ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLII, s.I.a, Matematică, 1996, f2.

PRIMARY AND SEMIPRIMARY Γ–SEMIGROUP

BY

NARAYAN CHANDRA ADHIKARI

Abstract. In this paper we introduce the notions of primary and semiprimary ideals is S where (S, Γ) is a Γ–semigroup and study it via operator semigroup of (S, Γ). We also introduce primary and semiprimary Γ–semigroup. We prove that in a Γ–semigroup (S, Γ) in which S is weakly regular the notions of prime, (left, right) primary and semiprimary ideals coincide.

1. Introduction. Let S and Γ be two nonempty sets. S is called a Γ–semigroup [8] if there exists mapping $S\times\Gamma\times S \to S$ written as $(a, \alpha, b) \to a\alpha b$ and $\Gamma \times S \times \Gamma \to \Gamma$ written as $(\alpha, a, \beta) \to \alpha a\beta$, satisfying the following identities $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ for all $a, b, c, \in S$ and $\alpha\beta \in \Gamma$. We call a Γ–semigroup S both sided if it further satisfies the identities: $\alpha a(\beta b\gamma) = (\alpha a\beta)b\gamma = \alpha(a\beta b)\gamma$ for all $\alpha, \beta, \gamma \in \Gamma$ and $a, b \in S$. Throughout this paper we consider only both sided Γ–semigroup and call it simply Γ–semigroup. Some earlier work on both sided Γ–semigroup viz. embedding a Γ–semigroup in a semigroup with zero, on Γ–semigroup with the right and left unites, on prime radical, on Noetherian Γ–semigroup and the radicals of Γ–semigroup may be found respectively [3], [4], [5], [6] and [7]. We shall denote a both sided Γ–semigroup S by (S, Γ).

In this paper we introduce the notations of (left, right) primary and semiprimary ideal in S where (S, Γ) is a Γ–semigroup. We show that if an ideal I is semiprime then the notions of prime, primary and semiprimary on I coincide. We also introduce the notion of primary and semiprimary Γ–semigroup. We obtain some necessary and sufficient conditions for (S, Γ) to be primary or semiprimary.

2. Preliminaries. Let (S, Γ) be a Γ–semigroup. A nonempty subset I is called a left (right) ideal of S if $S\Gamma I \subseteq I$ ($I\Gamma S \subseteq I$). A nonempty subset

I of S is called a two sided ideal or simply an ideal of S if it is a left ideal as well as a right ideal of S. The ideals of Γ are defined analogously.

Let (S, Γ) be a Γ–semigroup. We define a relation l on $S \times \Gamma$ as follows: $(x, \alpha) l (y, \beta)$ if and only if $x\alpha s = y\beta s$ for all $s \in S$ and $\gamma x \alpha = \gamma y \beta$ for all $\gamma \in \Gamma$. Obviously l is an equivalence relation. Let $[x, \alpha]$ denote the equivalence class containing (x, α).

Let $M = \{[x, \alpha] : x \in S \text{ and } \alpha \in \Gamma\}$. We define a multiplication in M as follows: $[x, \alpha][y, \beta] = [x\alpha y, \beta] = [x, \alpha y \beta]$. M is a semigroup with respect to this multiplication. This semigroup M is called the left operator semigroup of the Γ–semigroup (S, Γ). Dually we can define the right operator semigroup $N = \{[\alpha, a] : \alpha \in \Gamma \text{ and } a \in S\}$ of the Γ–semigroup (S, Γ) where the multiplication in N is defined by $[\alpha, a][\beta, b] = [\alpha a \beta, b] = [\alpha, a\beta b]$ [4].

For $P \subseteq M$ we define $P^+ = \{x \in S : [x, \alpha] \in P \text{ for all } \alpha \in \Gamma\}$, ${}^+P = = \{\gamma \in \Gamma : [x, \gamma] \in P \text{ for all } x \in S\}$. It follows that if P is an ideal or a right ideal of M then P^+ is an ideal or a right ideal of S. If P is an ideal or a left ideal of M then ${}^+P$ is an ideal or a left ideal of Γ [4]. Similarly for $Q \subseteq S(\Gamma)$ we define $Q^{+'} = \{[x, \alpha] \in M : x\alpha s \in Q \text{ for all } s \in S\}$ (resp. ${}^{+'}Q = \{[x, \alpha] \in M : \gamma x \alpha \in Q \text{ for all } \gamma \in \Gamma\}$). It follows that if Q is an ideal or a right ideal of S (ideal or a left ideal of Γ) then $Q^{+'}$ is an ideal or a right ideal of M (${}^{+'}Q$ is an ideal or a left ideal of M) [4].

