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PERIODIC AND ALMOST-PERIODIC SOLUTIONS FOR A CLASS OF NONLINEAR HYPERBOLIC SYSTEMS

BY

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1. Introduction. In this paper we shall investigate the existence of periodic solutions and almost-periodic solutions of the hyperbolic partial differential system:

$$\text{(S)} \qquad \begin{cases} \frac{\partial i_k}{\partial t} + \frac{\partial v_k}{\partial x} + \alpha_k(x, i_k) = 0 \\ \frac{\partial v_k}{\partial t} + \frac{\partial i_k}{\partial x} + \beta_k(x, v_k) = 0 \end{cases}$$

$$0 < x < 1,\ t > 0,\ k = \overline{1, n}$$

with the boundary condition:

$$\begin{pmatrix} col(i_1(t,0), -i_1(t,1), \ldots, i_n(t,0), -i_n(t,1)) \\ S\left(col\left(\frac{dw_1}{dt}(t), \ldots, \frac{dw_m}{dt}(t)\right)\right) \end{pmatrix} \in$$

$$\text{(BC)} \qquad \in -G\begin{pmatrix} col(v_1(t,0), v_1(t,1), \ldots, v_n(t,0), v_n(t,1)) \\ col(w_1(t), \ldots, w_m(t)) \end{pmatrix} +$$

$$+\begin{pmatrix} col(b_1(t), b_2(t), \ldots, b_{2n-1}(t), b_{2n}(t)) \\ col(b_{2n+1}(t), \ldots, b_{2n+m}(t)) \end{pmatrix}, \quad t > 0.$$

This problem occurs in integrated circuits modelling. The existence, uniqueness and asymptotic behaviour of strong or weak solutions of the problem (S), (BC) with the initial data:

$$\text{(IC)} \qquad \begin{cases} i_k(0,x) = i_{k0}(x),\ v_k(0,x) = v_{k0}(x), & 0 < x < 1,\ k = \overline{1,n} \\ w_j(0) = w_{j0}, & j = \overline{1,m}. \end{cases}$$

have been investigated in [7,8]. Other nonlinear systems with boundary conditions of this type and different examples have been studied in [5,6].

2. Preliminary results. In all which follows we denote by (A1), (A2), (A3) the following assumptions:

$$\text{(A1)} \quad \begin{cases} \text{(a) The functions } x \to \alpha_k(x,p) \text{ and } x \to \beta_k(x,p) \text{ are in} \\ \quad L^2(0,1), \text{ for any fixed } p \in I\!R. \text{ Besides the functions} \\ \quad p \to \alpha_k(x,p) \text{ and } p \to \beta_k(x,p) \text{ are continuous} \\ \quad \text{on } I\!R, \text{ for a.e. } x \in (0,1),\ k \in \overline{1,n}. \\ \text{(b) The functions } p \to \alpha_k(x,p) \text{ and } p \to \beta_k(x,p) \text{ are} \\ \quad \text{nondecreasing from } I\!R \text{ into } I\!R \text{ , for a.e. } x \in (0,1),\ k \in \overline{1,n}. \\ \text{(c) There exist } a_1 \text{ and } a_2 > 0 \text{ such that:} \\ \qquad (\alpha_k(x,p_1) - \alpha_k(x,p_2))(p_1 - p_2) \geq a_1(p_1 - p_2)^2, \\ \qquad (\beta_k(x,p_1) - \beta_k(x,p_2))(p_1 - p_2) \geq a_2(p_1 - p_2)^2, \\ \quad \text{for a.e. } x \in (0,1),\ \forall p_1, p_2 \in I\!R,\ k = \overline{1,n}. \end{cases}$$

(A2)

$$\begin{cases} \text{(a) } G\colon D(G) \subset I\!R^{2n+m} \to I\!R^{2n+m} \text{ is a maximal monotone mapping} \\ \quad \text{(possibly multivalued), } D(G) \neq \emptyset. \text{ Moreover, } G \text{ can be split in:} \\ \qquad G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \\ \quad \text{where} \quad \begin{array}{l} G_{11}\colon D(G_{11}) \subset I\!R^{2n} \to I\!R^{2n}, \quad G_{12}\colon D(G_{12}) \subset I\!R^m \to I\!R^{2n}, \\ G_{21}\colon D(G_{21}) \subset I\!R^{2n} \to I\!R^m, \quad G_{22}\colon D(G_{22}) \subset I\!R^m \to I\!R^m. \end{array} \\ \text{(b) There exist } K_0 > 0 \text{ such that for all } x, y \in D(G), \\ \quad \text{and all } w_1 \in G(x),\ w_2 \in G(y): \\ \qquad < w_1 - w_2, x - y >_{I\!R^{2n+m}} \geq K_0 \|x - y\|^2_{I\!R^{2n+m}}. \end{cases}$$

$$\text{(A3)} \qquad S = diag(s_1, \ldots, s_m) \quad \text{ with } s_j > 0,\ j = \overline{1,m}.$$

