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INVARIANT AND SEMI-INVARIANT SUBMANIFOLDS OF A METRIC PARA-φ-MANIFOLD WITH PARALLELIZABLE KERNEL.

BY

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Abstract. We study invariant and semi-invariant submanifolds of metric para-φ-manifolds with parallelizable kernel. Also we study semi-invariant submanifolds of a Riemannian almost-product manifold. Finally, examples of this kind of submanifolds are given.

1. Introduction. The G-structures on a manifold which both generalize the Riemannian almost product structures and the metric almost paracontact structures, are those characterized by a tensor field φ of type (1,1) satisfying $\varphi^3 - \varphi = 0$ and a riemannian metric which makes orthogonal the distributions associated to φ.

B u c k i and M i e r n o w s k i [1] defined these structures and introduced the notion of para-φ-manifold with parallelizable kernel. This latter structure appears in a natural way when studying the lift of almost product structures to the tangent bundle.

In this paper we study invariant and semi-invariant submanifolds of metric para-φ-manifolds with parallelizable kernel, and the induced structures which appear on this kind of submanifolds. To do this, we use our classification of metric para-φ-manifolds with parallelizable kernel [4], and Naveira's classification of Riemannian almost product manifolds [3].

Also we study semi-invariant submanifolds of a Riemannian almost-product manifold. Examples of this kind of submanifolds are given.

2. Preliminaries. In this paragraph we summarize all the basic definitions that are used later. Let M be a C^∞ manifold of dimension m. We will denote $\mathcal{F}(M)$ the algebra of real smooth functions on M, $\chi(M)$ the Lie algebra of tangent vector fields.

Definition 2.1. *We say that (M, P, g) is a Riemannian almost product manifold if P is a tensor field of type (1,1) such that $P^2 = I$, and g is a riemannian metric satisfying $g(PA, PB) = g(A, B)$ for all $A, B \in \chi(M)$.*

In a Riemannian almost product manifold (M, P, g) two orthogonal distributions appear in a natural way: $\mathcal{V} = \{U \in \chi(M) \mid P(U) = U\}$ and $\mathcal{H} = \{X \in \chi(M) \mid P(X) = -X\}$, called the vertical and the horizontal distribution respectively.

In [3] N a v e i r a has given a classification of Riemannian almost product manifolds. Each class is characterized by some property of $\mathcal{V}$ and $\mathcal{H}$. The relevant for us are the following.

Definition 2.2. *Let $\mathcal{D}$ be a p-dimensional distribution on (M, P, g). We say that*

1. *$\mathcal{D}$ is a totally geodesic foliation (in short, TGF) iff $\nabla_A(P) = 0$ for all $A \in \mathcal{D}$;*
2. *$\mathcal{D}$ is a foliation (F) iff $\nabla_A(P)B = \nabla_B(P)A$ for all $A, B \in \mathcal{D}$;*
3. *$\mathcal{D}$ is an antifoliation (AF) iff $\nabla_A(P)A = 0$ for all $A \in \mathcal{D}$.*

We will say that (M, P, g) is in the class (P_1, P_2) if $\mathcal{V}$ satisfies condition P_1 and $\mathcal{H}$ satisfies condition P_2, for P_1 and P_2 some of the conditions above.

The folloing definition is due to B u c k i [1]:

Definition 2.3. *1. A para-φ-structure on the manifold M is a tensor field φ of type (1,1) and rank $r \leq m$ everywhere, satisfying $\varphi^3 - \varphi = 0$.*

2. *We say that φ has parallelizable kernel if there exist global vector fields $E_1, \ldots, E_s$, $s = m - r$, and 1-forms $\eta^1, \ldots, \eta^s$ on M such that $\eta^a(E_b) = \delta^a_b$ and $\varphi^2 = I - \eta^a \otimes E_a$.*
3. *If moreover there exists a riemannian metric g such that*

$$g(\varphi A, \varphi B) = g(A, B) - \sum_{a=1}^{s} \eta^a(A)\eta^a(B)$$

for any $A, B \in \chi(M)$, we say that M is a metric para-φ-manifold with parallelizable kernel.

