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Tomul XLII, s.I.a, Matematică, 1996, f2.

DIFFERENTIAL MEAN VALUE PROPERTIES FOR QUASI–CONTINUOUS FUNCTIONS (I)

BY

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1. Introduction. Let X be a (real) locally convex space and $F : X \to R$, a function.Denote, for $r, s \in R,\ r < s$,

(1D1) $R_F(r,s) = F(s) - f(r),\ \ Q_F(r,s) = \dfrac{1}{s-r} R_F(r,s).$

(These will be referred to as the *incrementary difference/quotient* of F on the interval $[r,s]$). The following statement involving such quantities is basic to considerations below. Take the points $a, b \in R,\ a < b$, in accordance with

(1H1) *F is continuous over $[a,b]$,*

and assume that,for a certain denumerable part A of $[a,b]$,one has

(1H2) *F has a right derivative $\sigma[d,(+)]F(t),\ for\ t \in]a,b[\backslash A$.*

Let also B stand for a zero Lebesgue measure part of $[a,b]$,which includes A.

Theorem 1. *Under these assumptions,*

$$\operatorname{cl}\operatorname{co}\{Q_F(r,s); a \leq r < s \leq b\} = \operatorname{cl}\operatorname{co}\{d^{(+)}F(t); t \in]a,b[\backslash B\}. \tag{1.1}$$

Hence,in particular,

$$Q_F(a,b) \in \operatorname{cl}\operatorname{co}\{d^{(+)}F(t); t \in]a,b[\backslash B\}. \tag{1.2}$$

Moreover,if

(1H3) $\left|\begin{array}{l}\text{co}\,\{d^{(+)}F(t);t\in]a,b[\backslash B\}\textit{has a nonempty interior}\\ \textit{in the minimal closed linear variety including it},\end{array}\right.$

then,(1.2) *may be written in the stronger form*

$$Q_F(a,b)\in\text{co}\,\{d^{(+)}F(t);t\in]a,b[\backslash B\}. \tag{1.3}$$

A left counterpart of this result,based on (1H2) and (1H3) substituted by, respectively

(1H2') *F has a left derivative* $d^{(-)}F(t)$ *for* $t\in]a,b[\backslash A$

(1H3') $\left|\begin{array}{l}\text{co}\,\{d^{(-)}F(t);t\in]a,b[\backslash B\}\textit{ has a nonempty interior}\\ \textit{in the minimal closed linear variety including it}\end{array}\right.$

is directly obtainable from the above one by simply reversing the (standard) ordering on R. Note that,both (1H2) and (1H2') hold under

(1H4) *F has a derivative* $dF(t)$, *for all t in* $]a,b[\backslash A$.

The basic relation (1.2)–usually referred to as an *equality mean value property* for F on the interval $[a,b]$–was firstly obtained in l964 by M c L e o d [7] under the lines in A u m a n n [1,ch.7, sect.2];see also S o v a [10]. A different proof of it (in the context of (1H4)) were given in the 1967 survey paper by A v e r b u k h and S m o l y a – n o v [2,ch.1,sect.3] (cf. N a s h e d [8,ch1,sect.5]). Its stronger version (1.3),expressed as

$$Q_F(a,b)=\sum_{i\in I}\lambda_i d^{(+)}F(t_i) \tag{1.4}$$

where I is a finite index set ,$\{\lambda_i;i\in I\}$ are positive numbers with $\sum_{i\in I}\lambda_i=1$ and $\{t_i;i\in I\}$ are points of $]a,b[\backslash B$,were also stated in the above quoted paper by McLeod; see also F l e t t [5,ch.1,sect.11]. Note that,if $X=R$ and (1H2) is substituted by (1H4),the obtained relation is nothing but the classical Lagrange mean value theorem.

It follows from this that further extensions of Theorem 1 –which, technically speaking,are naturally connected with (1H1) being removed– may be not without interest.It is the main aim of the present exposition to indicate a result of this type,over the class of quasi-continuous vector

functions; details will be given in Section 4. The basic tool in proving it is represented by a monotonicity principle for real-valued functions;this will be the content of Section 2. Some preliminaries involving convex subsets and cones in locally convex spaces are collected in Section 3. And,in Section 5,some particular aspects of the obtained results are discussed.

2. A monotonicity principle. Let $f : R \to R$ be a function. Denote,for each $t \in R$,

(2D1) $\Delta^{(-)}f(t) = \limsup_{r\to t-} Q_f(r,t),\ \Delta^{(+)}f(t) = \limsup_{s\to t+} Q_f(t,s)$

These will be referred to as the *left/right superior Dini derivatives* of f at t. Their associated quantities

(2D1') $\Delta_{(-)}f(t), \Delta_{(+)}f(t)$ (*with* $\liminf$ *in place of* $\limsup$)

will be called the *left/right inferior Dini derivatives* of f at t. Note that

$$\Delta^{(-)}f(t) = \Delta_{(-)}f(t) = d^{(-)}f(t),\quad \Delta^{(+)}f(t) = \Delta_{(+)}f(t) = d^{(+)}f(t), \quad (2.1)$$

whenever the left/right derivatives in question exist.Denote further,for each $t \in R$,

(2D2) $\Omega^{(-)}f(t) = \limsup_{r\to t-} R_f(r,t),\ \Omega^{(+)}f(t) = \limsup_{s\to t+} R_f(t,s)$

We shall term these,the *left/right superior Dini oscillation* of f at t.Their associate quantities

(2D2') $\Omega_{(-)}f(t),\ \Omega_{(+)}f(t)$(*with* $\liminf$ *in place of* $\limsup$)

will be referred to as the *left/right inferior Dini oscillations* of f at t.Note that

$$\left|\begin{array}{l}\Omega^{(-)}f(t) = \Omega_{(-)}f(t) = f(t) - f(t-0),\\ \Omega^{(+)}f(t) = \Omega_{(+)}f(t) = f(t+0) - f(t),\end{array}\right. \quad (2.2)$$

whenever the limits in question exist.

