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EXPONENTIALLY CONVEX FUNCTIONS ON FOUNDATION TOPOLOGICAL SEMIGROUPS

BY

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Abstract. The aim of this paper is to find the integral representation of the exponentially convex function on a commutative foundation topological $*$–semigroup. Namely, we find the necessary and sufficient condition for a w–bounded exponentially convex function to be represented in the integral form. The uniqueness of the corresponding measure is also discussed.

Introduction. Exponentially convex functions were first introduced by B e r n s t e i n, cf.[5]. Both B e r n s t e i n and W i d d e r obtained the integral representation of these functions in the one–dimensional case, cf.[1] A k h i e z e r, in [1], indicates how the result can be extended to k–dimensional spaces. See also [5]. Similar results were also obtained in the case of open convex sets, cf.[6] O k b E l–B a b et al, in [7,8,9] proved similar results in the case of infinite– dimensional spaces, free groups and ℓ_p–space. In this work we develop these results for an extensive class of topological semigroups, the so–called "Foundation topological semigroups". In section 1 of this paper we give the basic definitions and notions that we shall use later. In section 2, we begin by proving some lemmae and theorems which we use later in proving our main result. Finally, we obtain our main theorem dealing with the integral representation for the w–bounded exponentially convex functions on a foundation topological semigroup with a continuous identity involution and a Borel measurable weight function ω.

1. Definitions and notations. Let S be a locally compact semigroup, $M(S)$ be the algebra of the bounded Radon measures μ defined on S and $\widetilde{L}(S)$ be the algebra of all $\mu \in M(S)$ for which the mappings $x \to \overline{x} * |\mu|$ and $x \to |\mu| * \overline{x}$ of S to $M(S)$ are weakly continuous. Here $\overline{x}$ stands for the point mass at x. S is said to be a foundation semigroup if it coincides with the closure of the union of the supports of all measures $\mu \in \widetilde{L}(S)$, see [2,3,4]. There are many examples of foundation semigroups, for examples:

1 – Let S be additive semigroup of non–negative real numbers in the usual topology. The algebra $\widetilde{L}(S)$ is clearly $L^1(S)$, algebra of measures on S which are absolutely continuous w.r.t. the usual Haar measure on $\boldsymbol{R}$, so that S is a foundation semigroup.

2 – Let G be a locally compact, commutative group, and S be G together with an isolated adjoint zero, 0. Clearly $\widetilde{L}(S)$ is the direct sum of $L^1(S)$ and the scalar multiples of the point mass at 0. Thus S is a foundation semigroups.

3 – Let $S = [0, 1]$ with the restriction topology of $\boldsymbol{R}$ and multiplication defined by $xy = \min(x+y, 1)$ for all $x, y \in [0, 1]$. Then S is a foundation semigroup.

For more examples and details see [3,11].

A semigroup S is called a $*$–semigroup if there is mapping $* : S \to S$ such that $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in S$. If S is also topological, we assume that $*$ is continuous. In our study, we consider the involution $*$ as the identity involution.

A real–valued function ω on a $*$–semigroup S is said to be a weight function if $\omega(x) \geq 0$ and $\omega(xy) \leq \omega(x)\omega(y)$ for all $x, y \in S$. A function $f : S \to \boldsymbol{R}$ is ω–bounded if there exists $k > 0$ such that

$$|f(x)| \leq k\omega(x) \quad \text{for all } x \in S.$$

A function $\rho : S \to \boldsymbol{R}$ is called a semicharacter if

$$\rho(xy) = \rho(x)\rho(y) \quad \text{for every } x, y \in S.$$

Moreover, $\widetilde{L}(S, \omega)$, the space of all measures μ in $M(S, \omega)$ such that $\omega|\mu| \in \widetilde{L}(S)$ is a two–sided ideal of $M(S, \omega)$. Finally, for a locally compact Hausdorff space X, we denote by $C(x)$, $C_\circ(x), C_{\circ\circ}(x)$, the algebras of all continuous functions on X that are bounded, vanish at infinity, or have compact support, respectively.