We present some existence and uniqueness results for the solutions of the problem (S), (BC), (IC), obtained in [7]. We consider the following

spaces: $X = (L^2(0,1))^{2n}, I\!R^m$ and $Y = X \times I\!R^m$ with the corresponding scalar products:

$$< f, g >_X = \sum_{k=1}^{2n} \int_0^1 f_k(x) g_k(x) dx, f = col(f_1, .., f_{2n}), g = col(g_1, .., g_{2n}) \in X,$$
$$< x, y >_s = \sum_{i=1}^{m} s_i x_i y_i, \quad x, y \in I\!R^m$$
$$< \binom{f}{x}, \binom{g}{y} >_Y = < f, g >_X + < x, y >_s, \quad \binom{f}{x}, \binom{g}{y} \in Y.$$

We define the *operators*: $\mathcal{A}_1 \colon D(\mathcal{A}_1) \subset Y \to Y$ with

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in Y;\ i_k, v_k \in H^1(0,1),\ k = \overline{1,n}, \right.$$

$$\left. \begin{pmatrix} \gamma_0 v \\ w \end{pmatrix} \in D(G) \text{ and } \gamma_1 i \in -G_{11}(\gamma_0 v) - G_{12}(w) \right\}$$

where $i = col(i_1, \ldots, i_n), v = col(v_1, \ldots, v_n), w = col(w_1, \ldots w_m), \gamma_1 i = col(i_1(0), -i_1(1), \ldots, i_n(0), -i_n(1)), \gamma_0 v = col(v_1(0), v_1(1), \ldots, v_n(0), v_n(1))$,

$$\mathcal{A}_1 \begin{pmatrix} i \\ v \\ w \end{pmatrix} = \begin{pmatrix} col(v_1', v_2', \ldots, v_n') \\ col(i_1', i_2', \ldots, i_n') \\ S^{-1} G_{21}(\gamma_0 v) + S^{-1} G_{22}(w) \end{pmatrix}, \qquad \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in D(\mathcal{A}_1)$$

and $\mathcal{A}_2 \colon D(\mathcal{A}_2) \subset Y \to Y$ with $D(\mathcal{A}_2) = \left\{ \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in Y;\ \mathcal{A}_2 \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in Y \right\}$,

$$\mathcal{A}_2 \begin{pmatrix} i \\ v \\ w \end{pmatrix} = \begin{pmatrix} col(\alpha_1(\cdot, i_1), \ldots, \alpha_n(\cdot, i_n)) \\ col(\beta_1(\cdot, v_1), \ldots, \beta_n(\cdot, v_n)) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in D(\mathcal{A}_2).$$

Remark. Under the assumptions (A1)ab, (A2)a and (A3), $D(\mathcal{A}_1) \neq \emptyset$ and $D(\mathcal{A}_1) \subset D(\mathcal{A}_2)$.

Lemma 1. *If (A1)ab, (A2)a and (A3) hold, then the operator* $\mathcal{A}_1 + \mathcal{A}_2$ *is maximal monotone.*

With the change of functions $i_k = \tilde{\imath}_k + \tilde{\tilde{\imath}}_k$, where $\tilde{\tilde{\imath}}_k(t,x) = (1-x)b_{2k-1}(t) - xb_{2k}(t)$, $k = \overline{1,n}$, the problem (S), (BC), (IC) can be written as:

$$(\widetilde{\mathrm{S}}) \qquad \begin{cases} \frac{\partial \tilde{\imath}_k}{\partial t} + \frac{\partial v_k}{\partial x} + \alpha_k(x, \tilde{\imath}_k + \tilde{\tilde{\imath}}_k(t,x)) = \tilde{f}_k(t,x) \\ \frac{\partial v_k}{\partial t} + \frac{\partial \tilde{\imath}_k}{\partial x} + \beta_k(x, v_k) = \tilde{g}_k(t,x), \end{cases}$$

$$0 < x < 1,\ t > 0,\ k = \overline{1,n}$$

with the boundary value condition:

$$\begin{pmatrix} col(\tilde{\imath}_1(t,0), -\tilde{\imath}_1(t,1), \ldots, \tilde{\imath}_n(t,0), -\tilde{\imath}_n(t,1)) \\ S\left(col\left(\frac{dw_1}{dt}(t), \ldots, \frac{dw_m}{dt}(t)\right)\right) \end{pmatrix} \in$$