It also follows that in a Γ–semigroup (S, Γ) with $(P^+)^{+'} = P$ and $(Q^{+'})^+ = Q$.

Definition (Definition 3.1 [4]). *Let (S, Γ) be a Γ–semigroup and M and N be the left operator semigroup and the right operator semigroup of (S, Γ). If there exists an element $[e, \delta] \in M$ ($[(\delta, e] \in N$) such that $e\delta x = x$ ($x\delta e = x$) for all $x \in S$ then S is said to have the left unity $[e, \delta]$ (right unity $[\delta, e]$). Similarly we can define the left and right unity of Γ. A Γ–semigroup (S, Γ) is said to be a Γ–semigroup with unity if S has left unity and right unity which are also right unity and left unity of Γ respectively.*

Definition (Definition 3.1 [5]). *Let (S, Γ) be a Γ–semigroup. An ideal P of S is said to be prime if, for any two ideals A and B of S, $A\Gamma B \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$. The prime ideal of Γ is defined analogously.*

Definition (Definition 4.3 [3]). *Let (S, Γ) be a Γ–semigroup. An ideal Q of S is semiprime if for any ideal A of S, $A\Gamma A \subseteq Q$ implies that $A \subseteq Q$.*

Definition. *An ideal B of S where (S, Γ) is a Γ–semigroup is said to be a maximal ideal of S, if $B \neq S$ and there exists no ideal I of S such that $B \subset I \subset S$.*

Definition (Definition 3.7 [5]). *Let (S,Γ) be a Γ–semigroup. A subset H of S is said to be an m–system of S if and only if $c \in H$, $d \in H$ imply that exist elements $p \in S$ and $\alpha_1, \alpha_2 \in \Gamma$ such that $c\alpha_1 p\alpha_2 d \in H$. The empty set is to be considered as an m–system of S.*

Theorem (Theorem 4.7 [4]). *Let (S,Γ) be a Γ–semigroup with unity. Then there is an inclusion preserving bijection between the set of all ideals of S and the set of all ideals of M via the mapping $Q \to Q^{+'}$ where is an ideal of S.*

Theorem (Theorem 3.13 [5]). *Let (S,Γ) be a Γ–semigroup. Then there exists an inclusion preserving bijection $Q \to Q^{+'}$ between the set of all prime ideal of S and the set of all prime ideals of M.*

Throughout this paper we consider a Γ–semigroup (S,Γ) such that S has left unity and right unity which are also right unity and left unity of Γ respectively and viceversa.

3. Primary and semiprimary ideal.

Definition 3.1. *Let (S,Γ) be a Γ–semigroup. An ideal P of S is said to be left (right) primary if it satisfies the following conditions.*

(i) If A and B are ideals of S such that $A\Gamma B \subseteq P$ and $B \not\subseteq P (A \not\subseteq P)$, then $A \subseteq r(P)$ $(B \subseteq r(P))$, where $r(P)$ is the intersection of all prime ideals of S containing P.

(ii) $r(P)$ is a prime ideal of S.

It is easy to pove that an ideal P in S satisfies (i) of Definition 3.1 if and only if (i) $< a > \Gamma < b > \subseteq P$ and $b \notin P (a \notin P)$ imply that $a \in r(P) (b \in r(P))$. $(< a > = S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S \cup \{a\}$ is the principal ideal of S generated by a).

Definition 3.2. *Let (S,Γ) be a Γ–semigroup. An ideal P of S is said to be semiprimary if $r(P)$ is a prime ideal of S.*

It is clear that every left (right) primary ideal of S is a semiprimary ideal.

Definition 3.3. *An ideal P of S where (S,Γ) is a Γ–semigroup is said to be primary provided it is left primary as well as right primary.*

Lemma 3.4. *Let (S,Γ) be a Γ–semigroup. Let A be an ideal of S disjoint form $H = \{a, a\gamma a, (a\gamma)^2 a, \ldots\}$ where $a \in S$ and $\gamma \in \Gamma$. Then there exists an ideal $P \supseteq A$ which is maximal in the class of ideals of S containing A and disjoint from H. Moreover, P is prime.*

Proof. It is clear that H is an m–system of S. So the proof follows form the Proposition 3.10 [5].

