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BLUMBERG'S PROPERTY ON DENSITY TOPOLOGICAL SPACES

BY

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Introduction. In real function theory a natural study is to characterise functions in terms of Blumberg's property. This leads to investigate topological spaces which permit a Blumberg function. $\mathbb{R}^n$ with natural topology τ_0 is such a space. However it is seen that many common refinements of $(\mathbb{R}^n, \tau_0)$ such as density topology, its category analogues or abstract measure density sapces are quite pathological in respect of this property. In this short note our aim is to discuss the behaviour of density topologies for Blumberg's property.

Recall that an open set in the ordinary density topology τ_d on $\mathbb{R}$ are those Lebesgue mesurable ($\mathcal{L}$) sets $U \subset \mathbb{R}$ such for each $x \in U$ we have

$$\lim_{h \to 0} \frac{\mu(U \cap (x-h, x+h))}{2h} = 1.$$

This means that if we denote by $\phi(A)$ the set of all density points of a Lebesgue mesurable set A then τ_d is the family $\{A \in \mathcal{L} \colon A \subset \phi(A)\}$.

$(\mathbb{R}, \tau_d)$ is a C.R, non-normal [G o f f m a n et al; 1961] T_2-space which is co-compact, pseudo-complete [24] bur not topologically complete [12] and with more properties that compact sets are finite, μ-null sets are the first category closed and discrete sets. Also here the τ_d-Borel sets are precisely the measurable sets [21] and approximately continuous functions are continuous and of Baire class 1 and have a Darboux property (cf. [11], [13]).

1. Henry B l u m b e r g in 1922 [5] proved that for any real-valued function f defined on the real line $\mathbb{R}$ there is a dense subset D of $\mathbb{R}$ such that the restriction of f to D ($f|_D$) is continuous.

Definition. *A topological space X is called a* **Blumberg space** *if for every function $f: X \to \mathbb{R}$, there exists a dense subset $D \subset X$ such that $f|_D$ is continuous.*

In general, we refer the following statement as **Blumberg proposition** (abbreviated BP) :
(BP): If $f: X \to Y$, where X, Y are topological spaces, then there exists a dense subset $D \subset X$ such that $f|_D$ is continuous.
f is called as **Blumberg function** and the pair (X, Y) is said to be a **Blumberg pair**.

In [6] B l u m b e r g proved that BP is true if X and Y are both general Euclidian spaces. He also proved BP when X is any separable metric space and $Y = \mathbb{R}$. However C. G o f f m a n [9] gave an exemple to show that BP is not true for all pairs (X, Y) of metric spaces. That BP holds for fairely general topological spaces was shown by B l o c k and C a r g a l ([2]) by suitably restricting the categoric nature of the open sets in X. For an arbitrary topological space R.F. L e v y ([14]) proved the following.

Theorem 1.1. *The following are equivalent :*
(i) Every $f: X \to \mathbb{R}$ such that $\operatorname{card}(f[X]) \leq \aleph_0$ is Blumberg,
(ii) For every Y, and every $f: X \to Y$ such that $\operatorname{card}(f[X]) \leq \aleph_0$ is Blumberg.
(iii) Every $f: X \to \boldsymbol{N}$ is Blumberg.
(iv) X is Baire.

L e v y also gave an example of a zero-dimensional Baire space X which is not Blumberg (where X is an η_1 set with interval topology and $|X| = c$). B r a d f o r d et al ([4], 1960) proved that :

Theorem 1.2. *A metrizable space X is Blumberg iff X is a Baire space.*

In the light of the above result of [4], the following Theorem is proved by O.T. A l a s ([1]), which is also an easy extension of the work of H.R. B e n n e t t ([3]).

Theorem 1.3. *If X is a Baire semi-metrizable space, (Y, d) is a 2-countable metric space and $f: X \to Y$, then BP holds.*

The proposition of [4] that a metric space is Blumberg iff it is Baire'-naturally aroused questions which Baire spaces are Blumberg ? Is every compact Hausdorff space Blumberg ? These were partially answered by W h i t e ([24]), W e i s s ([23]), and L e v y ([14]). W e i s s showed that there exists a non-Blumberg compact Hausdorf space. W h i t e showed that the BP holds for a topological space X having a σ-disjoint pseudo-base iff X is Baire.

Definition. *A* **pseudo-base** *for a topological space (X, τ) is a subset π of τ such that every non-empty element of τ contains a non-empty element of π.*

Many interesting spaces have a pseudo-base, which is σ-disjoint, i.e. countable union of disjoint families.

W h i t e deduced the following Blumberg type theorem restricting the domain to pseudo-base and range to a 2-countable space.

