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# ON THE OSCILLATION OF PERTURBED SECOND ORDER NONLINEAR DAMPED DIFFERENTIAL EQUATIONS

BY

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**1. Introduction.** In this paper we are concerned with the problem of oscillation of nonlinear perturbed second order ordinary differential equation with damping

$$(1) \qquad (r(t)x'(t))' + h(t)f(x(t))x'(t) + \psi(t, x(t)) = H(t, x(t), x'(t))$$

where $('= d/dt)$, $f : \boldsymbol{R} \longrightarrow \boldsymbol{R}$, $r$, $h : [t_0, \infty) \longrightarrow \boldsymbol{R} = (-\infty, +\infty)$, $t_0 \geq 0$ and $\psi : [t_0, \infty) \times \boldsymbol{R} \longrightarrow \boldsymbol{R}$, $H : [t_0, \infty) \times \boldsymbol{R} \times \boldsymbol{R} \longrightarrow \boldsymbol{R}$ are continuous functions, $r(t) > 0$ for $t \geq t_0$. For all $x \neq 0$ and for $t \in [t_0, \infty)$ we assume that there exists continuous functions $g : \boldsymbol{R} \longrightarrow \boldsymbol{R}$ and $p, q : [t_0, \infty) \longrightarrow \boldsymbol{R}$ such that

$$(c_1) \qquad xg(x) > 0 \ \text{ and } \ g'(x) \geq k > 0 \ \text{ for } \ x \neq 0,$$

$$\psi(t, x)/g(x) \geq q(t) \ \text{ and } \ H(t, x, x')/g(x) \leq p(t) \text{ for } \ x \neq 0.$$

Throughout this paper, we restrict our attention only to the solutions of equation (1) which exist on some ray $[t_0, \infty)$. Such a solution is said to be oscillatory if it has an infinite number of zeros, and otherwise it said to be nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

In the last two decades the problem of finding sufficient conditions for the oscillaion of all solutions of ordinary differential equations has begun to receive more and more attention. An interesting case is that of establishing oscillation criteria for the perturbed equation (1) with damping and/or related equations which involve the average behaviour of the integral of the alternating coefficient $g(t)$. As contribution to this study we refer to the papers of B u t l e r [1], E l a b b a s y [2], G r a c e et al. [3], [4] and [5], G r a e f et al. [6], N a g a b u ch i et al. [7], W o n g [8], Y a n [9] and Y e h [10] and the references cited in.

We say that (1) is sublinear if $g(x)$ satisfies

$$(c_2) \qquad 0 < \int_0^{\varepsilon} \frac{du}{g(u)} < \infty,\ 0 < \int_0^{-\varepsilon} \frac{du}{g(u)} < \infty,\ \varepsilon > 0.$$

(1) is superlinear if $g(x)$ satisfies

$$(c_3) \qquad 0 < \int_{\varepsilon}^{\infty} \frac{du}{g(u)} < \infty,\ 0 < \int_{-\varepsilon}^{-\infty} \frac{du}{g(u)} < \infty,\ \varepsilon > 0.$$

(1) is a mixed type if $g(x)$ satisfies

$$(c_4) \qquad 0 < \int_0^{\infty} \frac{du}{g(u)} < \infty,\ 0 < \int_0^{-\infty} \frac{du}{g(u)} < \infty,\ \varepsilon > 0,$$

We see easily that, if $g(x) = g_1(x) + g_2(x)$ where $g_1$ is sublinear and $g_2$ is superlinear, then $(c_4)$ is satisfied.

Y e h [10] considered equation (1) with $r(t) \equiv 1,\ f(x) \equiv 1,\ p(t) \equiv 0$ that is

$$(2) \qquad x''(t) + h(t)x'(t) + q(t)g(x) = 0$$

where $h, q \in c[t_0, \infty)$ and $xg(x) > 0$ and $g'(x) \geq k > 0$ for $x \neq 0$ and proved that

$$(c_5) \qquad \limsup_{t\to\infty} \int_{t_0}^{t} (t-s)^{n-1} s\ q(s)ds = \infty,$$

$$(c_6) \qquad \lim_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} s[(t-s)(h(s) - \frac{1}{n}) + n - 1]^2 (t-s)^{n-3} ds < \infty$$

for some integer $n \geq 3$ are sufficient conditions for the oscillation of (2).