It is known that in a metric para-φ-manifold with parallelizable kernel there exist three mutually ortogonal distributions

$$\mathcal{V} = \{U \in \chi(M) \mid \varphi(U) = U\},$$

$$\mathcal{H} = \{X \in \chi(M) \mid \varphi(X) = -X\},$$

and

$$\mathcal{K} = \{E \in \chi(M) \mid \varphi(E) = 0\},$$

called the vertical, horizontal and kernel distribution respectively.

In [4] we have given a classification of metric para-φ-manifolds with parallelizable kernel. In the following Proposition we summarize the classes which are of interest for the present paper.

Definition 2.4. *Let A, B arbitrary vector fields on the metric para-φ-manifold M with parallelizable kernel $\{E_a,\ a = 1, \dots s\}$. We say that M is*

1. *of class (1,–) (respectively (–,1)) iff $\nabla_{A+\varphi A}(\varphi) = 0$ and $\nabla_{A+\varphi A}E_a = 0$ (respectively $\nabla_{A-\varphi A}(\varphi) = 0$ and $\nabla_{A-\varphi A}E_a = 0$);*
2. *of class (2, –) (respectively (–,2)) iff*

$$\nabla_{A+\varphi A}(\varphi)(A + \varphi A) + \eta^a(A)\nabla_{A+\varphi A}E_a = 0$$

(respectively $\nabla_{A-\varphi A}(\varphi)(A - \varphi A) - \eta^a(A)\nabla_{A-\varphi A}E_a = 0$);

3. *of class (3, –) (respectively (–,3)) iff*

$$\nabla_{A+\varphi A}(\varphi)(B + \varphi B) + \eta^a(B)\nabla_{A+\varphi A}E_a =$$

$$= \nabla_{B+\varphi B}(\varphi)(A + \varphi A) + \eta^a(A)\nabla_{B+\varphi B}E_a$$

(respectively

$$\nabla_{A-\varphi A}(\varphi)(B - \varphi B) - \eta^a(B)\nabla_{A-\varphi A}E_a =$$

$$= \nabla_{B-\varphi B}(\varphi)(A - \varphi A) - \eta^a(A)\nabla_{B-\varphi B}E_a\ );$$

4. *of class (1,1) iff $\nabla\varphi = 0$ and $\nabla\eta^a = 0$;*
5. *of class (3,3) or normal iff $[\varphi, \varphi] - 2d\eta^a \otimes E_a = 0$, where $[\varphi, \varphi]$ represents the Nijenhuis tensor associated with φ.*

In the sequel we will consider a n-dimensional metric para-φ-manifold with parallelizable kernel $(M', \varphi', E'_a, (\eta^a)', g')$ and a submanifod M of dimension $m < n$. We shall identify the vector fields tangent to M with their images by the immersion of M in M'. The vector fields tangent to M' will be denoted with a " $'$ ".

Let ∇', ∇ denote the Levi-Civita connections associated to g' and g respectively, where g is the restriction of g' to M. The relationship between these connections is well known:

Proposition 2.5. *(Gauss-Weingarten's formulae)*

$$\nabla'_A B = \nabla_A B + H(A,B),$$

$$\nabla'_A N = -\mathcal{A}_N A + \nabla^\perp_A N,$$

with A, B vector fields tangent to m and N a normal vector field.

Here $H(A,B)$ denotes the normal component of $\nabla'_A B$, $-\mathcal{A}_N A$ denotes the tangent component of $\nabla'_A N$ and $\nabla^\perp$ is a linear connection in the normal bundle. It is an immediate consequence that

$$g'(H(A,B),N) = g'(\mathcal{A}_N A, B). \tag{1}$$

3. Invariant submanifolds of a metric para-φ-manifold with parallelizable kernel. Let $(M', \varphi', E'_a, (\eta^a)', g')$ be a n-dimensional metric para-φ-manifold with parallelizable kernel, and M an immersed submanifold af M'.