$$(\widetilde{\mathrm{BC}}) \qquad \in -G\begin{pmatrix} col(v_1(t,0), v_1(t,1), \ldots, v_n(t,0), v_n(t,1)) \\ col(w_1(t), \ldots, w_m(t)) \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 \\ col(b_{2n+1}(t), \ldots, b_{2n+m}(t)) \end{pmatrix}, \quad t > 0$$

and the initial data:

$$(\widetilde{\mathrm{IC}}) \qquad \begin{cases} \tilde{\imath}_k(0,x) = \tilde{\imath}_{k0}(x),\ v_k(0,x) = v_{k0}(x), & 0 < x < 1,\ k = \overline{1,n} \\ w_j(0) = w_{j0}, & j = \overline{1,m}, \end{cases}$$

$$\text{where } \begin{cases} \tilde{f}_k(t,x) = -(1-x)b'_{2k-1}(t) + xb'_{2k}(t), \\ \tilde{g}_k(t,x) = b_{2k-1}(t) + b_{2k}(t), \quad 0 < x < 1,\ t > 0, \\ \tilde{\imath}_{k0}(x) = i_{k0}(x) - (1-x)b_{2k-1}(0) + xb_{2k}(0),\ 0 < x < 1,\ k = \overline{1,n}. \end{cases}$$

Using the operators $\mathcal{A}_1$ and $\mathcal{A}_2$, the problem $(\widetilde{\mathrm{S}})$, $(\widetilde{\mathrm{BC}})$, $(\widetilde{\mathrm{IC}})$ leads us to consider in the space Y the following time dependent Cauchy problem:

$$(\mathrm{P}) \qquad \begin{cases} \frac{d}{dt}\begin{pmatrix} \tilde{\imath} \\ v \\ w \end{pmatrix} + \mathcal{A}_1\begin{pmatrix} \tilde{\imath} \\ v \\ w \end{pmatrix} + \mathcal{A}_2\begin{pmatrix} \tilde{\imath} + \tilde{\tilde{\imath}}(t) \\ v \\ w \end{pmatrix} = \begin{pmatrix} \tilde{f}(t,\cdot) \\ \tilde{g}(t,\cdot) \\ S^{-1}B_2(t) \end{pmatrix}, \quad t > 0 \\ \begin{pmatrix} \tilde{\imath}(0) \\ v(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} \tilde{\imath}^0 \\ v^0 \\ w^0 \end{pmatrix}, \end{cases}$$

where $\tilde{f} = col(\tilde{f}_1, \ldots, \tilde{f}_n)$, $\tilde{g} = col(\tilde{g}_1, \cdots \tilde{g}_n)$, $\tilde{\imath}^0 = col(\tilde{\imath}_{10}, \ldots, \tilde{\imath}_{n0})$, $B_2(t) = col(b_{2n+1}(t), \ldots, b_{2n+m}(t))$.

Theorem 1. *Assume that (A1)ab, (A2)ab and (A3) hold. If* $b_k \in W^{1,2}(0,T)$, $k = \overline{1,2n+m}$ *(*$T > 0$ *fixed),* $i_{k0}, v_{k0} \in H^1(0,1)$, $k = \overline{1,n}$, $w^0 \in I\!R^m$, $\binom{\gamma_0 v^0}{w^0} \in D(G)$ *and* $B_1(0) \in \gamma_1 i^0 + G_{11}(\gamma_0 v^0) + G_{12}(w^0)$, *then the problem (S), (BC), (IC) has a unique strong solution* $col(i,v,w) \in W^{1,\infty}(0,T;Y)$. *Moreover,* $i_k, v_k \in L^\infty(0,T;H^1(0,1))$, *hence* $i_k, v_k \in L^\infty((0,T)\times(0,1))$, $k = \overline{1,n}$, $(B_1(t) = col(b_1(t),\dots,b_{2n}(t)))$.

Theorem 2. *Assume that (A1)ab, (A2)ab and (A3) hold. If* $b_k \in L^2(0,T)$, $k = \overline{1,2n+m}$ *(with* $T > 0$ *fixed),* $i_{k0}, v_{k0} \in L^2(0,1)$, $k = \overline{1,n}$, $w^0 \in \overline{D(G_{12}) \cap D(G_{22})}$, *then the problem (S), (BC), (IC) has a unique weak solution* $col(i,v,w) \in C([0,T];Y)$.

3. Periodic solutions. In this section we shall investigate the existence of the periodic solutions of the problem (S), (BC).