In [7] N a g a b u c h i et al. has extended and improves Yeh's result to the equation

$$(3) \qquad (r(t)x'(t))' + h(t)x'(t) + q(t)g(x) = 0$$

where $g'(x) \geq k > 0$ for $x \neq 0$ and proved that (3) is oscillatory if there exists a continuously differentiable function $\rho(t)$ on $[t_0, \infty)$ and a constant $\alpha \in (1, \infty)$ such that

$$(c_7) \qquad \lim_{t\to\infty} \frac{1}{t^\alpha} \int_{t_0}^{t} (t-s)^\alpha \rho(s)q(s) - \frac{1}{4k}[(t-s)\frac{\rho(s)h(s)}{r(s)} + \alpha\rho(s) - -(t-s)\rho'(s)]^2 (t-s)^{\alpha-2} \frac{r(s)}{\rho(s)} ds = \infty.$$

If $g(x) \equiv x$ and $k = 1$ the result of Nagabuchi imply Yan's result in [10].

The purpose of this paper is to contribute further in this direction and establishing sufficient conditions for the oscillation of a broad class of second order nonlinear equations of the type (1) and as a consequence we are able to extend and improve a number of previously known oscillation results. Our results are fairly sharp be illustrated by some examples.

## 2. Main results.

**Theorem 1.** *Suppose $(c_1)$ and $(c_2)$ hold. Furthermore, assume that*

$$(c_8) \qquad f(x) \geq -c, \; c > 0 \;\; \text{for} \;\; x \in R,$$

$(c_9)$ *$r(t)$ is bounded for $t \in [t_0, \infty)$ i.e. $, 0 < r(t) \leq a, \; a > 0,$*

$(c_{10})$ *there exists a continuously differentiable function $\rho(t)$ on $[t_0, \infty)$ such that $\rho(t) > 0, \; \rho'(t) \geq 0$ and $\rho''(t) \leq 0$ on $[t_0, \infty),$ and*

$$\gamma(t) = \rho'(t)r(t) + c\rho(t)h(t) \geq 0, \; \gamma'(t) \leq 0 \;\; \text{for} \;\; t \geq t_0,$$

$$(c_{11}) \qquad \liminf_{t\to\infty} \int_{t_0}^{t} \rho(s)(q(s) - p(s))ds > -\infty$$

$$(c_{12}) \qquad \limsup_{t\to\infty} \left( \int_{t_0}^{t} \frac{1}{\rho(s)} ds \right)^{-1} \int_{t_0}^{t} \frac{1}{\rho(s)} \int_{t_0}^{s} \rho(u)(q(u) - p(u))duds = \infty.$$

*Then equation (1) is oscillatory.*

**Proof.** Let $x$ be a nonoscillatory solution of the differential equation (1). Without loss of generality, we may assume that $x(t) > 0$ on $[T, \infty)$ for some $T \geq t_0$ (the case $x(t) < 0$ can be treated similarly). Define

$$w(t) = \rho(t)r(t)x'(t)/g(x(t)) \quad \text{for all} \quad t \geq T.$$

This and (1) imply

$$w'(t) \leq \rho(t)(p(t) - q(t)) + \frac{\gamma(t)x'(t)}{g(x)} - \rho(t)\frac{\rho(t)r(t)x'^2(t)g'(x)}{g^2(x)}.$$