Let $a, b \in R$ with $a < b$,be fixed. The following result is well known (see,for instance, A u m a n n [1,ch.7,sect.2]):

Proposition 1. *Let the part A of $]a,b[$ be denumerable. Then,a function $h = h_A$ from R into itself exists with the properties of being increasing over $[a,b]$ and*

$$\min(h(t+0) - h(t),\ h(t) - h(t-0)) > 0,\ t \in A. \quad (2.3)$$

We also need the following statement (to be found,e.g.,in F l e t t [5,ch.1,sect.10]). Remember that a part B of R is with *zero Lebesgue measure* when

(2D3) $\left| \begin{array}{l} for\ each\ \varepsilon > 0\ there\ exists\ a\ denumerable\ family\ (]a_n, b_n[) \\ of\ open\ intervals\ which\ covers\ B\ and\ \sum_n (b_n - a_n) < \varepsilon. \end{array} \right.$

Proposition 2. *Let the part B of $]a,b[$ be of zero Lebesgue measure. Then,a function $k = k_B$ from R into itself exists with the properties of being increasing and continuous over $[a,b]$ and*

$$k'(t) = \infty,\ for\ all\ t \in B. \tag{2.4}$$

Having these precised,let $f : R \to R$ be a function.We assume that,for a certain denumerable part A of $]a,b[$

(2H1) $\Delta_{(+)} f(t) < \infty,\ for\ all\ t \in]a,b[\backslash A.$

Note that,under such a condition we have

$$\Omega_{(+)} f(t) \leq 0,\ for\ all\ t \in]a,b[\backslash A. \tag{2.5}$$

(In fact,given t in $]a,b[\backslash A$,there must be,via (2H1),some $\mu > 0, \theta > 0$ with

$$\inf\{Q_f(t,s); t < s < t + \varepsilon\} < \mu,\ 0 < \varepsilon < \theta.$$

And,from this,we find some sequence (s_n) in $]a,b[$ with $s_n \downarrow t$ and

$$f(s_n) - f(t) \leq \mu(s_n - t),\ \text{for all } n.$$

Hence,passing to $\liminf$ as $n \to \infty$,the conclusion is clear,by the very definition of the involved quantity).Further,let B stand for a zero Lebesgue measure part of $]a,b[$ which includes A.

Theorem 2. *Suppose that*

(2H2) $\Omega^{(-)} f(t) \leq 0,\ t \in]a,b[$

(2H3) $\Delta_{(+)} f(t) \leq 0,\ t \in]a,b[\backslash B$

(2H4) $\Omega_{(+)}f(t) \leq 0,\ t \in [a,b[$

(2H5) $\Omega_{(-)}f(t) \leq 0,\ t \in]a,b]$.

Then.necessarily,

$$f(b) - f(a) \leq 0. \tag{2.6}$$

Moreover,this property is valid over all sub–intervals of $[a,b]$*; i.e.,*

$$f \text{ is decreasing on } [a,b]. \tag{2.7}$$

Proof. Let $\varepsilon > 0$ be arbitrary fixed and put

$$g_\varepsilon(t) = \varepsilon(h(t) + k(t) + t),\ t \in R.$$

Here,$h = h_A$ and $k = k_B$ are the functions from R into itself introduced by Propositions 1 and 2 respectively.Define an ordering $\preceq$ on R by

(2D4) $t \preceq s$ iff $t \leq s$ and $f(s) - f(t) \leq g_\varepsilon(s) - g_\varepsilon(t)$.

Let also c, d be arbitrary fixed in $]a,b[$ with $c < d$. It is not hard to see,via (2H5), that

$$\left| \begin{array}{l} \textit{each ascending (modulo } \preceq) \textit{ sequence in } [c,d] \\ \textit{is a Cauchy sequence bounded from above (modulo } \preceq). \end{array} \right. \tag{2.8}$$

So,by the maximality principle in T u r i n i c i [11],there exists, for the starting point c in $[c,d]$,some maximal (modulo $\preceq$) point r in $[c,d]$ with $c \preceq r$. Assume $r < d$ (hence,r is in $]a,b[$). For each t in $]r,d[$,a relation like $r \preceq t$ is impossible.This,by the very definition of our ordering,yields

$$f(t) - f(r) > \varepsilon(h(t) - h(r) + k(t) - k(r) + t - r)\ , r < t < d. \tag{2.9}$$

Three alternatives are now open before us.