If ρ is a semicharacter on a $*$ –semigroup S such that $\rho(x^*) = \rho(x)$ for all $x \in S$, then ρ is said to be a $*$ –semicharacter. We denote the set of all bounded continuous $*$–semicharacters on S by S^*. Let S^*_+ denote the space of all positive $*$-semicharacters in S^*, i.e., the space of all $\rho(x) \geq 0$ for all $x \in S$.

If S has a weight function ω, then the space of all bounded continuous semicharacters on S will be denoted by Γ_ω. It is clear that for $\rho \in \Gamma_\omega$ we have

$$|\rho(x)| \leq \omega(x) \quad \text{for all } x \in S.$$

If S has an involution, then we denote by Γ_ω^* the space of all $*$–semicharacters in Γ_ω.

Let $M(S,\omega)$ be the space of all regular Borel measures μ on S such that $\omega|\mu| \in M(S)$ with the norm,

$$||\mu||_\omega = \int\limits_S \omega d|\mu|.$$

Clearly $M(S,\omega)$ can be identified with the dual of the Banach space $\ell_0(S,\omega)$ of all Borel measurable functions f on S such that $f/\omega \in \ell_0(S,\omega)$ under the norm given by

$$||f||_\omega = \sup\{|f/\omega(x)| : x \in S\},$$

via pairing

$$< \mu, f >= \mu(f) = \int\limits_S f(x)d\mu(x)$$

for every $\mu \in M(S,\omega)$ and $f \in \ell_0(S,\omega)$.

2. Exponentially convex functions on foundation semigroups. Let S be a $*$–semigroup. A continuous function $\varphi : S \to \boldsymbol{R}$ is said to be exponentially convex (E.C.) iff

$$\sum_{i,j=1}^{n} \varphi(x_i x_j) a_i a_j \geq 0$$

for every finite subset $\{x_1, \ldots, x_n\}$ of S and $\{a_1, \ldots, a_n\}$ of $\boldsymbol{R}$ [5].

Lemma 2.1. *Let S be a topological $*$–semigroup with identity. Then each (E.C.) function φ on S satisfies:*
(i) $\varphi(s) = \varphi(s^*)$,
(ii) $\varphi(ss^*) \geq 0$,
(iii) $|\varphi(st^*)| \leq \varphi(ss^*)\varphi(tt^*)$,
for $s,t \in S$.

Proof. We omit the proof of this lemma because it is standard, see for example [5].

Lemma 2.2. *Let S be a topological $*$–semigroup with identity. If φ is a ω– bounded (E.C.) function on S, then the formula*

$$T_\varphi(\mu) = \int_S \varphi(x) d\mu(x),$$

where μ belogs to the space $M(S,\omega)$ of all regular Borel measures on S such that $\omega|\mu| \in M(S)$ with norm

$$||\mu||_\omega = \int_S \omega d|\mu|,$$

defines a bounded positive functional T_φ on the Banach $*$–algebra $M(S,\omega)$ such that
(i) $T_\varphi(\mu^*) = \overline{T_\varphi(\mu)}, \qquad \mu \in M(S,\omega),$
(ii) $|T_\varphi(\mu)|^2 \leq k T_\varphi(\mu * \mu^*), \ \mu \in M(S,\omega),$
for some positive constant k.

Proof. Since φ is ω–bounded, then $|\varphi(x)| \leq k\omega(x)$ for all $x \in S$. Taking $k = 1$ we get $|\varphi(x)| \leq \omega(x)$ for all $x \in S$. Hence,

$$|T_\varphi(\mu)| = \left| \int_S \varphi(x) d\mu(x) \right| \leq \int_S |\varphi(x)| d\mu(x) \leq \int_S \omega(x) d\mu(x) = ||\mu||_\omega,$$

for every μ in $M(S,\omega)$. Thus T_φ defines a bounded linear functional on $M(S,\omega)$. Now we prove that T_φ is a positive functional; i.e., $T_\varphi(\nu * \nu^*) \geq 0$ for all $\nu \in M(S,\omega)$.

