leq \frac{M^2C^2}{2\omega}(1+\beta)^n e^{-2\omega(n-1)}||u||^2_{L^2(0,1;Z)} +$$

$$+\frac{M^2C^2}{2\omega}\left(1+\frac{1}{\beta}\right)\sum_{k=1}^{n-1}(1+\beta)^{n-k}e^{-2\omega(n-k-1)}||u||^2_{L^2(k,k+1;Z)} +$$

$$+C^2\left(1+\frac{1}{\beta)}\right)||u||^2_{L^2(n,n+1;Z)}.$$

Now, choosing $\beta > 0$ such that $\frac{(1+\beta)}{e^{2\omega}} < 1$ and summing we obtain

$$\int_0^\infty |h(t)|_X^2dt = \sum_{n=0}^\infty \int_n^{n+1}|h(t)|_X^2dt \leq$$

$$\leq \left[C^2 + \frac{M^2C^2}{2\omega}(1+\beta)\sum_{n=0}^\infty\left(\frac{1+\beta}{e^{2\omega}}\right)^n\right]$$

$$\left[||u||^2_{L^2(0,1;Z)} + \left(1+\frac{1}{\beta}\right)\sum_{n=1}^{\infty}||u||^2_{L^2(n,n+1;Z)}\right]$$

which proves the lemma.

Thus we have proved that $A + B_2F$ generates an exponentially stable semigroup on X. Now, we associate with system (1.1)– (1.2) the differential game

$$\sup_{w\in L^2(\boldsymbol{R}^+;W)}\ \inf_{u\in L^2(\boldsymbol{R}^+;U)} K(u,w) \tag{4.7}$$

where

$$K(u,w) = \int_0^{\infty}(|Cx(t)|^2_Z + |u(t)|^2_U - \gamma^2|w(t)|^2_W)dt \tag{4.8}$$

x being the solution of (1.1)–(1.2). Firstly consider the problem

$$Min\ \{K(u,w); u\in L^2(\boldsymbol{R}^+;W)\} \tag{4.9}$$

where $w\in L^2(\boldsymbol{R}^+;W)$ is arbitrary but fixed.

It results in a standard way that there exists a unique optimal solution of the problem (4.9), denoted by $\overline{u} = \Gamma w$. Also it can be proved that: $\Gamma : L^2(\boldsymbol{R}^+;W)\to L^2(\boldsymbol{R}^+;U)$ may be represented as

$$\Gamma w = \Gamma_0 w + f_0,\ \forall w\in L^2(\boldsymbol{R}^+;W)$$

where $\Gamma_0\in L(L^2(\boldsymbol{R}^+;W), L^2(\boldsymbol{R}^+;U))$ and $f_0\in L^2(\boldsymbol{R}^+;U)$, ($f_0 = arg\ inf\ K(u,0)$) (see [1]).

For this we denote by $\overline{x}^w$ the solution of (1.1) with $u = \overline{u} = \Gamma w$ and by $\widetilde{x}$ the solution of the equation

$$\begin{cases}\widetilde{x}' = (A+B_2F)\widetilde{x} + B_1w,\ t\in\boldsymbol{R}^+\\ \widetilde{x}(0) = x_0.\end{cases}$$

From the definition of Γw and from (1.4) follows

$$\int_0^{\infty}(|C\overline{x}^w|^2_Z + |\Gamma w|^2_U)dt \le \int_0^{\infty}(|C(e^{(A+B_2F)t}x_0 +$$
$$+\int_0^t e^{(A+B_2F)(t-s)}B_1w(s)ds|^2_Z + |B_2F\widetilde{x}|^2_U)dt \le \rho^2||w||^2_{L^2(\boldsymbol{R}^+;W)} + \alpha|x_0|^2_X$$

which implies $\Gamma_0 \in L(L^2(\boldsymbol{R}^+;W), L^2(\boldsymbol{R}^+;U))$.

From our assumptions system (1.1) is internally exponentially stabilizable by a state feedback $u = -B_2^* P_0 x$, where $P_0 \in L(X), P_0 = P_0^* \geq 0$ (see [9]).