Case 1. r *is outside* B. We have,by the above

$$Q_f(r,t) > \varepsilon,\ r < t < d.$$

Hence,passing to $\liminf$ as $t \to r+$,

$$\Delta_{(+)}f(r) \geq \varepsilon > 0 \text{ (in contradiction to (2H3))}.$$

Case 2. *r is in* $B \setminus A$. By (2.9) again,

$$Q_f(r,t) > \varepsilon Q_k(r,t),\ r < t < d.$$

And so.passing to lim inf as $t \to r+$,one gets (by Proposition 2)

$$\Delta_{(+)}f(r) \geq \varepsilon\infty = \infty\ (\textit{hence}\ \Delta_{(+)}f(r) = \infty),$$

in contradiction to (2H1).
Case 3. *r is in A*. We have by (2.9)

$$R_f(r,t) > \varepsilon R_h(r,t),\ r < t < d.$$

So,passing to lim inf as $t \to r+$,one has (via Proposition 1)

$$\Omega_{(+)}f(r) \geq \varepsilon(h(r+0) - h(r)) > 0 \text{ (in contradiction to (2H4))}.$$

From the discussion above it follows that r=d; i.e.,

$$f(d) - f(c) \leq \varepsilon(h(d) - h(c) + k(d) - k(c) + d - c). \tag{2.10}$$

Hence,passing to limit as $\varepsilon \to 0+$,

$$f(d) - f(c) \leq 0,\ a < c < d < b \tag{2.11}$$

or,equivalently,

$$f(b) - f(a) \leq f(b) - f(d) + f(c) - f(a),\ a < c < d < b. \tag{2.11$'$}$$

It suffices now passing to lim inf as $c \to a+$, $d \to b-$ in this last relation to get (via (2H4)+(2H5)) relation (2.6).Finally,note that the specific hypotheses (2H2)–(2H5) have a hereditary character with respect to the sub–intervals $[r,s]$ of $[a,b]$. Hence,so is the conclusion (2.6) above; and,from this,(2.7) follows .This ends the argument. q.e.d.

The following completion of this result is retainable.Let the general assumptions (2H1) as well as the specific assumptions (2H2)–(2H5) be in use.

Theorem 3. *Suppose that*

(2H6) *at least one of the inequalities in* (2H2)–(2H5) *is strict.*

Then, (2.6) *holds in the stronger form*

$$f(b) - f(a) < 0. \tag{2.6'}$$

Proof. Suppose by contradiction (2.6') is not true; i.e.,

(2H7) $f(b) - f(a) \geq 0.$

Combining with (2.6) yields $f(a) = f(b)$; and so,by (2.7),

$$f \textit{ is constant over } [a, b].$$

But then,all relations in (2H2)–(2H5) hold with equality;so that,(2H6) is contradicted. Hence,(2.6') must be true,as claimed. q.e.d.

Some remarks are in order.Call the function f, *quasi–continuous* on $[a, b]$ when

(2D5) conditions (2H4)+(2H5) hold simultaneously.

Note that

(2H8) f is continuous on $[a, b]$

is a sufficient condition for such a property (which also motivates the terminology).But this does not exhaust the class of all such functions; because,e.g.,

(2H9) f is decreasing on $[a, b]$

also gives us such a property.(We refer to the paper by T u r i n i c i [op.cit.] for other technical aspects). Finally,it is worth noting that the substitution

(2D6) $f(t) = g(-t), t \in R$

allows us obtaining the corresponding left versions of these statements; we do not give other details.

3.Convex sets and cones in l.c.s. Let X be a (real) locally convex space (l.c.s.).Call the subset A of X, *convex*,when

(3D1) $\lambda A + (1 - \lambda)A \subseteq A,\ 0 \leq \lambda \leq 1.$

It is simply to verify that the family of all convex sets is closed for arbitrary intersections. So,for each part M of X,the subset

(3D2) $\text{co}\,(M) = \cap\{A \supseteq M; A = \text{convex}\}$

is necessarily convex; it will be termed the *convex cover* of M. Note that $\text{co}\,(M)$ can be also defined as the family of all convex combinations

(3D3) $x = \sum_{i \in I} \lambda_i x_i$

where I is a finite index set, $\{\lambda_i; i \in I\}$ is a part of $[0, \infty[$ with $\sum_{i \in I} \lambda_i = 1$ and $\{x_i; i \in I\}$, points of M. In particular,when

(3H1) $\dim(X) = m$, for some $m \in N$,

we can arrange for $\text{card}\,(I) \leq m + 1$ in the representation (3D3).And,if in addition,

(3H2) M has at most m connected components

the index set I of this representation may be taken so that $\text{card}\,(I) \leq m$. See E g g l e s t o n [4,ch.2,sect.2] for all necessary (technical) details.

Further,call the subset B of X,a *cone* when

(3D4) $B + B \subseteq B,\ \lambda B \subseteq B$, for all $\lambda \geq 0$.

As before,the family of all cones is closed with respect to arbitrary intersections.So,for each part M of X,the subset

(3D5) $\text{cone}\,(M) = \cap\{B \supseteq M; B = \text{cone}\}$

is necessarily a cone;it will be termed the *conical cover* of M. Note that $\text{cone}\,(M)$ can be written as the set of all (conical) combinations (3D3) where I is a finite index set, $\{\lambda_i; i \in I\}$ a part of $[0.\infty[$ and $\{x_i; i \in I\}$,points of M. In other words

$$\text{cone}\,(M) = \cup\{\lambda \text{co}\,(M); \lambda \geq 0\},\ M \subseteq X. \tag{3.1}$$

So,the cardinality of I may be also reduced to $m + 1$ under (3H1) and, respectively,to m under (3H2).We also note the useful relation

$$\left| \begin{array}{l} \text{cone}\,(M_1 \cup M_2) = \text{cone}\,(M_1) + \text{cone}\,(M_2), \\ \text{for each couple } M_1, M_2 \text{ of parts in } X. \end{array} \right. \tag{3.2}$$

This is an immediate consequence of the representation above;we do not give details.

The following result is useful. Let d be a point in X and G, H,parts of X.