This is can be done by proving this inequality for all $\nu \in M(S,\omega)$ which take the form $f \cdot \mu$ where $f \in C_{\circ\circ}(S)$ and $\mu > 0$ in $M(S,\omega)$, by the ω–norm density of $M_k(S)$ (the algebra of all measures in $M(S)$ with compact supports) in $M(S,\omega)$. In fact,

$$T_\varphi(\nu * \nu^*) = \int_S \varphi(x) d(\nu * \nu^*)(x) = \int_S \int_S \varphi(xy) f(x) \overline{f(y)} d\mu(x) d\mu(y), \text{see [5]}.$$

Let K be the support of f, then $\varphi(xy)f(x)\overline{f(y)}$ will be uniformly continuous on $K \times K$ and K can be partitioned into disjoint sets $E_1, \ldots, E_n$ such that the sum

$$\sum_{i,j=1}^{n} \varphi(x_i y_j) f(x_i) \overline{f(y_j)} \mu(E_i) \mu(E_j) \quad (x_i \in E_i) \tag{2.1}$$

differs from the integral

$$\int_S \int_S \varphi(xy) f(x) f(y) d\mu(x) d\mu(y) \tag{2.2}$$

by as little we like. Since φ is E.C, we get that (2.1) is greater than or equal to zero and so is (2.2). Hence $T_\varphi(\nu * \nu^*) \geq 0$.
Clearly (i) is verified due to lemma (2.1). Now, we prove that (ii) is verified, which completes the proof. In fact, put $T_\varphi(\mu * \nu^*) \equiv <\mu, \nu>$ for every $\mu, \nu \in M(S, \omega)$. Hence $<\mu, \nu>$ is linear in μ and $<\mu, \mu> \geq 0$. This means that $<\mu, \nu>$ satisfies the Hilbert space inner product properties and it satisfies the Schwarz inequality which takes the form.

$$| <\mu, \nu> |^2 \leq <\mu, \mu><\nu, \nu>, (\mu, \nu \in M(S, \omega)). \tag{2.3}$$

Since $1 \in M(S, \omega)$, we have by using (2.3)

$$|T_\varphi(\mu)|^2 = | <\mu, 1> |^2 \leq <\mu, \mu><1, 1> = \varphi(1) T_\varphi(\mu * \mu^*)$$

for all $\mu \in M(S, \omega)$. Taking $\varphi(1) = k$ we arrive to statement (ii). ∎

Theorem 2.3. *Let S be a commutative topological $*$–semigroup with identity, S^* be locally compact (with the topology of uniform covergence on compact subsets of S) and Γ be a closed subset of S^*. If $\lambda \in M(\Gamma)$ satisfies $\int_\Gamma \rho(x) d\lambda(\rho) = 0$ for all $x \in S$, then $\lambda = 0$.*

Proof. Suppose that $\lambda \neq 0$. For each $x \in S$, define $\widetilde{x}$ on Γ by $\widetilde{x}(\rho) = \rho(x)$ for all $\rho \in \Gamma$. By the set $\{\widetilde{x} : x \in S\}$ let us generate an algebra, denoted A. This algebra defines a self–conjugate algebra of $C(\Gamma)$ which separates the points of Γ and contains the constant function 1. By lemma 3.5 of [4] we get that A is (L^1–norm) dense in $L^1(\Gamma, |\lambda|)$. Let f be a Borel function on Γ with $|f| = 1$, $d\lambda = f d|\lambda|$ and $d|\lambda| = \overline{f} d\lambda$. Choosing $\varepsilon = \frac{1}{2}\|\lambda\|$, we can find $g \in A$ such that

$$\int_\Gamma |f - g| d|\lambda| < \varepsilon.$$

Hence $|\int_\Gamma \overline{f} d\lambda - \int_\Gamma \overline{g} d\lambda| < \varepsilon$. Since $\int_\Gamma \overline{g} d\lambda = 0$ we get $|\int_\Gamma \overline{f} d\lambda| < \varepsilon$. Therefore $2\varepsilon = ||\lambda|| = \int_\Gamma \overline{f} d\lambda < \varepsilon$. From this contradiction we see that $\lambda = 0$. ∎