Let us denote by $A_{P_0} = A - B_2 B_2^* P_0$ and $M_1,\ \omega_1$ positive such that

$$||e^{A_{P_0}t}|| \leq M_1 e^{-\omega_1 t},\ \text{for } t > 0.$$

Proposition 4.1. *There is a unique $p \in C(\boldsymbol{R}^+; D(A)') \cap L^2(\boldsymbol{R}^+; X)$ satisfying*

$$\begin{cases} p' = -(A - B_2 B_2^* P_0)^* p - P_0 B_2 \overline{u} + C^* C \overline{x} \text{ in } \boldsymbol{R}^+ \\ p(\infty) = 0 \end{cases} \tag{4.10}$$

$$B_2^* p(t) = \overline{u}(t) \quad a.e. \quad t > 0 \tag{4.11}$$

(Above $\overline{x}$ is the solution of (1.1) with $u = \overline{u} = \Gamma w$).

Proof. Let $p \in C(\boldsymbol{R}^+; D(A)')$ be the mild solution of (4.10). We show that $p \in L^2(\boldsymbol{R}^+; X)$. Writing the solution of (4.10) as

$$p(t) = -\int_t^\infty e^{(A - B_2 B_2^* P_0)^*(s-t)}[-P_0 B_2 \overline{u}(s) + C^* C \overline{x}(s)]ds$$

and making the scalar product with an arbitrary $h \in L^2(\boldsymbol{R}^+; X)$ one obtains

$$|\int_0^\infty (p(t), h(t)dt| \leq |\int_0^\infty \int_0^s (e^{A_{P_0}^*(s-t)} P_0 B_2 \overline{u}(s), h(t))dtds| + \tag{4.12}$$

$$+|\int_0^\infty \int_0^s (e^{A_{P_0}^*(s-t)} C^* C \overline{x}(s), h(t))dtds|.$$

The evaluation of the first right side term in (4.12) (after the application of Hölder's inequality) leads to the estimate below

$$
\begin{aligned}
&|\int_0^\infty \int_0^s (e^{A^*_{P_0}(s-t)} P_0 B_2 \overline{u}(s), h(t)) dt ds|_X \\
&\leq M_1 \int_0^\infty \int_0^s e^{-\omega_1(s-t)} |P_0 B_2 \overline{u}(s)|_X |h(t)|_X dt ds \\
&\leq M_1 \left(\int_0^\infty |P_0 B_2 \overline{u}(s)|_X^2 ds \right)^{1/2} \left(\int_0^\infty e^{-2\omega_1 s} \left(\int_0^s e^{\omega_1 t} |h(t)|_X dt \right)^2 ds \right)^{1/2} \\
&\leq M_2 \left(\int_0^\infty e^{-2\omega_1 s} \left(\int_0^s e^{\omega_1 t} dt \right) \left(\int_0^s e^{\omega_1 t} |h(t)|_X^2 dt \right) ds \right)^{1/2} \\
&\leq M_2 \omega_1^{-\frac{1}{2}} \left(\int_0^\infty \int_0^s e^{-\omega_1 s} e^{\omega_1 t} |h(t)|_X^2 dt ds \right)^{\frac{1}{2}} \\
&= M_2 \omega_1^{-\frac{1}{2}} \left(\int_0^\infty \int_t^\infty e^{\omega_1 t} |h(t)|_X^2 e^{-\omega_1 s} ds dt \right)^{\frac{1}{2}} \\
&\leq M_2 \omega_1^{-1} ||h||_{L^2(\boldsymbol{R}^+;X)}
\end{aligned}
$$

where $M_i, i = 1, 2$, are different constants.

As regard the second right side term in (4.12), since of (1.3) and (4.3) it follows in the same way as in Lemma 4.1 that it is majorized by a quantity of the norm $M_3 ||h||_{L^2(\boldsymbol{R}^+;X)}$ which together with the preceding one leads to the conclusion.

In order to prove (4.11) we need the following

Lemma 4.2. *Let $Q \in L(X, U)$. Then for every $T > 0$ there exists a constant $\overline{C}_T > 0$ such that*

$$
\int_0^T |e^{(A+B_2Q)^* t} C^* z|_X^2 dt \leq \overline{C}_T |z|_Z^2, \ \forall z \in Z \tag{4.13}
$$

Proof. We shall use assumption (i).