Proposition 3. *Let the above data be such that*

$$\text{(3H3)}\quad \left|\begin{array}{l} whenever\ x^*\in X^*,\lambda\in R\ satisfy\ x^*(x)\le\lambda, x\in G \\ and\ x^*(y)\le 0, y\in H,\ then\ x^*(d)\le\lambda. \end{array}\right.$$

Then,necessarily

$$d\in \mathrm{cl}\,(\mathrm{co}\,(G)+\mathrm{cone}\,(H)). \tag{3.3}$$

Proof. Denote for simplicity $M=\mathrm{co}\,(G)+\mathrm{cone}\,(H)$.It it not hard to see, via (3H3),that the implication below is holding

$$\left|\begin{array}{l} whenever\ x^*\in X^*,\lambda\in R\ satisfy\ x^*(z)\le\lambda, z\in M, \\ then\ x^*(d)\le\lambda. \end{array}\right. \tag{3.4}$$

But,in this case,a well known separation result (see,e.g., P r e c u p a n u [9,ch.5,sect.4]) assures us $d\in\mathrm{cl}\,(M)$.Hence the result. q.e.d.

Now,as simple examples show,the closure operator cl cannot be dropped, in general from (3.3).So,the question arises of which supplementary assumptions give such a conclusion.This needs some conventions.By a *linear variety* of X we mean,in the following,any translate of a subspace in X;hence,a closed linear variety is a translate of some closed subspace of X. As before,the family of all closed linear varieties is invariant to intersections;so,for each part M of X,

$$\text{(3D6)}\quad \mathrm{aff}\,(M)=\cap\{C\supseteq M; C=\text{closed linear variety}\}$$

is a closed linear variety ;it will be referred to as the *minimal closed linear variety* including M.Further,call the convex part P of X, *flat* whenever

$$\text{(3D7)}\quad \mathrm{int}\,(P)\ \text{is empty in}\ \mathrm{aff}\,(P)$$

and *non–flat*,in the opposite case.
We are now in position to give a useful answer to the question above.

Proposition 4. *Suppose that*

$$\text{(3H4)}\quad \mathrm{co}\,(G)+\mathrm{cone}\,(H)\ \textit{is non–flat.}$$

If the specific condition is also admitted

$$\text{(3H5)}\quad \left|\begin{array}{l} whenever\ x^*\in X^*,\lambda\in R\ satisfy\ the\ premises\ of\ \text{(3H3)} \\ with\ in\ addition,\ either\ x^*(u)<\lambda\ for\ some\ u\in G \\ or\ x^*(v)<0\ for\ some\ v\in H, then\ x^*(d)<\lambda \end{array}\right.$$

it necessarily follows that (3.3) *takes place in the stronger sense*

$$d \in \text{co}\,(G) + \text{cone}\,(H). \tag{3.3'}$$

Proof. Put $M = \text{co}\,(G) + \text{cone}\,(H)$. By (3H5),we have

$$\left| \begin{array}{l} \textit{whenever } x^* \in X^*, \lambda \in R \textit{ satisfy the premises of } (3.4) \textit{ with} \\ \textit{in addition, } x^*(w) < \lambda \textit{ for some } w \in M, \textit{ then } x^*(d) < \lambda. \end{array} \right. \tag{3.5}$$

(The proof being immediate,we omit the details).Now,it is not restrictive to assume

(3H6) $0 \in M$ and $\text{aff}\,(M) = X$.

Indeed,if we replace (M, d) by $(M - a, d - a)$ for some $a \in M$,the first part of this condition holds (as well as the working assumption above). On the other hand,the reasonings (to be developed further) are centered on the induced by $\text{aff}\,(M)$ locally convex structure; and so,the second part of (3H6) is realizable.In this case,(3H4) may be written as

(3H4') M has a nonempty interior in X;

and the conclusion to be obtained is $d \in M$.Suppose not;i.e.,(by the statement above)

(3H7) d is a boundary point of M.

By the classical Eidelheit's separation theorem (see,for instance, C r i s–t e s c u [3,ch.1,sect.2]), M and d can be separated by closed hyperplanes; i.e.,there may be determined $x^* \in X^*, \lambda \in R$ with

$$x^*(x) \leq \lambda \leq x^*(d),\ x \in M. \tag{3.6}$$

Now,an assumption like

(3H8) $x^*(x) = \lambda$, for all $x \in M$

is unacceptable;because,with the notation

$$N = \{x \in X;\ x^*(x) = \lambda\}$$

we should have

$$M \subseteq N\ (\text{hence } \emptyset \neq \text{int}\,(M) \subseteq \text{int}\,(N) = \emptyset),$$

a contradiction.Therefore,the premise of (3.5) must be true for the couple (x^*, λ). But then,by the quoted implication,

$$x^*(d) < \lambda \text{ (in contradiction to (3.6))}.$$

Hence,(3H7) cannot be accepted and the conclusion is clear. q.e.d.

Note that the structural condition (3H4) is always fulfilled under (3H1); see,e.g., E g g l e s t o n [op.cit.,ch.1,sect.5]).But,this is not the only possible case,as simple examples show.

It remains now to say what happens when (3H4) is not true. A partial answer to this may be given under the lines below.Denote,for each closed convex part M of X

(3D8) $\operatorname{asc}(M) = \cap\{\varepsilon(M - a);\ \varepsilon > 0\}$, for some $a \in M$.