Lemma 3.5. *Let (S,Γ) be a Γ–semigroup and P be an ideal of S. If $x \in r(P)$ then $(x\gamma)^{n-1}x \in P$ (where $(x\gamma)^{\circ}x = x$) for all $\gamma \in \Gamma$ and for some positive integer n.*

Proof. Let $x \in r(P)$. If possible, let for some $\gamma \in \Gamma$, $(x\gamma)^{n-1}x \notin P$ for all positive integers n. Then the m–system

$$H = \{x, x\gamma x, (x\gamma)^2 x, \ldots, (x\gamma)^n x, \ldots\}$$

is disjoint from P. So there exists a prime ideal $Q \supseteq P$ s.t. $Q \cap H = \emptyset$. This implies that $x \notin Q$. So $x \notin r(P)$, a contradiction. So $(x\gamma)^{n-1}x \in P$ for all $\gamma \in \Gamma$ and for some positive integer n.

Lemma 3.6. *Let M be the left operator semigroup of a Γ–semigroup (S,Γ). If P and Q are ideals of M and S respectively then $r(P^+) = r(P)^+$ and $r(Q)^{+'} = r(Q^{+'})$.*

Proof. Let $a \in r(P)^+$. Then $[a,\alpha] \in r(P)$ for all $\alpha \in \Gamma$. Let Q be any prime ideal of S containing P^+. Then $Q^{+'}$ is a prime ideal of M containing $(P^+)^{+'} = P$ [4]. So $[a,\alpha] \in r(P) \subseteq Q^{+'}$ for all $\alpha \in \Gamma$. Hence $a \in (Q^{+'})^+ = Q$. Thus $r(P)^+ \subseteq r(P^+)$. Again, let $b \in r(P^+)$. Let Q be any prime ideal of M containig P. Then Q^+ is a prime ideal of S containing P^+. So $b \in r(P^+) \subseteq Q^+$. Hence $[b,\beta] \in Q$ for all $\beta \in \Gamma$. So $[b,\beta] \in r(P)$ for all $\beta \in \Gamma$. Hence $b \in r(P)^+$. Thus $r(P^+) \subseteq r(P)^+$. Hence $r(P)^+ = r(P^+)$. Similarly we can prove that $r(Q)^{+'} = r(Q^{+'})$.

Proposition 3.7. *Let (S,Γ) be a Γ–semigroup. If P is a left (right, semi) primary ideal of the left operator semigroup M of (S,Γ) then P^+ is a left (right, semi) primary ideal of S.*

Proof. Since P is an ideal of M, P^+ is an ideal of S [4]. Let $A\Gamma B \subseteq P^+$ and $B \not\subseteq P^+$ where A and B are ideals of S. Then $[A,\Gamma][B,\Gamma] = [A\Gamma B,\Gamma] \subseteq P$ and $[B,\Gamma] \not\subseteq P$. Now since A and B are ideals of S, $[A,\Gamma]$ and $[B,\Gamma]$ are ideals of M. Since P is left primary, $[A,\Gamma] \subseteq r(P)$. Hence $A \subseteq r(P)^+ = r(P^+)$. Thus $A \subseteq r(P^+)$. Also $r(P^+) = r(P)^+$ is a prime ideal of S [5]. So P^+ is a left primary ideal of S.

Proposition 3.8. *Let (S,Γ) be a Γ–semigroup. If Q is a left (right, semi) primary ideal of S then $Q^{+'}$ is a left (right, semi) primary ideal of M where M is the left operator semigroup of the Γ–semigroup (S,Γ).*

Proof. Since Q is an ideal of S, then $Q^{+'}$ is an ideal of M [4]. Let $AB \subseteq Q^{+'}$, $B \not\subseteq Q^{+'}$ where A and B are ideals of M. Then $AMB \subseteq AB \subseteq Q^{+'}$ i.e., $AS\Gamma BS \subseteq Q^{+'}S \subseteq Q$. Since Q is left primary and $BS \not\subseteq Q$, $AS \subseteq r(Q)$. So $A \subseteq r(Q)^{+'} = r(Q^{+'})$. Also $r(Q^{+'})$ is prime. Hence $Q^{+'}$ is left primary.

Similarly we can prove the above proposition in case of Γ and M.

Lemma 3.9. *Let (S,Γ) be a Γ–semigroup. If S contains a proper ideal, then S contains a unique maximal ideal.*

Proof. The proof follows from [P.137, 1] and Theorem 4.7 [4].

Lemma 3.10. *Let (S,Γ) be a Γ–semigroup and P be an ideal of S. If $r(P)$ is the unique maximal ideal of S then P is a primary ideal of S.*

Proof. The proof follows from Lemma 3.1 [1] and Theorem 4.7 [4].