Let $h \in L^2(0, T; X)$ be arbitrary. Then we have

$$
\begin{aligned}
&\int_0^T (e^{A^* + Q^* B_2^*)t} C^* z, h(t)) dt = \int_0^T (z, C e^{(A+B_2Q)t} h(t))_Z dt \\
&= (z, \int_0^T C(e^{At} h(t) + \int_0^t e^{A(t-s)} B_2 Q e^{(A+B_2Q)s} h(t) ds) dt)_Z \\
&= \int_0^T (e^{A^* t} C^* z, h(t)) dt + \int_0^T \int_s^T (z, C e^{A(t-s)} B_2 Q e^{(A+B_2Q)s} h(t))_Z dt ds \\
&\leq M |z|_Z ||h||_{L^2(0,T;X)}.
\end{aligned}
$$

The last inequality is obtained by Hölder's inequality and using (1.3). The last inequality implies (4.13).

Let y be solution to the equation

$$\begin{cases} y' = (A - B_2B_2^*P_0)y + B_2\widetilde{v} \ \text{ in}\boldsymbol{R}^+ \\ y(0) = 0 \end{cases} \tag{4.14}$$

where $\widetilde{v} \in L^2(\boldsymbol{R}^+; U)$.

As $A_{P_0} = A - B_2B_2^*P_0$ generates an exponentially stable semigroup it follows $y \in L^2(\boldsymbol{R}^+; X)$. Observing that the solution of (4.14) can be put in the form

$$y(t) = \int_0^t e^{A(t-s)}B_2v(s)ds$$

where

$$v = \widetilde{v} - B_2^*P_0y \in L^2(\boldsymbol{R}^+; U) \tag{4.15}$$

assumptions (i) assures that $Cy \in L^2(\boldsymbol{R}^+; Z)$.

If we denote $\overline{u} = \Gamma w$ and choose v as in (4.15) it results

$$\int_0^\infty (|C\overline{x}|_Z^2 + |\overline{u}|_U^2)dt \le \int_0^\infty (|Cx^{\overline{u}+\varepsilon v}|_Z^2 + |\overline{u} + \varepsilon v|_U^2)dt \tag{4.16}$$

where

$$\begin{cases} (x^{\overline{u}+\varepsilon v})' = Ax^{\overline{u}+\varepsilon v} + B_2(\overline{u} + \varepsilon v) + B_1w, \ t \in \boldsymbol{R}^+ \\ x^{\overline{u}+\varepsilon v}(0) = x_0 \end{cases} \tag{4.17}$$

and

$$\begin{cases} \overline{x}' = A\overline{x} + B_2\overline{u} + B_1w, \ t \in \boldsymbol{R}^+ \\ \overline{x}(0) = x_0. \end{cases} \tag{4.18}$$

Since $Cy \in L^2(\boldsymbol{R}^+; Z)$ from (4.17) and (4.18) follows $Cx^{\overline{u}+\varepsilon v} \to C\overline{x}$, for $\varepsilon \to 0$ in $L^2(\boldsymbol{R}^+; Z)$ which together with (4.16) leads to

$$\int_0^\infty ((C\overline{x}, Cy)z + (\overline{u}, v)_U)dt \ge 0 \tag{4.19}$$

Multiplying the solution of (4.10) by $B_2\widetilde{v}$ and integrating one obtains

$$\begin{aligned}
&\int_0^\infty (p(t), B_2\widetilde{v}(t))dt = \\
&= -\int_0^\infty \left(\int_t^\infty e^{(A-B_2B_2^*P_0)^*(s-t)}(-P_0B_2\overline{u}(s) + C^*C\overline{x}(s))ds, B_2\widetilde{v}(t) \right) dt \\
&= -\int_0^\infty \left(-P_0B_2\overline{u}(s) + C^*C\overline{x}(s), \int_0^s e^{(A-B_2B_2^*P_0)(s-t)}B_2\widetilde{v}(t)dt \right) ds \\
&= -\int_0^\infty (C\overline{x}(s), Cy(s))_Z ds + \int_0^\infty (\overline{u}(s), B_2^*P_0y(s))_U ds,
\end{aligned}$$

which together with (4.15) and (4.19) gives

$$\int_0^\infty (\overline{u}(t) - B_2^*p(t), \widetilde{v}(t))_U dt \geq 0, \ \forall \widetilde{v} \in L^2(\boldsymbol{R}^+; U)$$