The definition is consistent (it does not depend on the point $a \in M$ above); and the resulting object is a cone (called the *asymptotic cone* of M).Further, call the cone N of X, *locally compact*,when

(3D9) $\left|\begin{array}{l} N \cap V \textit{ is compact (in } N\textit{), for some} \\ \textit{neighborhood } V \textit{ of the origin.} \end{array}\right.$

Now,as another completion of Proposition 3,we have

Proposition 5. *Let the assumptions of Proposition* 3 *be fulfilled,as well as*

(3H9) $\operatorname{asc}(\operatorname{cl}\operatorname{co}(G)) \cap (-\operatorname{cl}\operatorname{cone}(H))$ *is a linear subspace*

(3H10) *either* $\operatorname{asc}(\operatorname{cl}\operatorname{co}(G))$ *or* $\operatorname{cl}\operatorname{cone}(G))$ *is locally compact.*

Then,(3.3) *may be written in the simpler form*

$$d \in \operatorname{cl}\operatorname{co}(G) + \operatorname{cl}\operatorname{cone}(H). \qquad (3.3^*)$$

Proof. Denote for simplicity $A = \operatorname{co}(G), B = \operatorname{cone}(H)$.By the Gwinner-Dedieu closedness criterion (see,for instance, P r e c u p a n u [9,ch.5, sect.4]),

$$\operatorname{cl}(A) + \operatorname{cl}(B) \textit{ is closed } (\textit{hence } \operatorname{cl}(A + B) \subseteq \operatorname{cl}(A) + \operatorname{cl}(B)). \qquad (3.7)$$

On the other hand,let V be any neighborhood of the origin and W,another neighborhood of the same with $W+W \subseteq V$.By a well known representation of the closure operator (see,for instance, C r i s t e s c u [op.cit.])

$$\mathrm{cl}\,(A)+\mathrm{cl}\,(B) \subseteq A+W+B+W \subseteq A+B+W+W \subseteq A+B+V. \quad (3.8)$$

And so,as V is arbitrary,

$$\mathrm{cl}\,(A)+\mathrm{cl}\,(B) \subseteq \mathrm{cl}\,(A+B). \quad (3.7')$$

The obtained relations now give

$$\mathrm{cl}\,(A+B)=\mathrm{cl}\,(A)+\mathrm{cl}\,(B); \quad (3.9)$$

and this,by the meaning of these data,ends the argument. q.e.d.

Now,concrete situations under which (3H10) is assured may be derived from the following equivalence valid for an arbitrary cone N in X (see P r e c u p a n u [op.cit.] for details)

$$\left| \begin{array}{l} N \ \textit{is locally compact if and only if} \\ N \ \textit{has a compact (in } N\textit{) sole.} \end{array} \right. \quad (3.10)$$

(Here,the part N_0 of N will be called a *sole* provided

(3D10) $0 \notin N_0$ and $N=\cup\{\lambda N_0;\lambda \geq 0\}$).

For example,this always happens under (3H1).But,there are many other instances in which this holds;we do not give other details. Finally,(3H9) admits a purely algebraic treatment,as it can be directly seen.

4.Main results. We are now in position to make precise the consi–derations of the introductory part. Let X be a (real) locally convex space, and $F: R \to X$, a function.For the arbitrary fixed $t \in R$,call the vector $x \in X$,a *right/left derivative value* for F at t when

$$(4D1) \quad \left| \begin{array}{l} Q_F(t,s_n) \to x, \textit{for some } (s_n) \textit{ in } R \textit{ with } s_n \downarrow t \\ (Q_F(r_n,t) \to x, \textit{for some } (r_n) \textit{ in } R \textit{ with } r_n \uparrow t). \end{array} \right.$$

The set of all such elements will be denoted as $\mathcal{D}^{(+)}F(t)$ (respectively, $\mathcal{D}^{(-)}F(t)$). Note that

$$\left| \begin{array}{l} \mathcal{D}^{(+)}F(t)=\{d^{(+)}F(t)\}(\textit{and } \mathcal{D}^{(-)}F(t)=\{d^{(-)}F(t)\}), \\ \textit{whenever the right/left derivative exists.} \end{array} \right. \quad (4.1)$$

Further,call the vector $y \in X$,a *right/left oscillation value* for F at t,when

$$\text{(4D2)} \quad \left| \begin{array}{l} R_F(t, s_n) \to y, for\ some\ (s_n)\ in\ R\ with\ s_n \downarrow t \\ (R_F(r_n, t) \to y, for\ some\ (r_n)\ in\ R\ with\ r_n \uparrow t). \end{array} \right.$$

The set of all such elements will be denoted as $\mathcal{E}^{(+)}F(t)$ (respectively, $\mathcal{E}^{(-)}F(t)$). Note that

$$\left| \begin{array}{l} \mathcal{E}^{(+)}F(t) = f(t+0) - F(t)\ (and\ \mathcal{E}^{(-)}F(t) = F(t) - F(t-0)) \\ whenever\ the\ right(left)\ limit\ exists. \end{array} \right. \tag{4.2}$$

Fix in the following a couple of points $a, b \in R$ with $a < b$.Assume that conditions below hold:

(4H1) $F(t-0)$ exists for $t \in]a, b[$ and $\mathcal{E}^{(-)}F(b)$ is nonvoid

(4H2) $\mathcal{E}^{(+)}F(t) \neq \emptyset$, for all t in $[a, b[$

as well as (for some denumerable part A of $]a, b[$)

(4H3) $\mathcal{D}^{(+)}F(t)$ is nonvoid,for each t in $]a, b[\backslash A$.

