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A FIXED POINT THEOREM FOR MAPPINGS WITH CONTRACTIVE ITERATE IN PM–SPACES[1]

BY

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The notions which we use in this paper follow the book [13] (S c h w e i z e r and S k l a r, 1983). In the following two definitions $(S, \mathcal{F})$ is a PSM–space and $(b_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of positive numbers with $\lim\limits_{n\to\infty} b_n = 1$.

Definition 1.1. *([10]) We say that $(S, \mathcal{F})$ is a probabilistic (b_n)–structure if*

$$(n \in \mathbb{N}, F_{pq}(s) \geq b_n, \quad F_{qr}(t) \geq b_n) \Rightarrow F_{pr}(s+t) \geq b_n.$$

Definition 1.2. *The t–norm T is called a (b_n)–t norm if $T(b_n, b_n) = b_n$ for every $n \in \mathbb{N}$.*

For example, every t–norm of H a d ž i ć type (see [5] or [12]) is a (b_n)–t norm for some sequence (b_n). We also remark that Min is a (b_n)–t norm for every sequence (b_n).

Example 1.3. Every Menger space $(S, \mathcal{F}, T)$ under the (b_n)–t norm T is a probabilistic (b_n)–structure.

Proposition 1.4. *If $(S, \mathcal{F})$ is a probabilistic (b_n)–structure, then, for every $n \in \mathbb{N}$, the mapping r_n defined on S^2 by*

$$r_n(p, q) = \inf\{t \mid F_{pq}(t) \geq b_n\}$$

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is a pseudometric on S. Moreover, the uniformity generated by the family $(r_n)_{n\in\mathbb{N}}$ is the $\mathcal{F}$–uniformity.

The mapping r_n first appears in [11] (Radu,1992) and the proof of Proposition 1.4. in the case of Menger spaces $(S, \mathcal{F}, T)$ under the (b_n)–t norm T appears in [12] (Radu,1994).

The following proof of Proposition 1.4. is similar to that of Theorem 2.3.18 from [12]. We give it here just for the sake of completness.

Proof of Proposition 1.4. Since $F_{pp}(t) = 1$ for each $t > 0$, then $r_n(p, p) = 0$.

On the other hand, let us remark that $r_n(p, q) = 0$ iff $F_{pq}(t) \geq b_n$ for all $t > 0$. Since $\lim_{n\to\infty} b_n = 1$, then we see that $r_n(p, q) = 0\ \forall n$ iff $F_{pq}(0+) = 1$ i.e. iff $p = q$.

By the symmetry of $\mathcal{F}$ it follows that r_n is symmetric.

For the triangle inequality, if we suppose that $r_n(p, q) < a$ and $r_n(p, q) < b$, then $F_{pr}(a) \geq b_n, F_{rq}(b) \geq b_n$. Therefore we have $F_{pq}(a + b) \geq b_n$ and so $r_n(p, q) \leq a + b$.

It is clear that $r_n \leq r_{n+1}$ so that $(r_n)_{n\in\mathbb{N}}$ is an augmented family of ècarts. Obviously we have

$$r_n(p, q) < a \Rightarrow F_{pq}(a) \geq b_n$$

and

$$F_{pq}(\varepsilon) > 1 - \lambda \geq b_n \Rightarrow r_n(p, q) \leq \varepsilon,$$

for which the second part of the proposition easily follows.

In the second part of this note we will use a fixed point theorem in uniform spaces from [2] (Cain Jr., 1985).

Definition 2.1. *[2] Let X be a set, $\mathcal{D}$ be a separating family of pseudo–metrics and $d \in \mathcal{D}$. A self map f of X is said to have a d–contractive iterate at $x \in X$ if there exists a constant $k_d \in (0, 1)$ and a positive integer $n(x, d)$ such that*

$$d(f^{n(x,d)}(x), f^{n(x,d)}(y)) \leq k_d d(x, y)$$

for all $y \in X$.