Hence $\overline{u}(t) = B_2^*p(t)$ a.e. $t \geq 0$ and Proposition 4.1 is proved.
Define the function

$$\psi : L^2(\boldsymbol{R}^+; W) \to \boldsymbol{R}, \text{ by } \psi(w) = -K(\Gamma w, w) \tag{4.20}$$

where $\Gamma w = \overline{u}$ is the solution of (4.9) corresponding to w and x_0. Then

$$\begin{aligned}
K(\Gamma w, w) = K(\Gamma_0 w + f_0, w) &= \int_0^\infty (|C\overline{x}|_Z^2 + |\overline{u}|_U^2 - \gamma^2|w|_W^2)dt \\
&= \int_0^\infty (|C\overline{x}|_Z^2 + |\Gamma_0 w + f_0|_U^2 - \gamma^2|w|_W^2)dt
\end{aligned}$$

where $\overline{x}$ satisfies

$$\begin{cases} \overline{x}' = A\overline{x} + B_2(\Gamma_0 w + f_0) + B_1 w, \text{ in } \boldsymbol{R}^+ \\ \overline{x}(0) = x_0. \end{cases}$$

We write $\overline{x} = \widetilde{x} + \overline{x}_0$ wherre $\widetilde{x}$ satisfies the system

$$\begin{cases} \widetilde{x}' = A\widetilde{x} + b_2\Gamma_0 w + B_1 w \text{ in } \boldsymbol{R}^+ \\ \widetilde{x}(0) = 0 \end{cases}$$

(hence $\widetilde{x}(t) = \int_0^t e^{A(t-s)}(B_2\Gamma_0 w(s) + B_1 w(s))ds$ and $C\widetilde{x} = Hw$ with $H \in L(L^2(\boldsymbol{R}^+; W); L^2(\boldsymbol{R}^+; Z))$) and

$$\begin{cases} \overline{x}_0' = A\overline{x}_0 + f_0, \ t \in \boldsymbol{R}^+ \\ \overline{x}_0(0) = x_0. \end{cases}$$

Thus

$$\begin{aligned} K(\Gamma w, w) &= \int_0^\infty (|Hw + C\overline{x}|_Z^2 + |\Gamma_0 w + f_0|_U^2 - \gamma^2 |w|_W^2) dt \\ &= \int_0^\infty (|Hw|_Z^2 + |\Gamma_0 w|_U^2 - \gamma^2 |w|_W^2) dt + ((\overline{f}, w))_{L^2(\boldsymbol{R}^+;W)} + \delta \\ &= (((H^*H + \Gamma_0^*\Gamma_0 - \gamma^2 I)w, w))_{L^2(\boldsymbol{R}^+;W)} + ((\overline{f}, w))_{L^2(\boldsymbol{R}^+;W)} + \delta \end{aligned}$$

(We have denoted by $((.,.))_{L^2(\boldsymbol{R}^+;W)}$ the scalar product of $L^2(\boldsymbol{R}^+;W)$).

If we denote

$$\widetilde{T} = \gamma^2 I - H^*H - \Gamma_0^*\Gamma_0$$

it follows

$$\begin{aligned} &(((H^*H + \Gamma_0^*\Gamma_0 - \gamma^2 I)w, w))_{L^2(\boldsymbol{R}^+;W)} = \\ &= ((-\widetilde{T}w, w))_{L^2(\boldsymbol{R}^+;W)} \leq -(\gamma^2 - \rho^2)||w||^2_{L^2(\boldsymbol{R}^+;W)}. \end{aligned} \tag{4.21}$$

So, $\widetilde{T}$ is monotone, everywhere defined, single valued, linear hemicontinuous operator from W to W hence (cf.[3],pp.40), maximal monotone.

As $\widetilde{T}$ is sefl–adjoint it follows (cf.[4],pp.114) that $\widetilde{T}$ is cyclically monotone so

$$((-\widetilde{T}w, w))_{L^2(\boldsymbol{R}^+;W)} = -||\widetilde{T}^{1/2}w||^2_{L^2(\boldsymbol{R}^+;W)}$$

and

$$K(\Gamma w, w) = -((\widetilde{D}w, \widetilde{D}w))_{L^2(\boldsymbol{R}^+;W)} + ((\overline{f}, w))_{L^2(\boldsymbol{R}^+;W)} + \delta \tag{4.22}$$

where $\widetilde{D} \in L(L^2(\boldsymbol{R}^+;W), L^2(\boldsymbol{R}^+;W))$ and $\delta \in \boldsymbol{R}$. From (4.20)–(4.22) it follows that ψ is a lower–semicontinous, proper strictly–convex function on $L^2(\boldsymbol{R}^+;W)$ so it attains its infimum on $L^2(\boldsymbol{R}^+;W)$ in a unique point w^*, i.e.,