Denote further

$$\text{(4D3)} \quad \left| \begin{array}{l} e^{(-)}F(t) = F(t) - F(t-0),\ t \in]a, b[\\ e^{(-)}F(b) = an\ arbitrary\ point\ of\ \mathcal{E}^{(-)}F(b), \end{array} \right.$$

and let $t \vdash D^{(+)}F(t)$ stand for a *selection* of the multivalued map $t \vdash \mathcal{D}^{(+)}F(t)$. We also take a *selection* $t \vdash e^{(+)}F(t)$ of the multivalued map $t \vdash \mathcal{E}^{(+)}F(t)$.For example,noting that

$$0 \in \mathcal{E}^{(+)}F(t),\ t \in]a, b[\backslash A, \tag{4.3}$$

such a selection (denoted $e_A^{(+)}F$) may be defined as

$$\text{(4D4)} \quad \left| \begin{array}{l} e_A^{(+)}F(t) = 0,\ if\ t\ belongs\ to\ ]a, b[\backslash A \\ e_A^{(+)}F(t) = arbitrary\ in\ \mathcal{E}^{(+)}F(t),\ when\ t \in \{a\} \cup A. \end{array} \right.$$

Finally,let B stand for a zero Lebesgue measure part of $]a, b[$ including A. Our first main result is

Theorem 4. *Under the precised assumptions,one has the identity*

$$\left| \begin{array}{l} \mathrm{cl\,co}\,\{Q_F(r, s); a \leq r < s \leq b\} = \mathrm{cl}\,[\mathrm{co}\,\{D^{(+)}F(t); t \in]a, b[\backslash B\} + \\ \mathrm{cone}\,\{e^{(-)}F(t); t \in]a, b]\} + \mathrm{cone}\,\{e^{(+)}F(t); t \in [a, b[\}] \end{array} \right. \tag{4.4}$$

Hence,in particular,

$$\left| \begin{array}{l} Q_F(a,b) \in \mathrm{cl}\,[\mathrm{co}\,\{D^{(+)}F(t); t \in]a,b[\backslash B\} + \\ \mathrm{cone}\,\{e^{(-)}F(t); t \in]a,b]\} + \mathrm{cone}\,\{e^{(+)}F(t); t \in [a,b[\}. \end{array} \right. \tag{4.5}$$

Proof. We start with the verification of (4.5).To this end,it will suffice applying Proposition 3 to the quantities

$$\left| \begin{array}{l} d = Q_F(a,b), G = \{D^{(+)}F(t); t \in]a,b[\backslash B\} \\ H = \{e^{(-)}F(t); t \in]a,b]\} \cup \{e^{(+)}F(t); t \in [a,b[\}. \end{array} \right.$$

Let the continuous linear functional x^* and the real number λ be such that

(4H4) $x^*(D^{(+)}F(t)) \leq \lambda,\ t \in]a,b[\backslash B$

(4H5) $x^*(e^{(-)}F(t)) \leq 0, t \in]a,b]$ and $x^*(e^{(+)}F(t)) \leq 0, t \in [a,b[$.

By the admitted assumption,Theorem 2 is applicable to the function

(4D5) $f(t) = x^*(F(t)) - \lambda t,\ t \in R,$

in view of the evident relations

$$\begin{array}{l} \Omega^{(-)}f(t) = x^*(e^{(-)}F(t)),\ t \in]a,b[\\ \Delta_{(+)}f(t) \leq x^*(D^{(+)}F(t)) - \lambda,\ t \in]a,b[\backslash B \\ \Omega_{(+)}f(t) \leq x^*(e^{(+)}F(t)),\ t \in [a,b[\\ \Omega_{(-)}f(t) \leq x^*(e^{(-)}F(t)),\ t \in]a,b]. \end{array}$$

So,by the conclusion of that statement,

$$x^*(F(b) - F(a)) - \lambda(b-a) \leq 0\ (i.e., x^*(Q_F(a,b)) \leq \lambda).$$

In other words,the implication (3H3) of Proposition 3 is true (for these quantities); and,from this,(4.5) is clear. As an immediate consequence of these,the left member of (4.4) is included in the right member of the same. We now prove that the converse inclusion is also true.Suppose not;that is,

(4H6) $\left| \begin{array}{l} \textit{there exists } v \in \mathrm{co}\,(G) + \mathrm{cone}\,(H) \textit{ which does not} \\ \textit{belong to } \ \mathrm{cl}\,\mathrm{co}\,\{Q_F(r,s); a \leq r < s \leq b\}. \end{array} \right.$

By a standard separation theorem (see,e.g., P r e c u p a n u [9,ch.5,sect.4]) there must be a continuous linear functional x^* and a real number λ, such that

$$x^*(Q_F(r,s)) \leq \lambda < x^*(v),\ a \leq r < s \leq b. \tag{4.6}$$

The first half of this gives,in a direct way,

$$x^*(x) \leq \lambda, for\ all\ x \in G\ (hence, for\ all\ x \in \mathrm{co}\,(G))$$
$$x^*(y) \leq 0, for\ all\ y \in H\ (hence, for\ all\ y \in \mathrm{cone}\,(H))$$

Adding these inequalities,we obtain

$$x^*(z) \leq \lambda, for\ all\ z\ in\ \mathrm{co}\,(G) + \mathrm{cone}\,(H),$$

a contradiction to (4.6) (the second half).Hence,the converse inclusion above is true; and this proves (4.4).Hence the result. q.e.d.

Now,in general,the closure operator cannot be dropped in the mean value property (4.5).(A concrete example will be provided in the next section). So,the question arises of under which supplementary assumptions is this removable.An appropriate answer to this–which represents the second main result of the present exposition–is the following.(As before,the assumptions (4H1)–(4H3) prevail).