Theorem 3. *Assume that (A1)ac, (A2)ab, (A3) hold and* $b_k \in L^\infty(I\!R_+)$, $k = \overline{1,2n+m}$. *Then every weak solution of the problem (S), (BC) is bounded on* $I\!R_+$.

Proof. We assume, without loss of generality that G is single-valued. From the assumptions of the theorem we deduce that the operator $\mathcal{A}_1 + \mathcal{A}_2$ is strongly monotone. Then it is a coercive operator, cosequently $R(\mathcal{A}_1 + \mathcal{A}_2) = Y$. So, the stationary problem:

$$\mathcal{A}_1 \begin{pmatrix} i \\ v \\ w \end{pmatrix} + \mathcal{A}_2 \begin{pmatrix} i \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1}$$

has a unique solution, say $col(p,q,r) \in D(\mathcal{A}_1)$.

First, we shall prove that if $b_k \in W^{1,2}_{loc}(I\!R_+) \cap L^\infty(I\!R_+)$, then every strong solution of the problem (S), (BC) is bounded on $I\!R_+$. For, let $T > 0$ be arbitrary, but fixed for the moment, $b_k \in W^{1,2}(0,T) \cap L^\infty(I\!R_+)$, $k = \overline{1,2n+m}$, $i_{k0}, v_{k0} \in H^1(0,1)$, $k = \overline{1,n}$, $w^0 \in I\!R^m$, $\binom{\gamma_0 v^0}{w^0} \in D(G)$ and $B_1(0) \in \gamma_1 i^0 + G_{11}(\gamma_0 v^0) + G_{12}(w^0)$. Then the strong solution $col(i(t), v(t), w(t))$ of the problem (S), (BC), (IC) corresponding to above

data, satisfies:

$$(2)\quad \begin{cases} \frac{d}{dt}\begin{pmatrix} i(t)\\ v(t)\\ w(t)\end{pmatrix} + \mathcal{A}_1\begin{pmatrix} i(t)\\ v(t)\\ w(t)\end{pmatrix} + \mathcal{A}_2\begin{pmatrix} i(t)\\ v(t)\\ w(t)\end{pmatrix} = \begin{pmatrix} 0\\ 0\\ S^{-1}B_2(t)\end{pmatrix}, \\ (\gamma_1 i)(t) = -G_{11}((\gamma_0 v)(t)) - G_{12}(w(t)) + B_1(t), \quad 0 \le t < T \\ \begin{pmatrix} i(0)\\ v(0)\\ w(0)\end{pmatrix} = \begin{pmatrix} i^0\\ v^0\\ w^0\end{pmatrix}. \end{cases}$$

We subtract from the equation $(2)_1$ the equation (1) and we multiply the obtained relation by $\begin{pmatrix} i(t)-p\\ v(t)-q\\ w(t)-r\end{pmatrix}$, in Y. We obtain:

$$\frac{1}{2}\frac{d}{dt}\left\|\begin{pmatrix} i(t)-p\\ v(t)-q\\ w(t)-r\end{pmatrix}\right\|_Y^2 +$$

$$+ < G\begin{pmatrix}(\gamma_0 v)(t)\\ w(t)\end{pmatrix} - G\begin{pmatrix}\gamma_0 q\\ r\end{pmatrix}, \quad \begin{pmatrix}\gamma_0(v(t)-q)\\ w(t)-r\end{pmatrix} >_{\mathbb{R}^{2n+m}} +$$

$$+\sum_{k=1}^{n}\int_0^1 [\alpha_k(x, i_k(t,x)) - \alpha_k(x, p_k(x))][i_k(t,x) - p_k(x)]dx +$$

$$+\sum_{k=1}^{n}\int_0^1 [\beta_k(x, v_k(t,x)) - \beta_k(x, q_k(x))][v_k(t,x) - q_k(x)]dx =$$

$$=< B_1(t), (\gamma_0 v)(t) - \gamma_0 q >_{\mathbb{R}^{2n}} + < B_2(t), w(t) - r >_{\mathbb{R}^m},\ 0 \le t < T.$$

Therefore:

$$\frac{1}{2}\frac{d}{dt}\left\|\begin{pmatrix} i(t)-p\\ v(t)-q\\ w(t)-r\end{pmatrix}\right\|_Y^2 + a_1\|i(t)-p\|^2_{(L^2(0,1))^n} + a_2\|v(t)-q\|^2_{(L^2(0,1))^n} +$$

$$+K_0\|(\gamma_0 v)(t) - \gamma_0 q\|^2_{\mathbb{R}^{2n}} + K_0\|w(t) - r\|^2_{\mathbb{R}^m} \le C_1\|B_1(t)\|^2_{\mathbb{R}^{2n}} +$$

$$+C_2\|(\gamma_0 v)(t) - \gamma_0 q\|^2_{\mathbb{R}^{2n}} + C_3\|B_2(t)\|^2_{\mathbb{R}^m} + C_4\|w(t) - r\|^2_{\mathbb{R}^m},\ 0 \le t < T$$

(we choose the constant $C_i > 0,\ i = \overline{1,4}$ such that $C_2 < K_0$ and $C_4 < K_0$).