Theorem 1.4. *Let X have a σ-disjoint pseudo-base and Y be a 2-countable space that contains an infinite discrete subset. Then X is a Baire space iff it has Blumberg property with respect to Y.*

This is extended by O.T. A l a s ([1]) in :

Theorem 1.5. *If X is a Baire Hausdorff space with σ-disjoint pseudo-base, Y is a Hausdorff 2-countable space and $f: X \to Y$, then there is a dense subset D of X such that $f|_D$ is continuous.*

L e v y introduced the notion of strongly non-Blumberg spaces and investigated its relation with Baire spaces. A topological space X such that $|X| \leq 2^{\aleph_0} = c$ is called **strongly non-Blumberg** if for every 1–1 function $f: X \to \mathbb{R}$ and for every dense subset D of X, $f|_D$ fails to be continuous.

Next we come to our point of discussion.

2. Let the density topology on $\mathbb{R}$ be τ_d and let μ, μ^* be the Lebesgue and Lebesgue outer measures respectively. Let $\mathcal{L}$ be the set of Lebesgue measurable subsets of $\mathbb{R}$, and τ_0-Euclidean topology on $\mathbb{R}$.

Lemma. [24] *Let $f: (\mathbb{R}, \tau_d) \to (\mathbb{R}, \tau_0)$ and $A \subset \mathbb{R}$ such that $\mu^*(A) \neq 0$. If $f|_A$ is continuous on A, then there exists a set $B \subset A$ such that $\mu^*(B) > 0$, and $f|_B$ is continuous on B as a subspace of the Euclidean topology on $\mathbb{R}$.*

Theorem 2.1. *Under continuum hypothesis, $(\mathbb{R}, \tau_d)$ is not a Blumberg space.*

Proof. In $(\mathbb{R}, \tau_d)$, $\mu^*(A) > 0 \Rightarrow A$ is uncountable.

C. G o f f m a n in [9] has proved that under CH, there exists a function $f: \mathbb{R} \to \mathbb{R}$ such that if $A \subset \mathbb{R}$ and $f|_A$ is $(\tau_0|_A - \tau_0)$ continuous, then A is countable.

Hence if CH is assumed, the above lemma gives that there exists a $f: \mathbb{R} \to \mathbb{R}$. Such that if $A \subset \mathbb{R}$ and if $f|_A$ is $(\tau_d - \tau_0)$ continuous, then $\mu^*(A) = 0$. Hence it follows from this that if CH is assumed, there does not exist any set A such that $f|_A$ is continuous where $\mu^*(A) > 0$, i.e., $(\mathbb{R}, d)$ is not a Blumberg space.

It follows that $(\mathbb{R}, \tau_d)$ has neither a σ-disjoint or locally countable pseudo-base nor it has a dense pseudo metrizable subspace D ([24]). In fact $(\mathbb{R}, \tau_d)$ is not semi-metrizable. Theorem 2.1 also shows us that there exists co-compact, α-favourable, completely regular Hausdorff, pseudo-complete space which is not Blumberg.

Indeed, because that $(\mathbb{R}, \tau_d)$ is a category measure space its non-Blumberg property follows also from the following theorem ([25]).

Theorem 2.2. *Let X be a Baire space of cardinality $2^{\aleph_0}$ such that*
(i) X satisfies C.C.C.
(ii) wt. of X = $2^{\aleph_0}$ = density character of X.
(iii) every 1-cat. set in X is nowhere dense in X. Then X does not have the Blumberg property.

Without even considering the CH, non-Blumbergness in case of $(\mathbb{R}, \tau_d)$ can be derived from this theorem, since any subset $A \subset \mathbb{R}$ such that $|A| < 2^{\aleph_0}$ has $\mu(A) = 0$. If $(Y, \mathcal{U})$ is a compactification of $(\mathbb{R}, \tau_d)$, then any subset of Y of cardinality $2^{\aleph_0}$ is a Baire space satisfying the hypothesis of the above theorem.

Another condition for non-Blumbergness of τ_d-topology is due to J. L u k e s and L. Z a j i c e k ([16]).

Theorem 2.3. *Let X be a topological space without isolated points such that any dense subset of X is of cardinality $2^{\aleph_0}$. If the cardinality of the set $\{b(A): A \subset X\}$, where $b(A)$ = derived set of A, is $\leq 2^{\aleph_0}$, then X is not Blumberg.*

We have that in $(\mathbb{R}^n, \tau_d)$, (a) any derived set is of type $G_{\delta\sigma}$ in the Euclidean topology, (b) for every $A \subset \mathbb{R}^n$ such that $|A| \leq 2^{\aleph_0}$ has $\mu(A) = 0$ and (c) $\mathbb{R}^n$ is a Baire space without isolated point.

3. The necessary conditions for which Blumberg's property is true for a Baire space in general, make also a density space a Blumberg space. This requires suitable restrictions on domain or range or even the function. For instance if in any density space (X, τ_d) we consider $f: X \to Y$ be a function such that for every open set B of X , there exists a somewhere dense subset C (i.e. $\overline{C}^{\circ} \neq \emptyset$) of B such that $f|_C$ is continuous, then it can be seen that f is Blumberg.