Hence, for all $t \geq T$, we have

$$\int_T^t \rho(s)(q(s) - p(s))ds \leq -w(t) + w(T) + \int_T^t \frac{\gamma(s)x'(s)ds}{g(x(s))} -$$

$$-\int_T^t \frac{w^2(s)g'(x(s))}{r(s)\rho(s)}ds. \tag{4}$$

The first integral in R.H.S of the above in equality is bounded above. This can be seen by applying the Bonnet theorem, for each $t \geq T$ there exists $\xi \in [T, t]$ such that

$$\int_T^t \frac{\gamma(s)x'(s)ds}{g(x(s))} = \gamma(T)\int_{x(T)}^{x(\xi)} \frac{du}{g(u)}.$$

Since

$$\int_{x(T)}^{x(\xi)} \frac{du}{g(u)} \leq \begin{cases} 0, & \text{if } x(\xi) < x(t) \\ \displaystyle\int_{x(T)}^{\infty} \frac{du}{g(u)}, & \text{if } x(\xi) \geq x(T) \end{cases}$$

and $\gamma(T) > 0$, it follows that

$$\int_T^t \frac{\gamma(s)x'(s)ds}{g(x(s))} \leq k_1, \quad \text{where} \quad k_1 = \gamma(T)\int_{x(T)}^{\infty} \frac{du}{g(u)}.$$

Hence, we have from (4)

$$\int_T^t \rho(s)(q(s) - p(s))ds \leq -w(t) + k_2 - \int_T^t \frac{w^2(s)g'(x(s))}{r(s)\rho(s)}ds \tag{5}$$

where $k_2 = k_1 + w(T)$.
Or, by virtue of condition $(c_1)$

$$(6) \qquad \int_T^t \rho(s)(q(s)-p(s))ds \leq -w(t) + k_2 - k\int_T^t \frac{w^2(s)}{r(s)\rho(s)}ds.$$

Now, we consider the behaviour of $x'$:

**Case 1.** $x'$ is oscillatory. Then, there exists an infinite sequence $\{t_n\}$ such that $t \to \infty$ as $n \to \infty$ and $x'(t_n) = 0$. Thus, (6) gives

$$\int_T^{t_n} \rho(s)(q(s)-p(s))ds \leq k_2 - k\int_T^{t_n} \frac{w^2(s)}{r(s)\rho(s)}ds.$$

By virtue of condition $(c_{11})$, wee get $w^2(t)/r(t)\rho(t) \in L^2(T,\infty)$. Thus, there exists a positive constant $N$ such that

$$\int_T^t \frac{w^2(s)}{r(s)\rho(s)}ds \leq N \qquad \text{for every } t \geq T.$$

For $t \geq T$, we may use the Schwarz inequality to obtain

$$\left|-\int_T^t \frac{w(s)}{\rho(s)}ds\right|^2 = \left|\int_T^t \left(\frac{r(s)}{\rho(s)}\right)^{1/2} [w(s)(r(s)\rho(s))^{1/2}]ds\right|^2 \leq$$

$$\leq \int_T^t \frac{r(s)}{\rho(s)}ds \int_T^t \frac{w^2(s)}{r(s)\rho(s)}ds \leq N\int_T^t \frac{r(s)}{\rho(S)}ds \leq Na\int_T^t \frac{ds}{\rho(s)}.$$

Hence, for every $t \geq T$ we have

$$-\int_T^t \frac{w(s)}{\rho(s)}ds \leq \sqrt{Na}\left(\int_T^t \frac{ds}{\rho(s)}\right)^{1/2}$$

Furthermore, (6) gives

$$\int_T^t \rho(s)(q(s)-p(s))ds \leq -w(t) + k_2$$

hence,

$$(7) \qquad \int_T^t \frac{1}{\rho(s)}\int_T^s \rho(u)(q(u)-p(u))duds \leq -\int_T^t \frac{w(s)}{\rho(s)}ds + k_2\int_T^t \frac{ds}{\rho(s)}.$$