From the above inequality we deduce:

$$\frac{d}{dt}\left\|\begin{pmatrix} i(t)-p \\ v(t)-q \\ w(t)-r \end{pmatrix}\right\|_Y^2 + C_5\left\|\begin{pmatrix} i(t)-p \\ v(t)-q \\ w(t)-r \end{pmatrix}\right\|_Y^2 \le C_6\|B(t)\|^2_{I\!R^{2n+m}},$$

$$C_5, C_6 = const. > 0, \quad 0 \le t < T,$$

($B(t) = col(B_1(t), B_2(t))$).

Integrating over $[0,t]$ the obtained inequality, we get:

$$\left\|\begin{pmatrix} i(t)-p \\ v(t)-q \\ w(t)-r \end{pmatrix}\right\|_Y^2 \le e^{-C_5 t}\left\|\begin{pmatrix} i^0-p \\ v^0-q \\ w^0-r \end{pmatrix}\right\|_Y^2 + \tag{3}$$
$$+C_6\int_0^t e^{-C_5(t-s)}\|B(s)\|^2_{I\!R^{2n+m}}ds, \quad 0 \le t \le T.$$

Because $b_k \in L^\infty(I\!R_+),\ k = \overline{1, 2n+m}$, we deduce that there exist $M_1, M_2 > 0$ independent of T such that:

$$\int_0^t e^{-C_5(t-s)}\|B(s)\|^2_{I\!R^{2n+m}}ds \le M_1\int_0^t e^{-C_5(t-s)}ds = \tag{4}$$
$$= M_1\frac{1-e^{-C_5 t}}{C_5} \le M_2, \quad 0 \le t \le T.$$

Now, using (3) and (4) we obtain:

$$\left\|\begin{pmatrix} i(t) \\ v(t) \\ w(t) \end{pmatrix}\right\|_Y \le const., \quad 0 \le t \le T,$$

where *const.* is a positive constant independent of T. Because T is arbitrary we conclude that the solution $col(i, v, w)$ is bounded on $I\!R_+$.

With the observation that the inequality (3) remains true also for the weak solutions, we deduce the conclusion of the theorem. ∎

Theorem 4. *Suppose the assumptions* (A1)ab, (A2)ab, (A3) *hold and let* $b_k \in L^2_{loc}(I\!R_+),\ k = \overline{1, 2n+m}$ *be* T*-periodic functions* ($T > 0$). *If the problem* (S), (BC) *has at least one bounded solution on* $I\!R_+$ *, then it has also a* T*-periodic (weak) solution.*

Proof. Let $col(i,v,w)$ a bounded solution on $\mathbb{R}_+$ of the problem (S), (BC). Then, using the inequality:

$$\left\| \begin{pmatrix} i(t) \\ v(t) \\ w(t) \end{pmatrix} - \begin{pmatrix} \bar{\iota}(t) \\ \bar{v}(t) \\ \bar{w}(t) \end{pmatrix} \right\|_Y \leq \left\| \begin{pmatrix} i(0) \\ v(0) \\ w(0) \end{pmatrix} - \begin{pmatrix} \bar{\iota}(0) \\ \bar{v}(0) \\ \bar{w}(0) \end{pmatrix} \right\|_Y , \ t \geq 0,$$

we deduce that all the solutions of the problem (S), (BC) are bounded on $\mathbb{R}_+$.

Now , we define the operator $Q: \overline{D(\mathcal{A}_1)} \to \overline{D(\mathcal{A}_1)}$, $Q \begin{pmatrix} i^0 \\ v^0 \\ w^0 \end{pmatrix} = \begin{pmatrix} i \\ v \\ w \end{pmatrix} \left(T; \begin{pmatrix} i^0 \\ v^0 \\ w^0 \end{pmatrix} \right)$, where $\begin{pmatrix} i \\ v \\ w \end{pmatrix} \left(t; \begin{pmatrix} i^0 \\ v^0 \\ w^0 \end{pmatrix} \right)$, $t \geq 0$ is the weak solution of the problem (S), (BC) with the initial data $col(i^0, v^0, w^0)$.