A coarser topology than a density topology can also serve for the Blumberg property.

Definition. *Let $(\mathbb{R}^n, \tau_d)$ be the density topological space and $B = \{U \subset \mathbb{R}^n: U$ is τ_d-open and U is both F_σ and $G_\delta\}$. The topology $\mathcal{T}$ generated*

by B as its basis is called **the r-modification** *of ordinary density topology τ_d. It is also called the r-***topology** *or* **O'Malley topology**.

The following theorem was proved in [19].

Theorem. *The topology $(\mathbb{R}, \mathcal{T})$ is a Blumberg space.*

A generalised version of this theorem will be proved in the next section.

4. We denote A_T° and $\bar{A}_T$ for interior and closure of A in topology T.

Definition. [22] *Two topologies T and D on a set X are said to be* **S-related** *if for every $A \subset X$, $A_D^\circ \neq \emptyset$ if and only if $A_T^\circ \neq \emptyset$. We write $T \underset{S}{\sim} D$*

The definition easily gives the following deductions :

Proposition 1. *If a topology D is finer than T and $T \underset{S}{\sim} D$ then*

a) $\bar{A}_D^\circ = \emptyset$ *implies* $\bar{A}_T^\circ = \emptyset$ *and conversely.*
b) $\bar{A}_D^\circ \backslash A_T^\circ$ *is both T and D nowhere dense.*
c) Denseness and Baire property are preserved under S-relation.

Theorem 4.1. *Let (X, T) be a Baire space, and D be a topology on X such that,*

$$(i)\ D \text{ is finer than } T \qquad (ii)\ D \underset{S}{\sim} T.$$

Then if (X, T) is Blumberg, so is (X, D).

Proof. (i) First we show that every T-dense subset of X is D-dense. Let Y be a T-dense set in X, and U be any D-open set ($U \neq \emptyset$). Then

$$U_T^\circ \neq \emptyset \quad (T \underset{S}{\sim} D)$$

But $U_T^\circ \cap Y \neq \emptyset$. This implies $U \cap Y \neq \emptyset$. Hence Y is D-dense.

(ii) Next, suppose $f\colon X \to \mathbb{R}$. For any $r \in \boldsymbol{Q}$ let $M_r = \{x \in X\colon f(x) > r\}$ and $N_r = \{x \in X\colon f(x) < r\}$. Consider the set,

$$F_r = [(M_r)_D^\circ \backslash (M_r)_T^\circ] \cup [(N_r)_D^\circ \backslash (N_r)_T^\circ].$$

By Proposition 1 (b), F_r is a 1-cat set in T.

If f is D-continuous at a point x where f is not T-continuous, then there exists a $r \in \boldsymbol{Q}$ such that $x \in F_r$.

So the points of discontinuity of f is an F_σ-set P such that

$$P = \bigcup_{r \in \boldsymbol{Q}} F_r.$$

Hence P is a T-1st cat. set and since X is Baire, the residual set P^c is dense (see [20, 7.4]). Thus every D-continuous function is T-continuous almost everywhere on X and the converse is also true.

Above (i) and (ii) show that if (X, T) is Blumberg, so is (X, D). The proof is completed

It is an immediate concequence of the above theorem that

Corollary 1. *If (X, τ_d) is a density topology on a topological space (X, T) such that τ_d is finer than T and if $\mathcal{T}$ is any 'r' or a.e. modification 'a' of τ_d ([18]) then $(X, \mathcal{T})$ is Blumberg if (X, T) is so. ('a' and 'r' are both S-related to T).*

Corollary 2. *$(\mathbb{R}^n, a)$ and $(\mathbb{R}^n, r)$ are both Blumberg.*

5. Variations of definitions of τ_d-topologies are seen in literature. In addition to the ordinary τ_d-topologies (measure-theoretic), other most common examples of such topologies are their category analogues called the $\mathcal{I}$-density topology $T_{\mathcal{I}}$, deep-$\mathcal{I}$-density topology $T_{\mathcal{D}}$ constructed by W. W i l c z y n s k i ([26]) and developed and generalised in many succeeding papers. The topologies are refinements of the natural topology on $\mathbb{R}$, and have many rich and complicated structures mostly useful for study of the characterisation of real functions in Analysis. We recall some definitions and properties below with standard references.

