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# ON THE MONOTONE ITERATIVE TECHNIQUE FOR THE SECOND ORDER NONLINEAR VOLTERRA TYPE BOUNDARY VALUE PROBLEM

BY

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**Abstract.** We use the method of upper and lower solution to develop a monotone iterative scheme to prove the existence of solutions of the second order nonlinear boundary value integro–differential equations of the Volterra type with nonlinear boundary conditions and the nonlinearity of the right hand side satisfying the Caratheodory conditions.

**1. Introduction.** Let $I = [0,1]$, $I^\circ = (0,1)$ and $\alpha, \beta : I \to R$ are continuous real valued functions such that $\alpha(t) \leq \beta(t)$ for all $t \in I$. Let $L^2(I)$ and $W^{2,2}(I)$ be the usual Lebesgue and Sobolev spaces of real functions defined on $I$ respectively and $T$ is any linear Volterra operator on $C(I,R)$ into $C(I,R)$ which is assumed to be monotone nondecreasing, that is, $\phi, \psi \in C(I,R), \phi(s) \leq \psi(s)$ and $0 \leq s \leq t \leq 1$ implies $[T\phi](t) \leq [T\psi](t)$.

Let $E = \{(t, Tx, x, y) : t \in I, \alpha(t) \leq x \leq \beta(t), Tx, y \in R\}$ and $f : E \to R$ is a Caratheodory function for $L^2(I)$. We shall consider the second order nonlinear boundary value integro–differential equations of Volterra type,

$$x''(t) = f(t, Tx(t), x(t), x'(t)), \text{ a.e. } t \in I \tag{1}$$

$$L_\mu(x(\mu), x'(\mu)) = 0 \quad \mu = 0, 1 \tag{2}$$

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where $L_\mu$ is a continuous numerical function satisfying monotonic conditions which we shall specify later. If the $BVP$ (1), (2) does not include the term $Tx$ then it is clearly the $BVP$

$$x''(t) = f(t, x(t), x'(t)) \text{ a.e. } t \in I \tag{3}$$

$$L_\mu(x(\mu), x'(\mu)) = 0 \quad \mu = 0, 1 \tag{4}$$

where the nonlinearity satisfies Caratheodory's conditions. In this case the properties of solutions have been studied by Adjei [1,2,3]. Moreover the literature in both the scalar and the vectorial version of the $BVP$ (3), (4) is very extensive (see [5,6,7,8,9,10,11] and the references there in).
However, nothing seems to be available in the literature concerning the properties of solutions of the $BVP$ (1), (2).

In this paper we develop monotone iterative schemes using the method of upper and lower solutions to find solutions of the $BVP$ (1), (2) satisfying nonlinear boundary conditions with the nonlinearity $f$ also satisfying the Caratheodory conditions. It will be interesting to see if our results can be extended to the case where $T$ is a nonlinear Volterra operator.

**2. Preliminaries.** Let $J_1 = [\min_I T\alpha, \max_I T\beta]$, $J_2 = [\min_I \alpha, \max_I \beta]$ then since $f$ is a Caratheodory function for $L^2(I)$, $f(t, Tx, x, y)$ is measurable for each $(Tx, x, y) \in J_1 \times J_2 \times \boldsymbol{R}$. Besides $f(t, ., ., .)$ is continuous for almost every $t \in I$, and for any real constant $k$ there exists a function $\phi_k \in L^2(I)$ such that $|f(t, Tx, x, y)| \leq \phi_k(t)$ for a.e. $t \in I$ and all $(Tx, x, y) \in J_1 \times J_2 \times \boldsymbol{R}$ with $|Tx| \leq k$, $|x| \leq k$, $|y| \leq k$.

In what follows we shall assume that the function $f(t, u, v, w)$ is non-decreasing in $u$ for each fixed $(t, v, w)$ and set

$Q = \max\left\{\max_I |\alpha'|, \max_I |\beta'|\right\}$.