This operator is nonexpansive and if $\begin{pmatrix} i^0 \\ v^0 \\ w^0 \end{pmatrix} \in \overline{D(\mathcal{A}_1)}$, then the sequence $\left\{ Q^n \begin{pmatrix} i^0 \\ v^0 \\ w^0 \end{pmatrix} \right\}_{n \geq 1}$ is bounded, because $Q^n \begin{pmatrix} i^0 \\ v^0 \\ w^0 \end{pmatrix} = \begin{pmatrix} i \\ v \\ w \end{pmatrix} \left(nT; \begin{pmatrix} i^0 \\ v^0 \\ w^0 \end{pmatrix} \right)$.

Using a fixed point theorem, due to F. B r o w d e r and W.V. P e t r y s h y n [3]we deduce that the operator Q has at least one fixed point. Therefore, the problem (S), (BC) has at least one T-periodic (weak) solution. ∎

By Theorem 3 and Theorem 4 we obtain sufficient conditions for the existence of T-periodic solutions of the problem (S), (BC).

Corollary. *Assume that* (A1)ac, (A2)ab, (A3) *hold and let* $b_k \in L^\infty(\mathbb{R}_+)$, $k = \overline{1, 2n+m}$ *be* T*-periodic functions. Then, the problem* (S), (BC) *has a (unique)* T*-periodic weak solution.*

4. Almost-periodic solutions. We begin this section with a version of Theorem 1, Lecture 34 [4] for our problem (S), (BC).

Theorem 5. *Let* b_k *be almost-periodic real functions* ($b_k \in AP(\mathbb{R})$), $k = \overline{1, 2n+m}$. *If the problem* (S),(BC) *has a solution* $col(i,v,w)$ *which is uniformly continuous on* $\mathbb{R}_+$ *with a precompact range, then there exists an almost-periodic (weak) solution of the problem* (S), (BC).

Theorem 6. *Assume that* (A1)ac, (A2)ab, (A3) *hold,* $b_k \in W^{1,\infty}(\mathbb{R}_+)$, $k = \overline{1,2n+m}$ *and* $col(i^0, v^0, w^0) \in D(\mathcal{A}_1)$. *Then the unique strong solution* $col(i, v, w)$ *of the problem* (S), (BC), (IC) *is in the space* $W^{1,\infty}(\mathbb{R}_+; Y)$. *Moreover,* $i_k, v_k \in L^\infty(\mathbb{R}_+; H^1(0,1))$, $k = \overline{1,n}$.

Proof. Because $b_k \in W^{1,\infty}(\mathbb{R}_+)$, it follows that $b_k \in W^{1,2}_{loc}(\mathbb{R}_+)$, hence the strong solution $col(i, v, w)$ of the problem (S), (BC), (IC), according to Theorem 1, is in the space $W^{1,\infty}_{loc}(\mathbb{R}_+; Y)$. Moreover, applying Theorem 3, we deduce that $col(i, v, w)$ is bounded on $\mathbb{R}_+$.

Using a similar argument as in the proof of Theorem 3, we obtain the inequality:

$$\frac{d}{dt}\left\|\begin{pmatrix} i(t+h)-i(t) \\ v(t+h)-v(t) \\ w(t+h)-w(t)\end{pmatrix}\right\|^2_Y + C_5\left\|\begin{pmatrix} i(t+h)-i(t) \\ v(t+h)-v(t) \\ w(t+h)-w(t)\end{pmatrix}\right\|^2_Y \leq$$

$$\leq C_6\|B(t+h)-B(t)\|^2_{\mathbb{R}^{2n+m}}, \quad 0 \leq t < t+h, \quad C_5, C_6 = const. > 0.$$

Integrating over $[0,t]$ the above inequality, we have:

$$\left\|\begin{pmatrix} i(t+h)-i(t) \\ v(t+h)-v(t) \\ w(t+h)-w(t)\end{pmatrix}\right\|^2_Y \leq e^{-C_5 t}\left\|\begin{pmatrix} i(h)-i^0 \\ v(h)-v^0 \\ w(h)-w^0\end{pmatrix}\right\|^2_Y +$$

$$+C_6\int_0^t e^{-C_5(t-s)}\|B(s+h)-B(s)\|^2_{\mathbb{R}^{2n+m}}ds, \ t \geq 0.$$

We divide the above relation by h^2 and, by letting $h \to 0$, we get:

$$\left\|\frac{d}{dt}\begin{pmatrix} i \\ v \\ w\end{pmatrix}(t)\right\|^2_Y \leq e^{-C_5 t}\left\|\frac{d}{dt}\begin{pmatrix} i \\ v \\ w\end{pmatrix}(0)\right\|^2_Y +$$

$$+C_6\int_0^t e^{-C_5(t-s)}\left\|\frac{dB}{ds}(s)\right\|^2_{\mathbb{R}^{2n+m}}ds, \quad t \geq 0.$$

Because $b_k' \in L^\infty(\mathbb{R}_+)$ we obtain $\left\|\frac{d}{dt}\begin{pmatrix} i \\ v \\ w\end{pmatrix}(t)\right\|_Y \leq const., \ \forall t \geq 0.$

Therefore $col(i, v, w) \in W^{1,\infty}(\mathbb{R}_+; Y)$.