Condition $(c_{10})$ implies that $\rho(t) \leq \mu(t)$ for all large $t$, where $\mu$ is positive constant. This ensures that $\int_{t_0}^{\infty} \frac{dt}{\rho(t)} = \infty$. Hence, we get from (7) for $t \geq T$

$$\int_T^t \frac{1}{\rho(s)} \rho(u)(q(u) - p(u))duds \leq \sqrt{Na} \left( \int_T^t \frac{ds}{\rho(s)} \right)^{1/2} +$$

$$+k_2 \int_T^t \frac{ds}{\rho(s)} \leq \left( \sqrt{Na} + k_2 \right) \int_T^t \frac{ds}{\rho(s)}.$$

Dividing by $\int_T^t \frac{ds}{\rho(s)}$ and then taking the upper limit gives a contradiction to $(c_{12})$.

**Case 2.** $x'(t) > 0$ for $t \geq T_1 > t_0$. Then, it follows from (6) that

$$\int_{T_1}^t \rho(s)(q(s) - p(s))ds \leq k_2$$

and consequently

$$\left( \int_T^t \frac{ds}{\rho(s)} \right)^{-1} \int_T^t \frac{1}{\rho(s)} \int_T^s \rho(u)(q(u) - p(u))duds \leq k_2$$

contradicts $(c_{12})$.

**Case 3.** $x'(t) < 0$ for $t \geq T_2 > t_0$. If $w^2(t)/r(t)\rho(t) \in L^2(T_2, \infty)$, then we can follow the procedure of Case 1 to arrive at a contradiction to $(c_{12})$. Suppose now $w^2(t)/r(t)\rho(t) \notin L^2(T_2, \infty)$. By virtue of $(c_{11})$, we get from (5) for some constant $\beta > 0$

$$-w(t) \geq \beta + \int_{T_2}^t \frac{w^2(s)g'(x(s))}{r(s)\rho(s)} ds \quad \text{for every} \quad t \geq T_2. \tag{8}$$

Since $w^2(t)/r(t)\rho(t) \notin L^2(T_2, \infty)$, there exists $T_3 \geq T_2$ that

$$M = \beta + \int_{T_2}^{T_3} \frac{w^2(s)g'(x(s))}{r(s)\rho(s)} ds > 0.$$

Thus (8) ensures $w(t)$ is negative on $[T_2, \infty)$. By using (8), we have from (5) for every $t \geq T_3$

$$-w(t) \left\{ \beta + \int_{T_2}^t \frac{w^2(s)g'(x(s))}{r(s)\rho(s)} ds \right\}^{-1} \geq 1$$

or

$$\frac{w^2(t)g'(s)}{r(t)\rho(t)}\left\{\beta+\int_{T_2}^{t}\frac{w^2(s)g'(x(s))}{r(s)\rho(s)}ds\right\}^{-1}\geq\frac{x'(s)g'(x)}{g(x)}.$$

An integration from $T_3$ to $t$ yields

$$\log\left[\left\{\beta+\int_{T_2}^{t}\frac{w^2(s)g'(x(s))}{r(s)\rho(s)}\right\}/M\right]\geq\log\left[\frac{g(x(T_3))}{g(x)}\right].$$

Then,

$$\beta+\int_{T_2}^{t}\frac{w^2(s)g'(x(s))}{r(s)\rho(s)}ds\geq M'g(x)\quad\text{for every}\quad t\geq T_3$$

where $M'=Mg(x(T_3))>0$.
Hence, from (8) we get

$$-w(t)\geq M'g(x)$$

i.e.

$$x'(t)\leq -M'/r(t)\rho(t).$$

So,

$$x(t)\leq x(T_3)-M'\int_{T_3}^{t}\frac{ds}{r(s)\rho(s)}\leq x(T_3)-\frac{M'}{a}\int_{T_3}^{t}\frac{ds}{\rho(s)},$$

it follows, that $x(t)\to-\infty$ as $t\to\infty$, a contradiction. This completes the proof.

The function $\rho$ satifies $(c_{10})$ may be taken as $\rho(t)=t^{\alpha},\ \alpha\in[0,1]$. Thus we have the following corollary of theorem 1.