Definition 3.11. *Let (S,Γ) be a Γ–semigroup and P be a prime ideal of S. A primary ideal A of S is said to be P–primary or a primary ideal belonging to P if $r(A) = P$.*

Proposition 3.12. *Let (S,Γ) be a Γ–semigroup and P be any prime ideal of S. If A is a P–primary ideal of S, then $A^{+'}$ is a $P^{+'}$–primary ideal of M.*

Proposition 3.13. *Let (S,Γ) be a Γ–semigroup and Q be any prime ideal of the left operator semigroup M of (S,Γ). If B is a Q–primary ideal of M, then B^{+} is a Q^{+}– primary ideal of S.*

Theorem 3.14. *If $A_1, A_2, \ldots, A_n$ are P-primary ideals in S where (S,Γ) is a Γ– semigroup, then $\bigcap_{i=1}^{n} A_i$ is also a P–primary ideal.*

Proof. Let $A = \bigcap_{i=1}^{n} A_i$. Now using Lemma 3.5. we can show that $r(A) = \left(\bigcap_{i=1}^{n} A_i\right) = \bigcap_{i=1}^{n} r(A_i) = P$. So $r(A)$ is a prime ideal. Suppose $< a > \Gamma < b > \subseteq A$ and $b \notin A$. Since $b \notin A$, $b \notin A_i$ for some i. Now $< a > \Gamma < b > \subseteq A_i$ and $b \notin A_i$. Since A_i is primary, we have $a \in r(A_i) = P = r(A)$. So A is left primary. Similarly we can show that A is right primary.

Lemma 3.15. *Let (S,Γ) be a Γ–semigroup (not necessary with unity) and let P be any ideal in S such that $r(P)$ is a maximal ideal in S. Then P is semiprimary.*

Proof. Obviously $P \subseteq r(P) \subseteq P_1$ where P_1 is a prime ideal containing P. Since $r(P)$ is maximal, we have $r(P) = P_1$. Therefore $r(P)$ is prime and hence P is a semiprimary ideal.

Lemma 3.16. *Let (S,Γ) be a Γ–semigroup and let Q be a semiprime ideal in S. Then $r(Q) = Q$.*

Theorem 3.17. *Let (S,Γ) be a Γ–semigroup. If P is a semiprime ideal in S, then the following condition are equivalent.*

(1) P is prime,
(2) P is primary,
(3) P is left primary,
(4) P is right primary,
(5) P is semiprimary.

Proof. Since P is semiprime, from the Lemma 3.16, we have $r(P) = = P$. Now (1)$\Longrightarrow$ (2)$\Longrightarrow$ (3)$\Longrightarrow$ (5) and (2)$\Longrightarrow$ (4)$\Longrightarrow$ (5) are clear. Since $r(P) = P$ it follows from the definition of semiprimary ideal that P is prime. So (5)$\Longrightarrow$ (1).

4. Primary and semiprimary Γ–semigroup.

Definition 4.1. *Let (S, Γ) be a Γ–semigroup. S is said to be (left, right, semi) primary if each ideal of S is (left, right, semi) primary.*

Theorem 4.2. *Let (S, Γ) be a Γ–semigroup. Then S is (left, right, semi) primary if and only if the left operator semigroup M of S is (left, right, semi) primary.*

Proof. The proof follows from Proposition 3.7 and Proposition 3.8 and Theorem 4.7 [4].

Theorem 4.3. *Let (S, Γ) be a Γ–semigroup. Then Γ is (left, right, semi) primary if and only if the left operator semigroup M is (left, right, semi) primary.*

Combining Theorem 4.2 and Theorem 4.3 we have

Theorem 4.4. *Let (S, Γ) be a Γ–semigroup. Then S is (left, right, semi) primary if and only if Γ is (left, right, semi) primary.*

Definition 4.5. *A Γ–semigroup is said to be (left, right, semi) primary if S or equivalently Γ is (left, right, semi) primary.*

Theorem 4.6. *Let (S, Γ) be a Γ–semigroup. (S, Γ) is left primary if and only if every ideal Q of S satisfies the following condition. $A\Gamma B \subseteq Q$ and $B \not\subseteq Q$ then $A \subseteq r(Q)$ for any ideals A, B of S.*

Proof. Suppose that (S, Γ) is left primary. Then from the definition it follows that S satisfies the given condition.

Conversely suppose that S satisfies the condition. Then we have to show that $r(Q)$ is prime. It is easy to prove that $r(r(Q)) = r(Q)$. Suppose that $< a > \Gamma < b > \subseteq r(Q)$ and $b \notin r(Q)$. Then, by our assumption, $a \in < a > \subseteq r(r(Q)) = r(Q)$. So $r(Q)$ is prime (Theorem 3.4 [5]). So Q is left primary.