Definition 3.1. *M is an invariant submanifold of M' if for all $p \in M$ the tangent space T_pM is invariant by φ'.*

Examples. We give now somme examples of invariant submanifolds.

1. Let (M, P, g) be a riemannian almost product manifold of dimension $n = p+q$, with a local adapted basis $\{U_1, \ldots, U_p, X_1, \ldots X_q\}$ such that $P(U_i) = U_i,\ P(X_j) = -X_j$. Let $\widetilde{M}$ be a parallelizable $(n+s)$-dimensional submanifold, with a parallelism $\{W_1, \ldots, W_p, Z_1, \ldots, Z_q, E_1, \ldots, E_s\}$ and dual basis $\{\omega^1, \ldots, \omega^p, \zeta^1, \ldots, \zeta^q, \eta^1, \ldots, \eta^s\}$ of 1-forms. On $M \times M$ we define

$$\begin{gathered}\varphi'(U_i, 0) = (U_i, 0),\\ \varphi'(0, W_j) = (0, W_j),\\ \varphi'(X_i, 0) = -(X_i, 0),\\ \varphi'(0, Z_j) = -(0, Z_j),\\ \varphi'(0, E_a) = 0,\end{gathered}$$

and the riemannian metric

$$g'((A, \widetilde{A}), (B, \widetilde{B})) = g(A,B)\pi_1 + (\sum \eta_a(\widetilde{A})\eta_a(\widetilde{B}))\pi_2 \tag{2}$$

with $A, B \in \chi(M),\ \widetilde{A}, \widetilde{B} \in \chi(\widetilde{M})$.

It is easy to prove that $(M \times \widetilde{M}, \varphi', (0, E_a), (0, \eta^a), g')$ is a metric para-φ-manifold with parallelizable kernel; and (M, P, g) is an invariant submanifold of $M \times \widetilde{M}$, with $(0, E_a)$ never tangent to M.

2. Take $(M, \varphi, E_a, \eta^a, g)$ a metric para-φ-manifold with parallelizable kernel and $\mathcal{B} = \{U_1, \ldots, U_p, X_1, \ldots, X_q, E_1, \ldots, E_s\}$ an adapted basis, that is an orthonormal system of local references such that $\{U_i,\ i = 1, \ldots, p\}$ generate the vertical distribution $\mathcal{V}$ and $\{X_j,\ j = 1, \ldots, q\}$ generate the horizontal distribution $\mathcal{H}$. Let us consider a parallelizable manifold $\widetilde{M}$ of dimension $n = p+q$ with parallelism $\{V_1, \ldots, V_p, Y_1, \ldots Y_q\}$. Then on $M \times \widetilde{M}$ it is possible to define a para-φ-structure with parallelizable kernel

$$\begin{gathered}\varphi'(A, 0) = (\varphi A, 0),\\ \varphi'(0, V_i) = (0, V_i),\\ \varphi'(0, X_j) = -(0, X_j).\end{gathered}$$

If we take g' as in (2) then $(\varphi', (0, E_a), (0, \eta^a), g')$ is a metric para-φ-structure with parallelizable kernel on $M \times \widetilde{M}$, and $(M, \varphi, E_a, \eta^a, g)$ is an invariant submanifold of $M \times \widetilde{M}$.

Proposition 3.2. *Let M be an invariant submanifold of the metric para-φ-manifold with parallelizable kernel $(M', \varphi', E'_a, (\eta^a)', g')$. Let us denote $\varphi = \varphi' \mid M$.*

1. *If E'_a are never tangent to M then (M, φ, g) is a Riemannian almost product manifold ;*
2. *If E'_a are tangent to M then $(M, \varphi, E'_a, (\eta^a)', g)$ is a metric para-φ-manifold with parallelizable kernel .*

Proof. 1) If $A \in \chi(M)$ then

$$\varphi^2 A = (\varphi')^2 A = A - (\eta^a)'(A) E'_a. \tag{3}$$

Since $E_a{}'$ are never tangent to M,

$$\begin{gathered}\varphi^2 = I,\\ (\eta^a)'(A) = 0;\end{gathered} \tag{4}$$

from (4) it follows that φ defines an almost product structure on M.