Theorem 5. *Suppose that*

$$\text{(4H7)} \quad \left| \begin{array}{l} \mathrm{co}\,\{D^{(+)}F(t); t \in]a,b[\backslash B\} + \mathrm{cone}\,\{e^{(-)}F(t); t \in]a,b]\} + \\ \mathrm{cone}\,\{e^{(+)}F(t); t \in [a,b[\}\ \textit{has a nonempty interior} \\ \textit{in the minimal closed linear variety including it.} \end{array} \right.$$

Then,the mean value property (4.5) *holds in the stronger form*

$$\left| \begin{array}{l} Q_F(a,b) \in \mathrm{co}\,\{D^{(+)}F(t); t \in]a,b[\backslash B\} + \\ \mathrm{cone}\,\{e^{(-)}F(t); t \in]a,b]\} + \mathrm{cone}\,\{e^{(+)}F(t); t \in [a,b[\}. \end{array} \right. \tag{4.5'}$$

Proof. We show that Proposition 4 is applicable to the triplet (d, G, H) introduced as in Theorem 4. Let the continuous linear functional x^* and the real number λ be such that (4H4)+(4H5) be valid,as well as one of the conditions below

(4H4') $x^*(D^{(+)}F(t)) < \lambda$,for some t in $]a,b[\backslash B$

$$\text{(4H5')} \quad \left| \begin{array}{l} either\ x^*(e^{(-)}F(t)) < 0\ for\ some\ t \in]a,b]\ or \\ x^*(e^{(+)}F(t)) < 0\ for\ some\ t \in [a,b[\end{array} \right.$$

By the admitted assumptions,Theorem 3 is applicable to the function f introduced by (4D5).So,by the conclusion of that statement,

$$x^*(F(b) - F(a)) - \lambda(b-a) < 0\ (i.e.,\ x^*(Q_F(a,b)) < \lambda).$$

In other words,the implication (3H5) of Proposition 4 is retainable. And, from this,conclusion (4.5') is clear. q.e.d.

In particular,the regularity conditions (4H1)+(4H2) are fulfilled under (1H1). So,the couple of the main results above comprises the (subsumed to Theorem 1) mean value M c L e o d 's theorem [7];see also S o v a [10]. For a number of historical aspects we refer to the survey paper by H i r i a r t – U r r u t y [6].

It remains now to see what happens when (4H7) is not true.The following answer to this (in the context of (4H1)–(4H3) being accepted) is the third main result of this exposition:

Theorem 6. *Suppose that*

$$\text{(4H8)} \quad \left| \begin{array}{l} \textit{the intersection between } \operatorname{asc}\operatorname{cl}\operatorname{co}\{D^{(+)}F(t);t\in]a,b[\backslash B\} \textit{ and} \\ -\operatorname{cl}[\operatorname{cone}\{e^{(-)}F(t);t\in]a,b]\}+\operatorname{cone}\{e^{(+)}F(t);t\in[a,b[\}] \\ \textit{is a linear subspace} \end{array} \right.$$

$$\text{(4H9)} \quad \left| \begin{array}{l} \textit{either } \operatorname{asc}\operatorname{cl}\operatorname{co}\{D^{(+)}F(t);t\in]a,b[\backslash B\} \textit{ or} \\ \operatorname{cl}[\operatorname{cone}\{e^{(-)}F(t);t\in]a,b]\}+\operatorname{cone}\{e^{(+)}F(t);t\in[a,b[\}] \\ \textit{is locally compact.} \end{array} \right.$$

Then,(4.4) *may be written in the weaker form*

$$\left| \begin{array}{l} \operatorname{cl}\operatorname{co}\{Q_F(r,s);a\le r<s\le b\}=\operatorname{cl}\operatorname{co}\{D^{(+)}F(t);t\in]a,b[\backslash B\}+ \\ \operatorname{cl}[\operatorname{cone}\{e^{(-)}F(t);t\in]a,b]\}+\operatorname{cone}\{e^{(+)}F(t);t\in[a,b[\}]. \end{array} \right. \tag{4.4*}$$

Hence,in particular,

$$\left| \begin{array}{l} Q_F(a,b)\in\operatorname{cl}\operatorname{co}\{D^{(+)}F(t);t\in]a,b[\backslash B\}+ \\ \operatorname{cl}[\operatorname{cone}\{e^{(-)}F(t);t\in]a,b]\}+\operatorname{cone}\{e^{(+)}F(t);t\in[a,b[\}]. \end{array} \right. \tag{4.5*}$$

Proof. It will suffice applying Proposition 5 to the triplet (d,G,H) we already introduced. q.e.d.

Some remarks are in order. Concerning the possibility of extending these statements beyond the locally convex setting,the answer to this is negative as the construction in A v e r b u k h and S m o l y a n o v [2,ch.1,sect.3] shows. On the other hand,the denumerable subset A of $]a,b[$ cannot be,in general,substituted by a zero Lebesgue measure one in the framework of (4H3); see,in this direction,the example in M c L e o d

[op.cit.]. Further,a close analysis shows that the underlying assumption may be weakened, with the cost of imposing some extra conditions upon the denumerable set A of $]a,b[$;this will be discussed elsewhere. Finally,a left counterpart of these developments is possible by simply reversing the natural order of R and making use of the construction (2D6).We do not give further details.