Theorem 2.4. *Let S be a commutative topological $*$–semigroup with a weight ω. Let Γ_ω with the topology of uniform convergence on compact subsets of S, be a locally compact (Hausdorff) space and Γ_ω^1 be a closed subset of Γ_ω. Then*

$$\varphi(x) = \int_{\Gamma_\omega^1} \rho(x) d\lambda(\rho) \qquad (x \in S)$$

where λ is a non–negative measure in $M(\Gamma_\omega^1)$, is a ω–bounded E.C. function on S.

Proof. To see that φ is ω–bounded let ω_k be the weight associated with a compact subset K of Γ_ω^1 by $\omega_k = \sup\{|\rho(x)| : \rho \in K\},\ x \in S$. In the special case of $K = \mathrm{supp}(\lambda)$ we find

$$|\varphi(x)| \leq \lambda(K)\omega_k = \varphi(0)\omega_k \quad \text{for } x \in S,$$

which shows that φ is ω–bounded.

Evidently, φ is also continuous. To prove the exponential convexity of φ, let $\{x_1, \dots, x_n\} \subseteq S$ and $\{a_1, \dots, a_n\} \subseteq R$. Then

$$\sum_{i,j=1}^{n} a_i a_j \varphi(x_i x_j) = \int_{\Gamma_\omega^1} |\sum_{k=1}^{n} a_k \rho(x_k)|^2 d\lambda(\rho) \geq 0. \quad \blacksquare$$

Theorem 2.5. *Let S be a commutative foundation topological $*$–semigroup with identity and with a weight ω. Then there is a one–to–one correspondence between Γ_ω and $\widetilde{L}(S,\omega)^\wedge$ which is defined as follows:*
i) for each $\rho \in \Gamma_\omega$ the corresponding h in $\widetilde{L}(S,\omega)^\wedge$ is given by

$$h(\mu) = \int_S \rho(x) d\mu(x) \qquad (\mu \in \widetilde{L}(S,\omega)), \tag{2.4}$$

ii) for each $h \in \widetilde{L}(S,\omega)^\wedge$ the corresponding ρ in Γ_ω is given by

$$\rho(x) = \frac{h(\mu * x)}{h(\mu)} \qquad (x \in S), \tag{2.5}$$

where μ is any element of $\widetilde{L}(S,\omega)$ for which $h(\mu) \neq 0$.

The proof of this theorem follows from the proof of Theorem 4.4 of [4] with a very slight modification, so we omit it.

Theorem 2.6. *Let A be a commutative Banach $*$–algebra, $\widehat{A}$ be the set of all non–zero multiplicative linear functional on A and*

$$\Delta = \{\tau \in \widehat{A} : \tau(x^*) = \tau(x), \quad \textit{for all } x \in A\}.$$

Let p be a positive functional on A satisfying conditions i) and ii) of lemma (2.2.). Then there exists a non– negative measure $\lambda \in M(\Delta)$ such that

$$\rho(x) = \int\limits_{\Delta} \widehat{x}(\tau) d\lambda(\tau) \textit{ for all } x \in A. \tag{2.6}$$

The converse is also true.

Proof. The proof can be obtained by adjoining an identity to A and then applying the results given on page 120 of [10], and the integral form of the Krein–Milman theorem [10, p.6].