**Corollary 1.** *Equation is oscillatory if* $(c_1),\ (c_3)$ *hold,* $\gamma(t)=\alpha t^{\alpha-1}r(t)+ct^{\alpha}h(t)\geq 0$ *and* $\gamma'(t)\leq 0$ *for some* $\alpha\in[0,1]$,

$$(c_{11})'\qquad \liminf_{t\to\infty}\int_{t_0}^{t}s^{\alpha}(q(s)-p(s))ds>-\infty,$$

$$(c_{12})'\qquad \begin{cases}\lim\limits_{t\to\infty}\frac{1}{\log t}\int_{t_0}^{t}\frac{1}{s}\int_{t_0}^{s}u(q(u)-p(u))duds=\infty, \text{if } \alpha=1\\ \lim\limits_{t\to\infty}\frac{1}{t^{1-\alpha}}\int_{t_0}^{t}\frac{1}{s^{\alpha}}\int_{t_0}^{s}u^{\alpha}(q(u)-p(u))duds=\infty,\ \text{if } 0\leq\alpha<1.\end{cases}$$

**Example 1.** Clearly, with $\rho(t)\equiv 1$ for $t\geq t_0>0$, the Van der Pol Oscillator with nonlinear restoring force

$$x''(t)+N(x^2(t)-1)x'(t)+x^3(t)=0,\ N>0,\ t>0$$

satisfies the above hypotheses. This result can not be obtained by any other existing oscillation criteria.

In the following theorem we do not assume that $g(x)$ satisfies condition $(c_3)$. So, it may be applicable for linear, sublinear or superlinear differential equations.

**Theorem 2.** *Suppose that* $(c_8)$ *holds. Moreover, assume that*

$$(c_{13}) \qquad xg(x) > 0 \ \ \text{and} \ \ g'(x) \geq 0 \ \ \text{for all} \ \ x \neq 0,$$

$(c_{14})$ *there exists a differentiable function* $\phi : [t_0, \infty) \to (0, \infty)$ *and the continuous functions*

$$h, H: \ D \equiv \{(t,s): \ t \geq s \geq t_0\} \longrightarrow \boldsymbol{R}$$

*and* $H$ *has a continuous and nonpositive partial derivative on* $D$ *with respect to the second variable such that*

$$H(t,t) = 0 \ \ \text{for} \ \ t \geq t_0, \ \ H(t,s) > 0 \ \text{for} \ \ t > s \geq t_0$$

*and*

$$(c_{15}) \qquad -\frac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \ \ \text{for all} \ \ (t,s) \in D.$$

*Then equation (1) is oscillatory if*

$$(c_{16}) \qquad \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big\{ \phi(s)H(t,s)(q(s)-p(s)) -$$

$$-\frac{1}{4k}\left[r(s)\phi(s)(h(t,s) - \left(\frac{ch(s)}{r(s)} + \frac{\phi'(s)}{\phi(s)}\right)\sqrt{H(t,s)})\right]^2 \Big\} ds = \infty$$

**Proof.** Let $x(t)$ be a nonoscillatory solution of (1) and assume that $x(t) > 0$ for $t \geq t_0$. Define $w(t) = \frac{\phi(t)r(t)x'(t)}{g(x(t))}$ for $t \geq t_0$. This and equation (1) imply

$$w'(t) \leq \phi'(t)(p(t) - q(t)) + \left(\frac{ch(t)}{r(t)} + \frac{\phi'(t)}{\phi(t)}\right) w(t) - \frac{kw^2(t)}{r(t)\phi(t)}.$$