2) If E'_a are tangent to M, then using (3) we have

$$\begin{gathered}\varphi^2 = I - \eta^a \otimes E_a,\\ (\eta^a)(E_b) = \delta^b_a,\end{gathered} \tag{4}$$

that is $(\varphi, E'_a, (\eta^a)')$ is a para-φ-structure with parallelizable kernel on M.

Finally, the metric condition is immediate. ∎

In the next Theorems we study which type of Riemannian almost product manifold or metric para-φ-manifold with parallelizable kernel is induced

in the invariant submanifold M when we consider on M' different classes of metric para-φ-manifolds with parallelizable kernel . To do this study we use Naveira's classification of Riemannian almost product manifolds [3] cited in the Preliminaries, and the classes of metric para-φ-manifolds with parallelizable kernel given in Proposition 2.4. As in Proposition 3.2, we will analyze the two following possibilities:

1. The vector fields E'_a are never tangent to M;
2. The vector fields E'_a are tangent to M.

Theorem 3.3. *If the metric para-φ-manifold with parallelizable kernel $(M', \varphi', E'_a, (\eta^a)', g')$ is of the type (1,–) then*

1. The Riemannian almost product manifold (M, φ, g) is $(TGF,–)$;

2. the metric para-φ-manifold with parallelizable kernel $(M, \varphi, E_a, \eta^a, g)$ is of type (1,–).

Proof. If M' is of class (1,–), then using Proposition 2.4 we have

$$\nabla'_{A'+\varphi A'}(\varphi') = 0 \tag{5}$$

with A' an arbitrary vector field in M'. In particular, if A, B are vector fields in M, using Gauss-Weingarten formulae we have

$$0 = \nabla'_{A+\varphi A}(\varphi')B = \nabla_{A+\varphi A}(\varphi)B + H(A+\varphi A, \varphi B) - \varphi'(H(A+\varphi A, B)).$$

Taking in this expression the tangent component to M we obtain $\nabla_{A+\varphi A}(\varphi)B = 0$, that is the Riemannian almost product manifold (M, φ, g) is $(TGF,–)$.

2) We will denote $E'_a = E_a$, $(\eta^a)' = \eta^a$. Now, if we take $A' = A$ in (5), and taking into account that E_a are tangent to M, we have $\nabla_{A+\varphi A}E_a + H(A+\varphi A, E_a) = 0$; then it follows that $\nabla_{A+\varphi A}E_a = 0$. This condition and $\nabla_{A+\varphi A}(\varphi) = 0$ prove that $(M, \varphi, E_a, \eta^a, g)$ is of type (1,–). ∎

In an analogous way we obtain

Theorem 3.4. *If the metric para-φ-manifold with parallelizable kernel $(M', \varphi', E'_a, (\eta^a)', g')$ is of class (2,–) then*

1. The Riemannian almost product manifold (M, φ, g) is $(AF, -)$;

2. the metric para-φ-manifold with parallelizable kernel $(M, \varphi, E_a, \eta^a, g)$ is of class (2,–).

Theorem 3.5. *If the metric para-φ-manifold with parallelizable kernel $(M', \varphi', E'_a, (\eta^a)', g')$ is of class (3,–) then*

1. The Riemannian almost product manifold (M, φ, g) is $(F, -)$;

2. the metric para-φ-structure with parallelizable kernel $(M, \varphi, E_a, (\eta^a), g)$ is of class (3, –).

4. Semi-invariant submanifolds of a metric para-φ-manifold with parallelizable kernel . Let $(M', \varphi', E'_a, (\eta^a)')$ be a manifold with a para-φ-structure with parallelizable kernel . Let M be an immersed submanifold of M'.

Definition 4.1. *M is a semi-invariant submanifold of M' if E'_a are never tangent to M and there exist ω^a 1-forms on M and a tensor field P of type $(1,1)$ on M such that φ' can be expresed:*

$$\varphi'(A) = P(A) - \omega^a(A)E'_a, \quad a = 1, \dots, s \tag{6}$$

for any vector field A on M.