$$w^* = arg\ inf\{\psi(w); w \in L^2(\boldsymbol{R}^+;W)\}.$$

Lemma 4.3. *We have*

$$w^*(t) = -\gamma^{-2} B_1^* p(t) \quad a.e. \quad t > 0$$

Proof. As w^* is a minimum point for (4.20) we have

$$\int_0^\infty (\gamma^2(w^*,w)_W - (\Gamma_0^*(\Gamma_0 w^* + f_0), w)_W - (C^*Cx^*, \widetilde{x}))dt = 0 \tag{4.23}$$

for all $w \in L^2(\boldsymbol{R}^+; W)$ where

$$\begin{aligned} &\widetilde{x}' = A\widetilde{x} + B_2\Gamma_0 w + B_1 w,\ t \in \boldsymbol{R}^+ \\ &\widetilde{x}(0) = 0. \end{aligned}$$

Multiplying the solution of (4.10) by $B_2\Gamma_0 w + B_1 w$ and integrating over $[0,T]$ one obtains

$$\begin{aligned} &\int_0^T (p(t), B_2\Gamma_0 w(t) + B_1 w(t))dt = \\ =&(p(t), \widetilde{x}(T)) - \int_0^T (\int_t^T e^{A^*(s-t)} C^*Cx^*(s)ds, B_2\Gamma_0 w(t) + B_1 w(t))dt \\ =&(p(T), \widetilde{x}(T)) - \int_0^T (Cx^*(s), C\widetilde{x}(s))_Z ds. \end{aligned}$$

Hence

$$\int_0^T (C^*Cx^*(s), \widetilde{x}(s))ds = (p(T), \widetilde{x}(T)) - \tag{4.24}$$

$$- \int_0^T (w, \Gamma_0^*(\Gamma_0 w^* + f_0))_W dt - \int_0^T (B_1^* p, w)_W dt$$

where x^* is a solution to (1.1) corresponding to $w^*, \Gamma w^*$ and x_0.

Choosing $w \in L^2(\boldsymbol{R}^+; W)$ a function with compact support we get

$$(p(T), \widetilde{x}(T)) \to 0 \text{ for } T \to \infty.$$

Then (4.24) gives (letting $T \to \infty$)

$$\int_0^\infty (C^*Cx^*, \widetilde{x})dt = -\int_0^\infty (\Gamma_0 w, \Gamma w^*)_U dt - \int_0^\infty (B_1^* p, w)_W dt$$

which along with (4.23) leads to

$$\int_0^\infty (B_1^* p + \gamma^2 w^*, w)_W dt = 0$$

and the lemma is proved.

Thus, up to now we have proved tht the problem

$$\sup_{w\in L^2(\boldsymbol{R}^+;W)} \inf_{u\in L^2(\boldsymbol{R}^+;U)} K(u,w) \tag{4.25}$$

has a unique solution (u^*, w^*) charaterized by Euler– Lagrange system

$$(x^*)' = Ax^* + B_1w^* + B_2u^*,\ t\in \boldsymbol{R}^+, x^*(0) = x_0 \tag{4.26}$$

$$p' = -A^*p + C^*Cx^*,\ p(\infty) = 0,\ p\in L^2(\boldsymbol{R}^+;X) \tag{4.27}$$

$$u^*(t) = B_2^*p(t), w^*(t) = -\gamma^{-2}B_1^*p(t),\quad a.e.\ \ t>0 \tag{4.28}$$

For $x_0\in X$ we set

$$P_{x_0} = -p(0)\in D(A)'$$

where p is solutioon to (4.27), hence

$$P_{x_0} = -\int_0^\infty e^{A^*s}C^*Cx^*(s)ds.$$

Since, for every $t\geq 0, (u^*, w^*)$ is the solution to problem

$$\sup_{w\in L^2(\boldsymbol{R}_t^+;W)} \inf_{u\in L^2(\boldsymbol{R}_t^+;U)} \Big\{\int_t^\infty |Cx+Du|_Z^2 ds,$$

$$x' = Ax + B_1w + B_2u, x(t) = x^*(t)\Big\}$$

in $\boldsymbol{R}_t^+ = (t,\infty)$; we infer that

$$Px^*(t) = -p(t),\ \text{for all } t\geq 0 \tag{4.29}$$

If we denote by $e^{A_Pt}: X\to X$ the family of operators

$$e^{A_Pt}x_0 = x^*(t),\ \forall t\geq 0 \tag{4.30}$$

where x^* is defined by (4.26) (i.e., x^* is the optimal state in (4.25)), it is readily seen that e^{A_Pt} is a C_0– semigroup on X.