Assume that,

(I) For $\alpha, \beta \in W^{2,2}(I)$

$$\alpha''(t) \geq f(t, T\alpha(t), \alpha(t), \alpha'(t)) \text{ a.e. } t \in I,$$
$$\beta''(t) \leq f(t, T\beta(t), \beta(t), \beta'(t)) \text{ a.e. } t \in I,$$

(II) There exists numbers $N_1 > 0, N_2 > 0$ such that

$$f(t, Tx_1, x_1, y_1) - f(t, Tx_2, x_2, y_2) \geq -N_1(x_2 - x_1) - N_2(Tx_2 - Tx_1)$$
$$\text{for } t \in I, \alpha(t) \leq x_1 \leq x_2 \leq \beta(t), \ |y_i| \leq Q,$$

(III) There exists $R, L \in L^\infty(I)$ such that $R(t) > 0,\ L(t) > 0$ on $I$ and

$$-R(t)|y_1 - y_2| \le f(t, Tx, x, y_1) - f(t, Tx, x, y_2) \le L(t)|y_1 - y_2|$$
$$\alpha(t) \le x(t) \le \beta(t),\ y_i \in \boldsymbol{R},\ i = 1, 2, \text{ and } t \in I,$$

(IV) $L_0 : [\alpha(0), \beta(0)] \times \boldsymbol{R} \to \boldsymbol{R}$ is continuous, $L_0(s,t)$ is increasing in $s$ and nonincreasing in $t$. Moreover,

$$L_0(\alpha(0), \alpha'(0)) \le 0 \le L_0(\beta(0), \beta'(0)),$$

and

(V) $L_1 : [\alpha(1), \beta(1)] \times \boldsymbol{R} \to \boldsymbol{R}$ is continuous, $L_1(s,t)$ is increasing in $s$ and in $t$. Moreover,

$$L_1(\alpha(1), \alpha'(1)) \le 0 \le L_1(\beta(1), \beta'(1)).$$

## 3. Main results.

**Theorem 3.1.** *Let (I), (III), (IV) and (V) hold. Then the BVP (1), (2) has at least one solution $x$ such that $\alpha(t) \le x(t) \le \beta(t)$ for all $t \in I$.*

**Proof.** $f$ satisfies the Caratheodory condition so $\exists\ \ g \in L^2(I)$ such that $|f(t, Tx, x, 0)| \ \le g(t)$ for all $(t, Tx, x, 0) \in E$. By (III) we can fix $m(t) \in L^1(I)$ such that if $|y| \le 1$ for a.e. $t \in I$,

$$\max\{|f(t, T\alpha(t), \alpha(t), y)|, |f(t, T\beta(t), \beta(t), y)|,\ g(t)\} < m(t) \tag{5}$$

Define the modified function $F$ by setting,

$$F(t, Tx, x, y) = f(t, Tx, p(t,x), y) + m(t)(x - p(t,x))$$

where $p : I \times \boldsymbol{R} \to \boldsymbol{R}$ is defined as $p(t,x) = \max[\alpha(t), \min(x, \beta(t))]$. For $0 \le \lambda \le 1$ define $(BVP)_\lambda$

$$x''(t) = (1-\lambda)m(t)x(t) + \lambda F(t, Tx(t), x(t), x'(t)) \text{ a.e. } t \in I,$$

$$x(0) = \lambda[p(0, x(0)) - L_0(p(0, x(0)), x'(0))]$$
$$x(1) = \lambda[p(1, x(1)) - L_1(p(1, x(1)), x'(1))].$$

We now show that any solution $x$ of $(BVP)_1$ is such that

$$\alpha(t) \le x(t) \le \beta(t) \text{ for } \quad \text{all } t \in I. \tag{6}$$

Any such solution $x$ is in fact a solution of $BVP$ (1), (2) and therefore the proof is complete.
Suppose (6) is not true, then $(x-\beta)(t) > 0$ for some $t \in I$. Hence $\exists t^* \in I$ such that $(x-\beta)(t^*) = \max(x-\beta)(t) = \varepsilon > 0$.
Thus if $t^* \in I^\circ$, $x'(t^*) - \beta'(t^*) = 0$, and $x''(t^*) - \beta''(t^*) \leq 0$. Hence the functions $x-\beta$ and $x-\beta-|x'-\beta'|$ are strictly positive in some interval $I^* = (t^*-\varepsilon, t^*+\varepsilon) \subset I$. So for a.e. $t \in I^*$ and by (III),

$$\begin{aligned} x''(t) &= f(t, Tx(t), \beta(t), x'(t)) + m(t)(x(t)-\beta(t)) \\ &\geq f(t, T\beta(t), \beta(t), x'(t)) + m(t)(x(t)-\beta(t))) \\ &\geq f(t, T\beta(t), \beta(t), \beta'(t)) - m(t)|x'(t)-\beta'(t)| \\ &\qquad + m(t)(x(t)-\beta(t)) \\ &\geq f(t, T\beta(t), \beta(t), \beta'(t)) \geq \beta''(t), \end{aligned}$$