Example. We give an example of this kind of submanifolds . Let (M, P, h) be a Riemannian almost product manifold of dimension m, and $\widetilde{M}$ a s-dimensional parallelizable manifold with parallelism $\{E_1, \dots, E_s\}$ and dual basis of 1-forms $\{\eta^1, \dots, \eta^s\}$. We consider the manifold $M \times \widetilde{M}$.

If $A, B \in \chi(M)$, $\widetilde{A}, \widetilde{B} \in \chi(\widetilde{M})$ we define

$$\varphi(A, \widetilde{A}) = (P(A), 0)$$
$$g((A, \widetilde{A}), (B, \widetilde{B})) = h(A, B)\pi_1 + (\textstyle\sum_a \eta^a(\widetilde{A})\eta^a(\widetilde{B}))\pi_2.$$

It is easy to prove that $(M \times \widetilde{M}, \varphi, (0, E_a), (0, \eta^a), g)$ is a metric para-φ-manifold with parallelizable kernel and M is a semi-invariant submanifold.

Proposition 4.2. *A para-φ-structure with parallelizable kernel on M' induces an almost product structure on any semi-invariant submanifold M of M'.*

Proof. If we apply φ' in (6) then $P^2 = I$ and $(\eta^a)'(A) = \omega^a(PA)$, that is P is an almost product structure . ∎

Proposition 4.3. *If M' is a normal para-φ-manifold with parallelizable kernel , then the almost product structure induced on a semi-invariant submanifold M is of type (F, F).*

Proof. If M' is normal (see Definition 2.4) then

$$0 = \left([\varphi', \varphi'] - 2\sum\nolimits_a d(\eta^a)' \otimes E'_a\right)(A', B')$$

with A', B' vector fields on M'. If we take in this expression $A' = A$, $B' = B$ vector fields tangent to M, then

$$0 = ([P, P](A, B) - \textstyle\sum_a((d\omega^a(PA, B) + d\omega^a(A, PB))E'_a) - \\ -\omega^a(A)\left(\mathcal{L}_{E'_a}(\varphi')B\right) + \omega^a(B)\left(\mathcal{L}_{E'_a}(\varphi')A\right)$$

where $\mathcal{L}$ denotes the Lie derivative.

It is easy to prove that if M' is normal then

$$\mathcal{L}_{E'_a}(\varphi')A' = 0. \tag{7}$$

Taking into account (7) and the fact that E'_a are no tangent to M we obtain $[P, P](A, B) = 0$, that is M is (F, F). ∎

5. Semi-invariant submanifold of a Riemannian almost product manifold . Now we consider on M' a Riemannian almost product structure . If M is an invariant submanifold of M' then M has an induced Riemannian almost product structure which posseses the same properties. In this Section we undertake the study of a more general type of submanifolds .

Definition 5.1. *Let (M', P', g') be a Riemannian almost product manifold and M an immersed submanifold of M'. We say that M is a semi-invariant submanifold of M' if there exist non null vector fields $U'_1, \ldots, U'_s$ which are no tangent to M, such that $E_a = P'(U'_a)$ are tangent to M and*

$$P'(A) = \varphi(A) + \sum\nolimits_a \eta^a(A)U'_a, \quad for\ all A \in \chi(M), \tag{8}$$

where $\varphi(A)$ denotes the component of $P'(A)$ tangent to M.

Proposition 5.2. *If M is a semi-invariant submanifold of a Riemannian almost product manifold (M', P', g'), then M is a metric para-φ-manifold with parallelizable kernel .*

Proof. We must verify that $\varphi, E_a, \eta^a,\ a = 1, \ldots, s$ are respectively a tensor field of type (1,1), vector fields tangent to M and 1-forms on M.