Theorem 2.7. *Let S be a commutative foundation topological $*$–semigroup with identity and ω be a weight function on S. The necessary and sufficient condition for the function $\varphi : S \to \boldsymbol{R}$ to be ω–bounded E.C. is that it has the integral representation*

$$\varphi(x) = \int\limits_{\Gamma^*_\omega} \rho(x) d\lambda_\varphi(\rho) \qquad (x \in S) \tag{2.7}$$

where λ_φ is a non–negative measure from $M(\Gamma^*_\omega)$. The uniqueness of λ_φ is verified if either $\omega = 1$ on S or λ_φ has compact support.

Proof. Clearly, the necessary condition follows from theorem 2.4. Now to prove the sufficient condition, suppose that φ is a ω–bounded E.C. function on S. Define a functional T_φ in the form

$$T_\varphi(\mu) = \int\limits_{S} \varphi(x) d\mu(x) \qquad\qquad (\mu \in \widetilde{L}(S, \omega)). \tag{2.8}$$

It is easy to see that T_φ is a bounded positive linear functional defined on the Banach $*$–algebra $\widetilde{L}(S, \omega)$ and it satisfies i) and ii) of lemma 2.2. Thus by theorem 2.6 there exists a positive measure $\lambda_\varphi \in M(\Delta)$ such that

$$T_\varphi(\mu) = \int\limits_{\Delta} h(\mu) d\lambda_\varphi(h) \tag{2.9}$$

for all $\mu \in \widetilde{L}(S,\omega)$, where Δ is the set of all bounded non–zero $*$–multiplicative linear functionals on $\widetilde{L}(S,\omega)$. Now, applying formulae (2.4) and (2.5) of theorem 2.5 we get that Δ is homeomorphic to Γ^*_ω, and from formulae (2.8) and (2.9) we obtain, for every $\mu \in \widetilde{L}(S,\omega)$,

$$\int\limits_S \varphi(x)d\mu(x) = \int\limits_\Delta h(\mu)d\lambda_\varphi(h) = \int\limits_{\Gamma^*_\omega}\int\limits_S \rho(x)d\mu(x)d\lambda_\varphi(\rho) =$$

$$= \int\limits_S\int\limits_{\Gamma^*_\omega} \rho(x)d\lambda_\varphi(\rho)d\mu(x).$$

Comparing both sides of this equality we get

$$\varphi(x) = \int_{\Gamma^*_\omega} \rho(x)d\lambda_\varphi(\rho) \qquad (x \in S),$$

which is the formula (2.6). Note that φ is a ω–bounded continuous function and the right side is the same by Theorem 2.4. This proves the sufficient condition. By Theorem 2.3, λ_φ is unique if $\omega = 1$. In case when λ_φ has compact support, the uniqueness of λ is proved as follows. In fact, the set of all functions of the form $\rho \to T_\rho(e)$, $T \in \widetilde{L}(S,\omega))$ is clearly dense in the space of continuous functions on each compact subset of S^*. Suppose that there exists $\nu \in M(\Gamma^*_\omega)$ such that

$$\varphi(x) = \int\limits_{\Gamma^*_\omega} \rho(x)d\nu(\rho)x \in S \qquad \text{for each } x \in S.$$

Then we claim, $\mathrm{supp}(\nu) \subseteq \mathrm{supp}(\mu)$, where $\mu \in M(\Gamma^*_\omega)$ is any measure with compact support satisfying (2–6). For if not, there exists a compact set K, disjoint from $\mathrm{supp}(\mu)$, such that $\nu(K) > 0$. Choose $\varepsilon > 0$ such that

$$\varepsilon^2\varphi(e) < \nu(K).$$

We can find a real–valued function $\rho \to T_\rho(e)$ which is greater than 1 on K and is strictly between 0 and ε on $\mathrm{supp}(\mu)$. Then $\rho \to T^2_\rho(e)$ is non–negative and

$$\varepsilon^2\varphi(e) < \int T^2_\rho(e)d\nu = \int T^2_\rho(e)d\mu < \varepsilon^2\varphi(e),$$

which is a contradiction. Uniqueness now follows by density of the functions $\rho \to T_\rho(e)$ in the space of continuous functions on $\mathrm{supp}(\mu)$. ∎

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