which contradicts $x''(t^*) \leq \beta''(t^*)$. If $t^* = 0$, $x'(0) \leq \beta'(0)$ and by the property of $L_0$,

$$x(0) = \beta(0) - L_0(\beta(0), x'(0)) \leq \beta(0) - L_0(\beta(0), \beta'(0)) \leq \beta(0),$$

which is a contradiction. If $t^* = 1$, $x'(1) \geq \beta'(1)$ and the property of $L_1$ implies,

$$x(1) = \beta(1) - L_1(\beta(1), x'(1)) \leq \beta(1) - L_1(\beta(1), \beta'(1)) \leq \beta(1)$$

which again is a contradiction. So $x(t) \leq \beta(t)$ for all $t \in I$. A similar argument shows that $\alpha(t) \leq x(t)$ for all $t \in I$ and the proof is complete.

**Lemma 3.2.** *Let the assumptions of Theorem 3.1 hold. Then any solution $x$ of $BVP$ (1), (2) with $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in I$ is such that $|x|_{C(I)}$ is bounded.*

**Proof.** Let $M = \max\{|\alpha|_{C(I)}, |\beta|_{C(I)}\}$, and $t_0 \in I$ such that $x(t_0) = \max_I x(t) = \delta > 0$, then $x(t_0) \leq M+1$. Suppose not, then for any $\varepsilon > 0$, and $t \in I^{**} = (t_0-\varepsilon, t_0+\varepsilon) \subset I, x(t) > \beta(t)+1$ and $|x'(t)| < 1$. If $t_0 \in I^\circ$, $x'(t_0) = 0$ and for a.e. $t \in I^{**}$, and any possible solution of $(BVP)_\lambda$,

$$\begin{aligned} x''(t) &= (1-\lambda)m(t)x(t) + \lambda[f(t, Tx(t), \beta(t), x'(t)) + m(t)(x(t)-\beta(t))] \\ &\geq (1-\lambda)m(t)x(t) + \lambda(f(t, T\beta(t), \beta(t), x'(t)) + m(t)(x(t)-\beta(t)) \\ &\geq (1-\lambda)m(t)x(t) \geq 0, \end{aligned}$$

using (5). Hence $x(t_0)$ cannot be a maximum, which is a contradiction. If $t_0 = 0,\ x'(0) < 0$ and

$$\begin{aligned} x(0) &= \lambda[p(0,x(0)) - L_0(p(0,x(0)),x'(0)) \\ &\le p(0,x(0)) - L_0(p(0,x(0)),0) \le D_1 \end{aligned}$$

where $D_1 = \max_{s\in \boldsymbol{R}} |p(0,s) - L_0(p(0,s),0)|$. If $t_0 = 1,\ x'(1) \ge 0$ and

$$\begin{aligned} x(1) &= \lambda[p(1,x(1)) - L_1(p(1,x(1),x'(1)) \\ &\le \lambda[p(1,x(1)) - L_1(p(1,x(1)),0)] = D_2 \end{aligned}$$

where $D_2 = \max_{s\in \boldsymbol{R}} |p(1,s) - L_1(p(1,s),0)|$. Therefore $\max_{t\in I} x(t) = x(t_0) \le M + 1 = M_1$. A similar argument shows that there exists $M_2$ such that $\min_{t\in I} x(t) = -M_2$. Hence we can find $B > 0$ such that $|x|_{C(I)} < B$.

**Theorem 3.3.** *Let (I), (III), (IV), (V) hold. Then there exists $C > 0$ independent of $x$ a solution of $BVP$ (1),(2) such that $\alpha(t) \le x(t) \le \beta(t),\ t \in I$ with $|x|_{C'(I)} < C$.*