If we apply P' to the expression (8) then

$$P'(P'A) = P'(\varphi A) + \eta^a(A)E_a + \eta^a(\varphi A)U'_a. \tag{9}$$

Using the fact that $(P')^2 = I$ and considering in the expression above the tangent and normal components we obtain $\varphi^2 = I - \eta^a \otimes E_a$ and $\eta^a \circ \varphi = 0$. If we take $A = E_a$ in (9) we have $\varphi(E_a) = 0$ and $\eta^a(E_b) = \delta^a_b$. On the other hand, if g' satisfies $g'(P'A', P'B') = g'(A', B')$ with A', B' arbitrary vector fields on M', then using (8) we obtain

$$g(\varphi A, \varphi B) = g(A, B) - \sum\nolimits_{a=1}^{s} \eta^a(A)\eta^a(B)$$

for all $A, B \in \chi(M)$, where g is the restriction of g' to M. Then the semi-invariant submanifold $(M, \varphi, E_a, \eta^a, g)$ is a metric para-φ-manifold with parallelizable kernel . ∎

Example. We give an example of a semi-invariant submanifold of a Riemannian almost product manifold . Let (M, P, g) be a riemannian almost product manifold of dimension m. We consider $\mathbb{R}^{2t}$ with the usual metric and the almost product structure $\widetilde{P}$ given by

$$\begin{aligned} \widetilde{P}(\partial/\partial x^i) &= \partial/\partial x^{i+t} \qquad \text{if } 1 \le i \le t; \\ \widetilde{P}(\partial/\partial x^i) &= \partial/\partial x^{i-t} \qquad \text{if } 1 \le i \le 2t; \end{aligned}$$

where $\{\partial/\partial x^i\}$ is a basis of vector fields in $\mathbb{R}^{2t}$.

On $M \times \mathbb{R}^{2t}$ we define an almost product structure

$$P'(A, A') = (PA, \widetilde{P}A');$$

then $M \times \mathbb{R}^{2t}$ with P' and the product metric is a Riemannian almost product manifold .

We will prove now that $M \times \mathbb{R}^2$ is a semi-invariant submanifold of $M \times \mathbb{R}^{2t}$ with respect to $U'_1 = (0, \partial/\partial x^{t+1})$ and $U'_2 = (0, \partial/\partial x^{t+2})$. If $(A, a_1\partial/\partial x^1 + a_2\partial/\partial x^2)$ is a vector field tangent to $M \times \mathbb{R}^2$ then

$$\begin{aligned} P'(A, a_1\partial/\partial x^1 + a_2\partial/\partial x^2) &= (PA, a_1\partial/\partial x^{t+1} + a_2\partial/\partial x^{t+2}) \\ &= (PA, 0) + a_1(0, \partial/\partial x^{t+1}) + a_2(0, \partial/\partial x^{t+2}). \end{aligned}$$

On the other hand U_1' and U_2' are unitary vector fields, no tangent to $M\times I\!R^2$ and

$$E_1 = P'(U_1') = P'(0,\partial/\partial x^{t+1}) = (0,\partial/\partial x^1)$$
$$E_2 = P'(U_2') = P'(0,\partial/\partial x^{t+2}) = (0,\partial/\partial x^2).$$

The para-φ-structure with parallelizable kernel of rank $r = m$ is defined on the submanifold $M \times I\!R^2$ by

$$\begin{aligned}\varphi(A, a_1\partial/\partial x^1 + a_2\partial/\partial x^2) &= (PA, 0),\\ E_1 &= (0,\partial/\partial x^1),\\ E_2 &= (0,\partial/\partial x^2),\\ \eta^1(A, a_1\partial/\partial x^1 + a_2\partial/\partial x^2) &= a_1,\\ \eta^2(A, a_1\partial/\partial x^1 + a_2\partial/\partial x^2) &= a_2.\end{aligned}$$

and the metric h compatible with φ is given by

$$h\left((A, a_1\partial/\partial x^1 + a_2\partial/\partial x^2), (B, b_1\partial/\partial x^1 + b_2\partial/\partial x^2)\right) = g(A,B)+a_1b_1+a_2b_2.$$

It is possible to generalize the previous example to the manifold $M \times I\!R^{2t}$ and the submanifold $M \times I\!R^{2k}$, $0 \le k \le 2t$.

Now we study the relationship between the different classes of Riemannian almost product structures on M' and some classes of metric para-φ-structures with parallelizable kernel on M given in Definition 2.4. We need some previous results.