Hence, for all $t \geq t_0$, we have

$$\int_{t_0}^{t} \phi(s)H(t,s)(q(s)-p(s))ds \leq -\int_{t_0}^{t} H(t,s)w'(s)ds+$$

$$+\int_{t_0}^{t} H(t,s)\bigg(\frac{ch(s)}{r(s)}+\frac{\phi(s)}{\phi(s)}\bigg)w(s)ds - k\int_{t_0}^{t} \frac{H(t,s)w^2(s)}{r(s)\phi(s)}ds =$$

$$= H(t,t_0)w(t_0) - \int_{t_0}^{t} \left(\frac{-\partial H}{\partial s}\right) w(s)ds + \int_{t_0}^{t} H(t,s)\bigg(\frac{ch(s)}{r(s)}+$$

$$+\frac{\phi'(s)}{\phi(s)}\bigg)w(s)ds - k\int_{t_0}^{t} \frac{H(t,s)w^2(s)}{r(s)\phi(s)}ds =$$

$$= H(t,t_0)w(t_0) - \int_{t_0}^{t} \left[ \left( h(t,s)\sqrt{H(t,s)} - H(t,s)\left(\frac{ch(s)}{r(s)}+\frac{\phi'(s)}{\phi(s)}\right)\right) w(s)+\right.$$

$$\left. +\frac{kH(t,s)w^2(s)}{r(s)\phi(s)}\right] ds =$$

$$= H(t,t_0)w(t_0) - \int_{t_0}^{t} \left[ \sqrt{\frac{kH(t,s)}{r(s)\phi(s)}}w(s) - \frac{\sqrt{r(s)\phi(s)}}{2\sqrt{k}}\Bigg(h(t,s)-\right.$$

$$\left.\left. -\sqrt{H(t,s)}\left(\frac{ch(s)}{r(s)}+\frac{\phi'(s)}{\phi(s)}\right)\right)\right]^2 ds+$$

$$+\frac{1}{4k}\int_{t_0}^{t} r(s)\phi(s)\left(h(t,s) - \sqrt{H(t,s)}\left(\frac{ch(s)}{r(s)}+\frac{\phi'(s)}{\phi(s)}\right)\right)^2 ds \leq$$

$$\leq H(t,t_0)w(t_0) + \frac{1}{4k}\int_{t_0}^{t} r(s)\phi(s)\Bigg(h(t,s)-$$

$$\left. -\sqrt{H(t,s)}\left(\frac{ch(s)}{r(s)}+\frac{\phi'(s)}{\phi(s)}\right)\right)^2 ds.$$

Hence,

$$\frac{1}{H(t,t_0)}\int_{t_0}^{t} \left[ H(t,s)\phi(s)(q(s)-p(s)) - \frac{r(s)\phi(s)}{4k}\Bigg(h(t,s)-\right.$$

$$\left.\left. -\sqrt{H(t,s)}\left(\frac{ch(s)}{r(s)}+\frac{\phi'(s)}{\phi(s)}\right)\right)^2\right] ds \leq w(t_0)$$

a contradiction to $(c_{16})$. This completes the proof.

**Corollary 2.** *Suppose that condition* $(c_{16})$ *in Theorem 2 be replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \phi(s)H(t,s)(q(s)-p(s))ds = \infty, \quad \textit{and}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} r(s)\phi(s)\left(h(t,s)-\sqrt{H(t,s)}\left(\frac{ch(s)}{r(s)}+\frac{\phi'(s)}{\phi(s)}\right)\right)^2 ds \le \infty$$

*then the conclusion of Theorem 2 holds.*

**Remark.** The functions $H(t,s)=(t-s)^{n-1}$, $t\ge s\ge t_0$, where $n$ is an integer with $n>2$, and $h(t,s)=(n-1)(t-s)^{(n-3)/2}$, $t\ge s\ge t_0$ are continuous and satisfy $(c_{15})$. Therefore, the results in [7], [9] and [10] can be obtained from Theorem 2 as special cases.

The following theorem is devoted to equation (1) with $g(x)$ satisfies $(c_4)$ i.e. of mixed type.