5. Some particular aspects. As precised in a previous place,the closure operator cannot be,in general,dropped in the mean value property (4.5).This is shown from the following

Example. Let X be an infinite dimensional Banach space and Z,a cone in X with the property

$$\text{(5H1)}\quad \left| \begin{array}{l} \textit{there exists a sequence } (v_n) \textit{ in } Z \textit{ so that the series} \\ \sum_n v_n \textit{ converges and its sum } ,v, \textit{ is outside } Z. \end{array} \right.$$

(For example,Z may be taken as the nullspace of some discontinuous linear functional (z^*) over X.Indeed,as Z is dense in X,there exists a $v \in X$ and a sequence (x_n) in Z,with

$$v \notin Z \textit{ and } x_n \to v, \textit{ as } n \to \infty.$$

It suffices now putting

$$v_1 = x_1, v_n = x_{n+1} - x_n, \ n \geq 2,$$

to satisfy all the requirements in (5H1) above).Let the function $F : R \to X$ be defined as

$$F(t) = \begin{cases} 0, \text{ if } t \leq 0 \\ v_1, \text{ if } 0 < t \leq \frac{1}{2} \\ v_1 + v_2, \text{ if } \frac{1}{2} < t \leq \frac{3}{4} \\ ..... \\ v, \text{ if } t \geq 1. \end{cases}$$

Take also $a = 0, b = 1$,and

$$A = \{1 - 2^{-n}; n = 1, 2, ...\}, \ B = A.$$

Clearly,(4H1) is fulfilled by these data,because

$$F(t - 0) = F(t), \ \textit{for all } t \textit{ in }]0, 1].$$

This is also true for (4H2) and (4H3),in view of

$$\mathcal{D}^{(+)}F(t) = \{0\},\ for\ each\ t\ in\ ]0,1[\backslash A \tag{5.2}$$

$$\mathcal{E}^{(+)}F(1-2^{-n}) = \{v_{n+1}\},\ n = 0,1,... \ . \tag{5.3}$$

Hence,Theorem 4 applies in this context,under the form

$$v \in \mathrm{cl}\,\mathrm{cone}\,\{v_1, v_2, ...\}.$$

Now,in view of (5H1),it is clear that the closure operator cannot be dropped here;and this proves our assertion.

Returning to the general setting,remember that–as already precised in Theorem 5–a basic structural assumption under which the closure operator is to be removed in the mean value property (4.5) is (4H7). This,in particular,happens when the ambient locally convex space X fulfils (3H1).So,let $m \geq 1$ be a natural number and $F : R \to R^m$,a function.Fix a couple of points $a, b \in R$ with $a < b$ and assume that (4H1)+(4H2) are fulfilled,as well as (4H3) (for some denumerable part A of $]a, b[$).Let the mapping $t \vdash e^{(-)}F(t)$ be introduced as in (4D3). We also take a selection $t \vdash D^{(+)}F(t)$ of the multivalued map $t \vdash \mathcal{D}^{(+)}F(t)$ as well as a selection $t \vdash e^{(+)}F(t)$ of the multivalued map $t \vdash \mathcal{E}^{(+)}F(t)$. Finally,let B stand for a zero Lebesgue measure part of $]a, b[$ including A.We have,by the remarks above

Theorem 7. *Under these assumptions,one has the mean value property*

$$Q_F(a,b) = \sum_{i=1}^{m+1} \lambda_i D^{(+)}F(t_i) + \sum_{j=1}^{m+1} \mu_j e^{(-)}F(s_j) + \sum_{k=1}^{m+1} \nu_k e^{(+)}F(r_k) \tag{5.4}$$

where $\{\lambda_i; 1 \leq i \leq m+1\}, \{\mu_j; 1 \leq j \leq m+1\}, \{\nu_k; 1 \leq k \leq m+1\}$ *are positive numbers (with,in addition,* $\sum_{i=1}^{m+1} \lambda_i = 1$*) and* $\{t_i; 1 \leq i \leq m+1\}, \{s_j; 1 \leq j \leq m+1\}, \{r_k; 1 \leq k \leq m+1\}$ *are points in* $]a, b[\backslash B,]a, b]$ *and* $[a, b[$ *,respectively.*

Some remarks are in order.The regularity condition (4H3) is clearly fulfilled under (1H4). Note that,in such a case,Theorem 7 reduces to the result in M c L e o d [7];see also F l e t t [5,ch.1,sect.11] and the

references therein.Further,the number of terms in the convex combinations in the right hand side of (5.4) cannot be diminished,in general. This is clearly shown in the case $m = 1$,where the mean value property (5.4) becomes

$$\left| \begin{array}{l} Q_F(a,b) = \lambda_1 D^{(+)}F(t_1) + \lambda_2 D^{(+)}F(t_2) + \\ \mu_1 e^{(-)}F(s_1) + \mu_2 e^{(-)}F(s_2) + \nu_1 e^{(+)}F(r_1) + \nu_2 e^{(+)}F(r_2). \end{array} \right. \tag{5.5}$$

Indeed,the fact that

$$u = \lambda_1 D^{(+)}F(t_1) + \lambda_2 D^{(+)}F(t_2)$$

cannot be reduced to a single term of this type is to be verified upon many polygonal functions,such as

$$F(t) = |t|,\ t \in R,$$

and $a = -1, b = 1, A = B = \{0\}$.But,under some supplementary assumptions,this may be true.One of these is (cf. Section 3)

(5H2) $\{D^{(+)}F(t); t \in]a,b[\backslash B\}$ *has at most* m *connected components*,

when the mean value property (5.4) has the simpler form

$$Q_F(a,b) = \sum_{i=1}^{m} \lambda_i D^{(+)}F(t_i) + \sum_{j=1}^{m+1} \mu_j e^{(-)}F(s_j) + \sum_{k=1}^{m+1} \nu_k e^{(+)}F(r_k). \tag{5.4$'$}$$

On the other hand,if (1H1) were admitted,then (4H1)+(4H3) are surely fulfilled. In this case,the result above is reduced to the one due to M c L e o d [op.cit.];see also T u r i n i c i [12]. Finally, note that by simply reversing the natural ordering on R one may obtain a left counterpart of this statement,which is useful in practice.Some further aspects of these facts will be discussed elsewhere.

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Received: 13.VI.1996

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