Let $\mathcal{B}$ be the family of all subsets of $\mathbb{R}$ having the Baire property, $\mathcal{I}$ be the σ-ideal of sets of the first category in $\mathbb{R}$, and $\mathcal{S}$ be the set of all measurable sets. Then we define

Definition 1. [26] **0 is an $\mathcal{I}$-density of $A \in \mathcal{S}$** *iff for every increasing sequence $\{n_m\}_{m\in\boldsymbol{N}}$ of positive real numbers tending to infinity, there exists a subsequence $\{n_{m_p}\}_{p\in\boldsymbol{N}}$, such that the sequence $\{\chi_{(n_{m_p}\cdot A)\cap(-1,1)}\}$ converges to 1 excepts a set belonging to $\mathcal{I}$ (or, equivalently iff $\chi_{(nA)\cap(-1,1)} \xrightarrow[n\to\infty]{\mathcal{I}} 1$).*

Definition 2. *$x_0 \in \mathbb{R}$ is an* **$\mathcal{I}$-density point of $A \in \mathcal{B}$,** *iff 0 is an $\mathcal{I}$ point of $A - x_0 = \{x - x_0 \mid x \in A\}$.*

$\phi_{\mathcal{I}}(A)$ and $\mathcal{I}$-dispersion points are also defined similar to ordinary τ_d-topology. The $\mathcal{I}$-density operator $\phi_{\mathcal{I}}(A)$ is restricted to $\mathcal{B}$ satisfying the conditions of a lower density ([20], [26]) including the Lebesgue density theorem, viz., $\phi_{\mathcal{I}}(A)\Delta A \in \mathcal{I}$. Then (as usual) the topology $T_{\mathcal{I}}$ induced by this lower density is known as the $\mathcal{I}$-density topology; and

$$T_{\mathcal{I}} = \{\phi_{\mathcal{I}}(A) - I : \quad A \in \mathcal{S},\ I \in \mathcal{I}\}.$$

Properties. [26] *0 is an $\mathcal{I}$-density point of A iff $\liminf_{n\to\infty}[(nA)\cap(-1,1)]$ is residul in $(-1,1)$; 0 is an $\mathcal{I}$-dispersion point of A iff $\limsup_{n\to\infty}[(nA)\cap(-1,1)]$ is of first category.*

$T_{\mathcal{I}}$ is not separable, not Lindelöf, not regular and not a generated space; and it is such that every $C \subset \mathbb{R}$ is closed and discrete iff $C \in \mathcal{I}$. A set is compact in $T_{\mathcal{I}}$ iff A is finite. These properties along with the theorem 2.2 above gives us the following.

Theorem 5.1. *Under CH, $T_{\mathcal{I}}$ is not Blumberg.*

Definition 3. [26] *A point $p \in I\!R$ is called a* **deep-$\mathcal{I}$-density point** *of a set $A \in \mathcal{B}$, iff exists a closed set $F \subset A \cup \{p\}$ such that p is an $\mathcal{I}$-density of F.*

The lower density $\phi_{\mathcal{D}}(A) = \{x \in I\!R : x$ is a deep-$\mathcal{I}$-density point of $A\}$ induces a topology $T_{\mathcal{D}} = \{A \in \mathcal{B} : A \subset \phi_{\mathcal{D}}(A)\}$ called the deep-$\mathcal{I}$-density topology.

This topology is finer than the natural topology τ_0 but coarser than $T_{\mathcal{I}}$. $T_{\mathcal{D}}$ is non-normal but completely regular, and separable. We show

Theorem 5.2. *The deep-$\mathcal{I}$-density topology $T_{\mathcal{D}}$ is S-related to τ_0 and independent of CH it is Blumberg (τ_0-natural topology on $I\!R$).*

Proof. First we show $T_{\mathcal{D}} \underset{S}{\sim} \tau_0$. Let $D \subset I\!R$ be τ_0-dense. Let U be any $T_{\mathcal{I}}$-open set which contains D. U has a Baire property and by Theorem 4.4 in ([20], p.20) U can be expressed as disjoint union of a G_δ set G and a set N of 1st category. We observe that if x belongs to any $T_{\mathcal{I}}$-open set H, then x is an $\mathcal{I}$-density point of H, and for every $\delta > 0$, $(x - \delta, x + \delta) \cap H$ is of 2-category. Hence U^c is of 1-cat. If $x \in I\!R$ and $x \in U^c$, any $T_{\mathcal{I}}$-open set H containing x must intersect U.

But $\tau_0 \subset T_{\mathcal{D}} \subset T_{\mathcal{I}}$. Hence if U is $T_{\mathcal{D}}$-open containing D, then $\forall x$, any $T_{\mathcal{D}}$-open set H containing x must intersect U at a 2-cat. set. Since $T_{\mathcal{D}}$ is regular, $H \cap D \neq \emptyset$ and so D is dense in $T_{\mathcal{D}}$. Thus $T_{\mathcal{D}} \underset{S}{\sim} \tau_0$.

Hence by theorem 4.1, $T_{\mathcal{D}}$ is Blumberg since τ_0 is Blumberg.

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