**Theorem 3.** *Suppose that* $(c_4)$, $(c_8)$ *and* $(c_{13})$ *hold. Furthermore, assume that*

$(c_{17})$
$$r'(t)\le 0 \quad \textit{for} \quad t\ge t_0,$$

$(c_{18})$ *There exists a continuously differentiable function* $\rho:[t_0,\infty)\to(0,\infty)$, $\rho'(t)\ge 0$ *and* $\rho''(t)\le 0$; *and*

$(c_{19})$
$$\sigma(t)=2\rho'(t)r(t)+ch(t)\rho(t)\ge 0 \quad \textit{and} \quad \sigma'(t)\le 0 \quad \textit{for} \quad t\ge t_0.$$

*Then equation (1) is oscillatory if*

$(c_{20})$
$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1}\rho(s)(q(s)-\rho(s))ds = \infty, \quad \textit{for} \quad n\ge 2.$$

**Proof.** Let $x(t)$ be a nonoscillatory solution of (1) and assume that $x(t)>0$ for $t\ge 0$. Define $w(t)=\rho(t)\int_{t_0}^{t}\frac{r(s)x'(s)}{g(x(s))}ds$, $t\ge t_0$. Therefore $w(t)$ is well defined for $t\ge t_0$. From this and equation (1) imply

$$w''(t)\le \rho(t)(p(t)-q(t))+\frac{\rho(t)x'(t)}{g(x(t))}+\rho''(t)\int_{t_0}^{t}\frac{r(s)x'(s)ds}{g(x(s))}.$$

But, by the Bonnet theorem, for a fixed $t\ge t_0$ and some $\xi[t_0,t]$

$$\int_{t_0}^{t}\frac{r(s)x'(s)}{g(x(s))}ds = r(t_0)\int_{t_0}^{\xi}\frac{x'(s)ds}{g(x(s))} = r(t_0)\int_{x(t_0)}^{x(\xi)}\frac{du}{g(u)}$$

and hence, since $r(t_0) > 0$ and

$$\int_{x(t_0)}^{x(\xi)} \frac{du}{g(u)} = \int_{x(t_0)}^{0} \frac{du}{g(u)} + \int_{0}^{x(\xi)} \frac{du}{g(u)} > \int_{x(t_0)}^{0} \frac{du}{g(u)} = -k_1;$$

where $k_1 = \displaystyle\int_0^{x(t_0)} \frac{du}{g(u)}$, we have $\displaystyle\int_{t_0}^{t} \frac{r(s)x'(s)ds}{g(x(s))} \geq k_1 r(t_0) = -k_2$.
Therefore,

$$w''(t) \leq \rho(t)(p(t) - q(t)) + \frac{\sigma(t)x'(t)}{g(x(t))} - k_2\rho''(t).$$

Hence, for all $t \geq t_0$, we have

$$\int_{t_0}^{t} (t-s)^{n-1}\rho(s)(q(s) - p(s))ds \leq -\int_{t_0}^{t} (t-s)^{n-1} w''(s)ds +$$

$$+\int_{t_0}^{t} \frac{(t-s)^{n-1}\rho(s)x'(s)ds}{g(x(s))} - k_2\int_{t_0}^{t} (t-s)^{n-1}\rho''(s)ds =$$

$$= I_1 + I_2 + I_3.$$

The integrals $I_i,\ i = 1, 2, 3$ can be estimated as follows:
If $n = 2$, then

$$I_1 = -\int_{t_0}^{t} (t-s)w''(s)ds = (t-t_0)w'(t_0) - w(t) \leq$$

$$\leq (t-t_0)w'(t_0) + |w(t)| \leq (t-t_0)w'(t_0) + r(t_0)\rho(t)\int_0^{\infty} \frac{du}{g(u)}$$