By some computations , we can prove, like in the proof of Theorem 1 (see [7]), that:

$$\sum_{k=1}^{n}\left[\int_0^1\left|\frac{\partial i_k}{\partial x}(t,x)\right|dx+\int_0^1\left|\frac{\partial v_k}{\partial x}(t,x)\right|dx\right]\leq const.,\ \forall t\geq 0.$$

Therefore $\partial i_k/\partial x,\ \partial v_k/\partial x\in L^\infty(\mathbb{R}_+;L^1(0,1))$. This conclusion, combined with $i_k,v_k\in L^\infty(\mathbb{R}_+;L^2(0,1))\subset L^\infty(\mathbb{R}_+;L^1(0,1))$ yields $i_k,v_k\in L^\infty(\mathbb{R}_+;W^{1,1}(0,1))$, hence $i_k,v_k\in L^\infty(\mathbb{R}_+;L^\infty(0,1))\subset L^\infty(\mathbb{R}_+\times(0,1))$, $k=\overline{1,n}$.

Using once again the assumptions of the theorem we deduce that $\alpha_k(\cdot,i_k),\beta_k(\cdot,v_k)\in L^\infty(\mathbb{R}_+;L^2(0,1))$, $k=\overline{1,n}$. From the equations of the system (S) we obtain that $\partial i_k/\partial x,\ \partial v_k/\partial x\in L^\infty(\mathbb{R}_+;L^2(0,1))$, therefore $i_k,\ v_k\in L^\infty(\mathbb{R}_+;H^1(0,1))$, $k=\overline{1,n}$, as claimed. ∎

Lemma 2. *If $col(i,v,w)$ is a solution of the problem* (S), (BC) *with the properties:*

$$col(i,v,w)\in W^{1,\infty}(\mathbb{R}_+;Y);\ i_k,v_k\in L^\infty(\mathbb{R}_+;H^1(0,1)),\ k=\overline{1,n},$$

then it is uniformly continuous on $\mathbb{R}_+$ with precompact range in Y.

Proof. Because $col(i,v,w)\in W^{1,\infty}(\mathbb{R}_+;Y)$, we have:

$$\left\|\begin{pmatrix}i\\v\\w\end{pmatrix}(t)-\begin{pmatrix}i\\v\\w\end{pmatrix}(s)\right\|_Y=\left\|\int_s^t\frac{d}{d\tau}\begin{pmatrix}i\\v\\w\end{pmatrix}(\tau)d\tau\right\|_Y\leq$$

$$\leq\left|\int_s^t\left\|\frac{d}{d\tau}\begin{pmatrix}i\\v\\w\end{pmatrix}(\tau)\right\|_Y d\tau\right|\leq M|t-s|,\ \forall t,s\in\mathbb{R}_+,\ M=const>0.$$

Therefore $col(i,v,w)$ is uniformly continuous on $\mathbb{R}_+$.

Because the solution $col(i,v,w)$ is bounded in $Y=(L^2(0,1))^{2n}\times\mathbb{R}^m$, to prove that the set $\{col(i(t),v(t),w(t))\,;\,t\in\mathbb{R}_+\}$ is precompact it is sufficient to prove that the sets:

$A_k=\{i_k(t);t\in\mathbb{R}_+\}\subset L^2(0,1)$ and $B_k=\{v_k(t);t\in\mathbb{R}_+\}\subset L^2(0,1)$, $k=\overline{1,n}$ are precompact in $L^2(0,1)$.