Definition 5.3. [2]

1. *A vector field $C' \in \chi(M')$ is Killing iff $\mathcal{L}_{C'}g' = 0$; it is Killing on M iff $\mathcal{L}_{C'}(g')(A,B) = 0$ for all $A, B \in \chi(M)$.*
2. *A non-null normal vector field N is parallel along M iff $\nabla'_A N = 0$ for any $A \in \chi(M)$; and is parallel on the normal bundle iff $\nabla^\perp_A N = 0$ for any $A \in \chi(M)$.*
3. *A vector field $C' \in \chi(M')$ is analytic iff $\mathcal{L}_{C'}P' = 0$.*

It is easy to prove the following

Lemma 5.4. *1. $\mathcal{L}_{U_a'}(g')(A',B') = g'(\nabla'_{A'}U_a', B') + g'(\nabla'_{U_a'}, A')$,*

2. $\mathcal{L}_{U_a'}(g')(A,B) = 2g'(\nabla'_A U_a', B) = 2g'(\nabla'_B U_a', A)$,

3. U_a' is parallel along M if and only if U_a' is Killing on M and parallel on the normal bundle.

Lemma 5.5. *If U_a' is Killing along M, then for any vector fields A, B on M we have*

1. *$H(A,B)$ has null component in the direction of U_a';*

2. *$P'(H(A,B))$ has null component in the direction of U'_a and null component in the tangent direction;*
3. *$\nabla'_A U'_a$ has no tangent component.*

Proof. 1) Since $0 = \mathcal{L}_{U'_a}(g')(A,B) = 2g'(\nabla'_A U'_a, B)$, using (1) we obtain

$$g'(H(A,B),U'_a) = g'(\mathcal{A}_{U'_a}A,B) = -g'(\nabla'_A U'_a,B) = 0.$$

2) We compute

$$\begin{aligned} g'(P'(H(A,B)),C) &= g'(H(A,B),P'C) \\ &= g'(H(A,B),\varphi C + \eta^a(C)U'_a) \\ &= g'(H(A,B),\eta^a(C)U'_a) \\ &= 0 \end{aligned}$$

by 1). Finally, 3) is obtained from 1). ∎

Proposition 5.6. *If M' is (TGF,TGF) and U'_a are parallel along M then M is of type (1,1).*

Proof. We have $\nabla' P' = 0$; if we develop this expression for $A,B \in \chi(M)$, using Gauss-Weingarten's formulae we obtain

$$\nabla_A(\varphi)B + (\nabla_A(\eta^a)B)U'_a + H(A,\varphi(B)) - P'(H(A,B)) + \eta^a(B)\nabla'_A U'_a = 0.$$

Then $\nabla_A(\varphi)B = 0$. Taking into account Lemma 5.5 and the fact that U'_a are parallel along M, we have $\nabla_A(\eta^a)B = 0$ for $a = 1,\dots,s$, that is M is a metric para-φ-manifold with parallelizable kernel of type (1,1). ∎

Proposition 5.7. *If M' is (F,F) and U'_a are analytic and satisfy $[U'_a,U'_b] = 0$, then M is normal.*

Proof. Since M' is (F,F), if we take $A,B \in \chi(M)$ we have

$$[P'A,P'B] - P'[P'A,B] - P'[A,P'B] + [A,B] = 0$$

Then, using (8),

$$\begin{gathered} [P'A,P'B] - P'[P'A,B] - P'[A,P'B] + [A,B] = \\ = ([\varphi,\varphi] - 2d\eta^a \otimes E_a)(A,B) - \eta^a(A)\eta^a(B)[U'_a,U'_b] + \\ +\eta^a(A)\mathcal{L}_{U'_a}(p')B - \eta^a(B)\mathcal{L}_{U'_a}(P')A + \\ +(\mathcal{L}_{\varphi A}(\eta^a)B)U'_a - (\mathcal{L}_{\varphi B}(\eta^a)A)U'_a = 0. \end{gathered}$$

The result follows immediately. ∎

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