condition $(c_{18})$ implies that $\rho(t) \leq \mu t,\ \mu > 0$. Thus

$$I_1 \leq (t-t_0)w'(t_0) + (\mu r(t_0)\int_0^{\infty} \frac{du}{g(u)})t.$$

If $n > 2$, then $I_1 = -\int_{t_0}^{t} (t-s)^{n-1} w''(s)ds =$

$$= (t-t_0)^{n-1}w'(t_0) - (n-1)(n-2)\int_{t_0}^{t} (t-s)^{n-3} w(s)ds \leq$$

$$\leq (t-t_0)^{n-1}w'(t_0) + (n-1)(n-2)\int_{t_0}^{t}(t-s)^{n-3}|w(s)|ds.$$

Hence

$$I_1 \leq (t-t_0)^{n-1}w'(t_0) + (n-1)(n-2)\mu r(t_0)\int_0^\infty \frac{du}{g(u)}\int_{t_0}^{t} s(t-s)^{n-3}ds \leq$$

$$\leq (t-t_0)^{n-1}w'(t_0) + \mu r(t_0)\int_0^\infty \frac{du}{g(u)}\cdot (t-t_0)^{n-1} =$$

$$= (w'(t_0) + \mu r(t_0)\int_0^\infty \frac{du}{g(u)}(t-t_0)^{n-1}, \quad \text{for} \quad n > 2.$$

For $I_2$, by using the Bonnet theorem, for a fixed $t \geq t_0$ and some $\eta \in [t_0, t]$

$$I_2 = \int_{t_0}^{t}\sigma(s)(t-s)^{n-1}\frac{x'(s)}{g(x(s))}ds = \sigma(t_0)(t-t_0)^{n-1}\int_{x(t_0)}^{x(\eta)}\frac{du}{g(u)} \leq$$

$$\leq \sigma(t_0)(t-t_0)^{n-1}\int_0^\infty \frac{du}{g(u)}.$$

For $I_3 = -k_2\int_{t_0}^{t}(t-s)^{n-1}\rho''(s)ds=$

$$= k_2(t-t_0)^{n-1}\rho'(t_0) - k_2(n-1)\int_{t_0}^{t}(t-s)^{n-1}\rho'(s)ds \leq (t-t_0)^{n-1}\rho'(t_0).$$

Therefore, taking into account the above estimation for $I_i;\ i = 1,2,3$ we conclude

$$\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}\rho(s)(q(s)-p(s))ds \leq (w'(t_0) + \mu r(t_0)\int_0^\infty \frac{du}{g(u)} +$$

$$+\sigma(t_0)\int_0^\infty \frac{du}{g(u)} + k_2\rho'(t_0)) - \left(1 - \frac{t_0}{t}\right)^{n-1}.$$

Taking the upper limit as $t \to \infty$, a contradiction. This completes the proof.

Theorem 3 can be used to some cases for which some other oscillation criteria can not be used. For example, consider the differential equation

$$(t^\lambda x'(t))' + (t\sin t + \frac{1}{t} + t^2)(x^3(t) + x^{1/3}(t)) =$$

$$\frac{\cos t \sin x'(t)(x^5(t) + x^{7/3}(t))}{t(x^2(t)+1)}$$

where $\lambda \leq 0,\ t \geq t_0 = \pi/2$ let $g(x) = x^3(t) + x^{1/3}(t)$, we see that $\psi(t,x) > t\sin t + \dfrac{1}{t} = q(t)$, and $H(t,x,x')/g(x) \leq \dfrac{1}{t} = p(t)$. Indeed, we have from

$$\limsup_{t\to\infty} \frac{1}{t}\int_{\pi/2}^{t} (t-s)\cdot s\cdot s\sin s ds = \limsup_{t\to\infty}\left\{ s^2 \sin s ds - \frac{1}{t}\int_{\pi/2}^{t} s^3 \sin s ds \right\} =$$

$$= \limsup_{t\to\infty}\left\{ -t\sin t - 4\cos t - \frac{6}{t} + \frac{6\sin t}{t} - \pi + \frac{3\pi^2}{4}\right\} = \infty$$

and theorem 3 that the above equation is oscillatory. That this equation is oscillatory does not appear to be deducible from other known oscillation criteria.

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