Lemma 2.2. *([2], Theorem 1) If $(X, \mathcal{D})$ is sequentially complete and f is a continuous mapping on X having for each $d \in \mathcal{D}$ a d– contractive iterate at every $x \in X$, then f has exactly one fixed point u and $f^n(x) \to u$ for every $x \in X$.*

Definition 2.3. *Let $(S, \mathcal{F})$ be a PSM–space and (b_n) be a strictly increasing sequence of positive numbers with $\lim_{n\to\infty} b_n = 1$. We say that the mapping $A: S \to S$ have a (b_n)– contractive iterate at $p \in S$ if*

$$\forall n \in I\!N\ \exists k_n \in (0,1)\ \exists v = v(p,n) \in I\!N: (t > 0, q \in S,\quad F_{pq}(t) \geq b_n)$$
$$\Rightarrow F_{A^v pA^v q}(k_n t) \geq b_n.$$

Theorem 2.4. *Let $(S, \mathcal{F})$ be a complete probabilistic (b_n)– structure and suppose that $A: S \to S$ is a continuous mapping which has a (b_n)– contractive iterate at every $p \in S$. Then A has a unique fixed point which can be obtained by the successive approximation method, starting from an arbitrary point of S.*

Proof. We will show that for every $n \in I\!N$ and $p \in S$ we have

$$r_n(A^v p, A^v q) \leq k_n r_n(p,q) \quad for\ every \quad q \in S.$$

Indeed, let $n \in I\!N$ and $p \in S$ be fixed. Then, for every $\varepsilon > 0$ and every $q \in S$,

$$k_n r_n(p,q) < \varepsilon \Rightarrow r_n(p,q) < \frac{\varepsilon}{k_n} \Rightarrow F_{pq}\left(\frac{\varepsilon}{k_n}\right) \geq b_n \Rightarrow$$
$$F_{A^v pA^v q}(\varepsilon) \geq b_n \Rightarrow r_n(A^v p, A^v q) \leq \varepsilon.$$

Therefore $k_n r_n(p,q) \geq r_n(A^v p, A^v q)$ and we can apply Lemma 2.2.

Particular cases.

Corollary 3.5. *([1],Theorem 3) Let $(S, \mathcal{F}, Min)$ be a complete Menger space and f be a continuous map of S into it self. Suppose that there exists $\delta \in (0,1)$ such that for each $\alpha \in (0,\delta)$ there is a $k_\alpha \in (0,1)$ with the property that to each $x \in S$ there coresponds a positive integer $n(x,\alpha)$ for which, letting $u = f^{n(x,\alpha)}(x), v = f^{n(x,\alpha)}(y)$,*

$$F_{uv}(k_\alpha t) \geq F_{xy}(t) \quad for\ \ all \quad t > 0.$$

Then f has a unique fixed point w and $f^n(x) \to w$ for all $x \in X$.

Corollary 3.6. *([7]) If $(S, \mathcal{F}, Min)$ is a complete Menger space and A is a continuous map with the property that there is a $k \in (0,1)$ such that for each $x \in S$ there exists a $n(x) \in I\!N$ such that*

$$F_{uv}(kt) \geq F_{xy}(t) \quad for\ \ all \quad t > 0$$

where $u = A^{n(x)}(x), v = A^{n(x)}(y)$, then A has a unique fixed point u^* and $A^n(x) \to u^* \quad for\ \ each \quad x \in S$.

Corollary 3.7. *([6]) Let $(S, \mathcal{F}, T)$ be a complete Menger space under a (b_n)–t–norm T. If $f: S \to S$ is a continuous mapping with the property that for every $p \in S$ there exists a $n(p) \in \mathbb{N}$ such that, for every $q \in S$ and every $t > 0$,*

$$F_{f^{n(p)}(p) f^{n(p)}(q)}(kt) \geq F_{pq}(t),$$

with $k \in (0,1)$, then f has a unique fixed point $s \in S$ and, for any $r \in S$, $\lim_{n\to\infty} f^n(r) = s$.

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