Theorem 4.7. *Let (S, Γ) be a Γ–semigroup. If all prime ideals of S are maximal, then (S, Γ) is primary.*

Proof. Let P be a prime ideal of M. Then P^+ is a prime ideal of S. Since every prime ideal of S is maximal, so P^+ is maximal in S and hence $(P^+)^{+'} = P$ is maximal in M. Thus M is primary [1] which implies that S is primary.

Theorem 4.8. *A Γ–semigroup (S, Γ) is semiprimary if and only if prime ideals in S form a chain.*

Proof. The proof follows from the Theorem 4.4 [1] and Theorem 3.13 [5].

Definition 4.9. *Let (S, Γ) be a Γ–semigroup. S is said to be weakly regular if $I\Gamma I = I$ for every ideal I of S.*

Lemma 4.10. *Let (S, Γ) be a Γ–semigroup. If S is weakly regular then every ideal of S is semi prime.*

Lemma 4.11. *Let (S, Γ) be a Γ–semigroup. Then the following conditions are equivalent.*

(i) Principal ideals in S form a chain,

(ii) Ideals in S form a chain.

Theorem 4.12. *Let (S, Γ) be a Γ–semigroup with S weakly regular. Then the following conditions are equivalent*

(1) Every ideal in S is prime,

(2) S is primary,

(3) S is left primary,

(4) S is right primary,

(5) S is semi primary,

(6) Prime ideals in S form a chain,

(7) Principal ideals in S form a chain,

(8) Ideals in S form a chain.

Proof. Since S is weakly regular, it follows from the Lemma 4.10 that every ideal of S is semi prime. So by the Theorem 3.17, (1) to (5) are equivalent. By Theorem 4.8 (5) and (6) are equivalent. Assume (6). Let $< a >$ and $< b >$ be two principal ideals in S. Since prime ideals in S form a chain and (6) implies (1), either $< a > \subseteq < b >$ or $< b > \subseteq < a >$. Thus (6) implies (7). (7) implies (8) follows form Lemma 4.11. (8) $\Longrightarrow$ (6) is obvious. Thus (6) $\Longleftrightarrow$ (7) $\Longleftrightarrow$ (8).

Corollary 4.13. *Let (S, Γ) be a Γ–semigroup. Every ideal of S is prime if and only if S is weakly regular (semi) primary.*

Proof. If S is weakly regular (semi) primary, then from the Theorem 4.12 every ideal in S is prime. Conversely assume that every ideal in S is prime. Let A be an ideal of S. Now $A\Gamma A \subseteq A\Gamma A$ and since every ideal of S is prime, we have $A \subseteq A\Gamma A$. Again $A\Gamma A \subseteq A\Gamma S \subseteq A$. Hence $A = A\Gamma A$. Therefore S is weakly regular (semi) primary.

Corollary 4.14. *Every ideal in S where (S, Γ) is a Γ–semigroup, is prime if and only if S is weakly regular and the ideals in S form a chain.*

Proof. The proof follows from Theorem 4.12 and Corollary 4.13.

Acknowledgement. The autor wishes to express his sincere thanks to Dr. T.K. Dutta for his kind help in the preparation of this paper.

REFERENCES

1. A n j a n e y u l u, A. – *Primary ideals in semigroups*, Semi group forum, Vol.20, 1980, 129–144.
2. A n j a n e y u l u, A. – *On primary semigroups*, Czechoslovak Math. Journal, Vol.30, 1980,382–386.
3. D u t t a, T.K. and A d h i k a r i, N.C. – *Embedding a* Γ*–semigroup in a semigroup with zero*, Bull.Cal.Math.Soc., Vol.86, No.1, 1994, 1–6.
4. D u t t a, T.K. and A d h i k a r i, N.C. – *On* Γ*–semigroup with the right and left unities*, Soochow Journal of Math., Vol.19, No.4, 1993, 461–474.
5. D u t t a, T.K. and A d h i k a r i, N.C. – *On prime radical of* Γ*–semigroup*, Bull.Cal.Math.Soc., Vol. 86, No.5, 1994, 335–342.
6. D u t t a, T.K. and A d h i k a r i, N.C. – *On Noetherian* Γ*–semigroup*, Accepted for publication in Kyungpook Math. Journal.
7. D u t t a, T.K. and A d h i k a r i, N.C. – *On the Radicals of* Γ*–semigroup*, Communicated for publication.
8. S e n, M.K. and S a h a, N.K. – *On* Γ*–semigroup–*I, Bull.Cal.Math.Soc., Vol.78, 1986, 380.

Received: 06.V.1995

Kumar Ashutosh Institution (Main) Boys'
10/1, Dum Dum Road
Calcutta – 700 030
INDIA