First, we consider the set A_1. According to Theorem 2.21 [1] we shall prove that A_1 satisfies the condition:

$\forall \varepsilon > 0, \exists \delta > 0$ and $G \subset\subset (0,1)$, ($\overline{G}$ is compact, $\overline{G} \subset (0,1)$), $G = (\varepsilon', 1-\varepsilon')$, $\varepsilon' > 0$, such that $\forall t \in I\!R_+$ and $\forall h \in I\!R_+$, $|h| < \delta$, the inequalities:

(a) $\int_0^1 |\tilde{\imath}_1(t, x+h) - \tilde{\imath}_1(x,t)|^2 dx < \varepsilon^2$

and

(b) $\int_0^{\varepsilon'} |i_1(t,x)|^2 dx + \int_{1-\varepsilon'}^1 |i_1(t,x)|^2 dx < \varepsilon^2$

are satisfied, where $\tilde{\imath}_1(t,x) = \begin{cases} i_1(t,x) & , x \in (0,1) \\ 0 & , \text{in rest.} \end{cases}$

From the properties of the solution $col(i, v, w)$ it follows that

$$\sup\{|i_1(t,x)|;\ t \in I\!R_+,\ x \in (0,1)\} \leq M_1,$$

$$\sup\{\|\partial i_1/\partial x(t,\cdot)\|_{L^2(0,1)};\ t \in I\!R_+\} \leq M_2.$$

($M_1, M_2 = const. > 0$).

The above condition (b) is satisfied. Indeed, we have:

$$\int_0^{\varepsilon'} |i_1(t,x)|^2 dx + \int_{1-\varepsilon'}^1 |i_1(t,x)|^2 dx \leq 2\varepsilon' \cdot M_1^2 < \varepsilon^2,$$

whenever $\varepsilon' < \varepsilon^2/2M_1^2$.

For the condition (a) we first consider the case $h > 0$. We get:

$$\int_0^1 |\tilde{\imath}_1(t, x+h) - \tilde{\imath}_1(t,x)|^2 dx = \int_0^{1-h} |i_1(t, x+h) - i_1(t,x)|^2 dx +$$

$$+ \int_{1-h}^1 |i_1(t,x)|^2 dx = \int_0^{1-h} \left| \int_x^{x+h} \frac{\partial i_1}{\partial x}(t,\varphi) d\varphi \right|^2 dx + \int_{1-h}^1 |i_1(t,x)|^2 dx.$$

But:

$$\left| \int_x^{x+h} \frac{\partial i_1}{\partial x}(t,\varphi) d\varphi \right|^2 \leq \left(\int_x^{x+h} 1 d\varphi \right) \left(\int_x^{x+h} \left| \frac{\partial i_1}{\partial x}(t,\varphi) \right|^2 d\varphi \right) =$$

$$= h \int_x^{x+h} \left| \frac{\partial i_1}{\partial x}(t,\varphi) \right|^2 d\varphi \leq h \left(\int_0^1 \left| \frac{\partial i_1}{\partial x}(t,\varphi) \right|^2 d\varphi \right) \leq h M_2^2.$$

Then:

$$\int_0^{1-h}\left|\int_x^{x+h}\frac{\partial i_1}{\partial x}(t,\varphi)d\varphi\right|^2 dx \le \int_0^{1-h} hM_2^2dx \le \int_0^1 hM_2^2dx = hM_2^2.$$

This yields:

$$\int_0^1 |\tilde{\iota}(t,x+h)-\tilde{\iota}_1(t,x)|^2dx \le hM_2^2+hM_1^2 < \delta(M_1^2+M_2^2) < \varepsilon^2,$$

whenever $\delta < \varepsilon^2/(M_1^2+M_2^2)$.

In the same manner we treat the case $h<0$. So, the condition (a) is also satisfied, hence A_1 is a precompact set. Similarly, the sets $A_k,\ k=\overline{2,n}$ and $B_k,\ k=\overline{1,n}$ are precompact sets. Therefore, the set $\{col(i(t),v(t),$ $w(t));\ \ t\in I\!R_+\}$ is a precompact set in the space Y. ∎

Theorem 7. *Assume that* (A1)ac, (A2)ab, (A3) *hold,* $b_k \in AP(I\!R),\ b_k' \in L^\infty(I\!R_+),\ k=\overline{1,2n+m}$. *Then the problem* (S), (BC) *has at least one almost-periodic (weak) solution.*

Proof. Because $b_k \in AP(I\!R)$, it follows that $b_k \in L^\infty(I\!R_+)$. Then, by the assumption $b_k' \in L^\infty(I\!R_+)$, we deduce that $b_k \in W^{1,\infty}(I\!R_+),\ k=\overline{1,2n+m}$. Using Theorem 6 we obtain that the problem (S), (BC) has at least one solution $col(i,v,w) \in W^{1,\infty}(I\!R_+;Y)$ wiyh the properties $i_k, v_k \in L^\infty(I\!R_+;H^1(0,1)),\ k=\overline{1,n}$. Therefore, by Lemma 2 , the solution $col(i,v,$ $w)$ is uniformly continuous on $I\!R_+$ with precompact range in the space Y. Consequently, with Theorem 5, we deduce that there exists an almost-periodic (weak) solution of the problem (S), (BC). ∎

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