Because of (4.28) and (4.29) we may write (4.30) as

$$e^{A_P t}x_0 = e^{At}x_0 + \int_0^t e^{A(t-s)}[-B_2B_2^*Px^*(s) + \gamma^{-2}B_1B_1^*Px^*(s)]ds.$$

Let $x_0, z_0 \in D(A)$. Then

$$\begin{aligned}
(4.31)\quad (Px_0, z_0) &= \int_0^\infty (e^{A^*s}C^*Cx^*(s), z_0)ds \\
&= \int_0^\infty (Ce^{A_P s}x_0, Ce^{As}z_0)_Z ds = \int_0^\infty (Ce^{A_P s}x_0, Ce^{A_P s}z_0)_Z ds \\
&+ \int_0^\infty (Ce^{A_P s}x_0, C\int_0^s e^{A(s-\tau)}(B_2B_2^*Pe^{A_P^*\tau}z_0 - \gamma^{-2}B_1B_1^*Pe^{A_P^*\tau}z_0)d\tau)_Z ds \\
&= \int_0^\infty (Ce^{A_P s}x_0, Ce^{A_P s}z_0)_Z ds \\
&+ \int_0^\infty (\int_\tau^\infty (B_2^*e^{A^*(s-\tau)}C^*Ce^{A_P s}x_0, B_2^*Pe^{A_P^*\tau}z_0)_U ds)d\tau \\
&-\gamma^{-2}\int_0^\infty (\int_\tau^\infty (B_1^*C^*Ce^{A_P s}x_0, B_1^*Pe^{A_P^*\tau}z_0)_W ds)d\tau \\
&= \int_0^\infty (Cx^*(s), Cz^*(s))_Z ds + \int_0^\infty (u^*(s,x_0), u^*(s,z_0))_U ds \\
&-\gamma^2\int_0^\infty (w^*(s,x_0), w^*(s,z_0))_W ds = (x_0, Pz_0).
\end{aligned}$$

Now the later implies

$$\begin{aligned}
0 &\le (P(x_0 - z_0), x_0 - z_0) \\
&= (Px_0, x_0) + (Pz_0, z_0) - 2(Px_0, z_0),\ \forall x_0, z_0 \in D(A).
\end{aligned}$$

and since

$$(Px_0, x_0) \le \alpha |x_0|_X^2$$

from (4.31) follows

$$(Px_0, z_0) \le \alpha,\ \forall x_0, z_0 \in D(A),\quad |x_0|_X = |z_0|_X = 1$$

hence, P can be extended by linearity and continuity as a bounded operator in X.

Thus, we have proved that $P \in L(X,X), P = P^*$ and $||P|| \leq \alpha$. Next, by (4.29) and for $x \in D(A)$ we have

$$Px = \int_t^\infty e^{A_P^*(s-t)} C^* C e^{A(s-t)} x ds \tag{4.32}$$

Multiplying (4.32) by $y \in D(A)$ and differentiating with respect to t one obtains

$$\begin{aligned}
0 &= (Cx, Cy)_Z + \int_t^\infty (Ce^{A(s-t)}Ax, Ce^{A_P(s-t)}y)_Z ds \\
&+ \int_t^\infty (Ce^{A(s-t)}x, CA_P e^{A_P(s-t)}y)_Z ds \\
&= (Cx, Cy)_Z + (Ax, Py) \\
&+ \int_t^\infty ((A - B_2B_2^*P + \gamma^{-2}B_1B_1^*P)e^{A_P^*(s-t)}C^*Ce^{A(s-t)}x, y) ds \\
&= (Cx, Cy)_Z + (Ax, Py) + (Px, Ay) - (B_2^*Px, B_2^*Py)_U + \gamma^{-2}(B_1^*Px, B_1^*Py)_W
\end{aligned}$$

which proves (2.1). Now, from Proposition 4.1 and Lemma 4.3 (see also [5],[9]) it follows that $A_P := A - (B_2B_2^* - \gamma^{-2}B_1B_1^*)P$ generates an exponentially stable semigroup on X, thus concluding the proof of the only if part.

Remark 4.1. Our result can be easily extended to the case of complex Hilbert spaces. This analogus result can be applied to the Schrödinger equation.

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