**Proof.** Let $0 \le \lambda \le 1$, and $x$ any possible solution of $(BVP)_\lambda$ a.e. $t \in I$,

$$\begin{aligned} &|x''(t)| \le (1-\lambda)m(t)|x(t)| + \lambda|F(t,Tx(t),x(t),x'(t)| \le \\ &\le (1-\lambda)m(t)|x(t)| + \lambda m(t)(|x(t)| + M) + \lambda|f(t,Tx(t),\beta(t),x'(t))| \le \\ &\le (1-\lambda)m(t)|x(t)| + \lambda m(t)(|x(t)| + M) + \lambda|f(t,T\beta(t),\beta(t),x'(t))| \le \\ &\le (1-\lambda)m(t)|x(t)| + \lambda m(t)(|x(t)| + M) + \lambda f(t,T\beta(t),\beta(t),0) + \lambda m(t)|x'(t)|. \end{aligned}$$

By Lemma 3.2.,

$$|x''(t)| \le m(t)(1 + M + B) + m(t)|x'(t)|.$$

Gronwall's inequality implies the existence of $C > 0$ such that $|x|_{C'(I)} < C$.

We now establish a uniqueness result.

**Lemma 3.4.** *Assume that there exists $\alpha, \beta \in W^{2,2}(I)$ such that $\alpha(t) \le \beta(t)$ for all $t \in I$ and (I), (II), (III), (IV) and (V) are satisfied. Let $[\alpha,\beta] = \{x \in C(I) : \alpha(t) \le x(t) \le \beta(t),\ t \in I\}$. Then for any $\eta \in [\alpha,\beta]$, the $(BVP)_\eta$*

$$(BVP)_\eta \qquad \begin{aligned} &x''(t) = G_\eta(t,Tx(t),x(t),x'(t)) \quad \text{a.e.} \quad t \in I \\ &L_0(x(0),x'(0)) = 0 = L_1(x(1),x'(1)), \end{aligned}$$

*where* $G_\eta(t, Tu, u, v) = f(t, T\eta(t), \eta(t), v) + N_1(u - \eta(t)) + N_2(Tu - T\eta(t))$, *has a unique solution* $x(t)$.

**Proof.** For any $\eta \in [\alpha, \beta]$ we have for a.e. $t \in I$,

$$\begin{aligned} G_\eta(t, T\alpha(t), \alpha(t), \alpha'(t)) &= f(t, T\eta(t), \eta(t), \alpha'(t)) + N_1(\alpha(t) - \eta(t)) \\ &\quad + N_2(T\alpha(t) - T\eta(t)) \leq f(t, T\alpha(t), \alpha(t), \alpha'(t)) \\ &\leq \alpha''(t), \end{aligned}$$

using (I), (II) and the property of the operator $T$. Similarly we observe that also,

$$\begin{aligned} G_\eta(t, T\beta(t), \beta(t), \beta'(t)) &= f(t, T\eta(t), \eta(t), \beta'(t)) + N_1(\beta(t) - \eta(t)) \\ &\quad + N_2(T\beta(t) - T\eta(t)) \geq f(t, T\beta(t), \beta(t), \beta'(t)) \\ &\geq \beta''(t). \end{aligned}$$

The $(BVP)_\eta$ thus satisfies the hypothesis of Lemma 3.2 and Theorem 3.3 and so $(BVP)_\eta$ has at least one solution $x \in W^{2,2}(I)$ such that $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in I$ and $|x(t)|_{C'(I)} < C$ for some constant $C$ independent of $x$. We show it is unique. Assume that for some $\eta \in [\alpha, \beta]$, the $(BVP)_\eta$ has two solutions $x(t)$ and $y(t)$. Define $v(t) = x(t) - y(t)$, then for a.e. $t \in I$,

$$\begin{aligned} v''(t) &= G_\eta(t, Tx(t), x(t), x'(t)) - G_\eta(t, Ty(t), y(t), y'(t)) \\ &\geq -|R|_{L^\infty(I)}|x'(t) - y'(t)| + N_1 v(t) + N_2 Tv(t) \\ &\geq N_1 v(t) + N_2 Tv(t) - |R|_{L^\infty(I)}|v'(t)|. \end{aligned} \tag{7}$$

Suppose $v \neq 0$ a.e. on $I$ and assume $v$ has a positive maximum at some $t_0 \in I$, then if $t_0 \in (0, 1), v'(t_0) = 0$. Since $v \in W^{2,2}(I)$ there exists a sequence $\{t_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} t_n = t_0$ and $v''(t_n) \leq 0$ for all $n = 1, 2, 3, 4, ...$ If we set $t = t_n$ in (7), we have

$$v''(t_n) \geq N_1 v(t_n) + N_2 Tv(t_n) - |R|_{L^\infty(I)}|v'(t_n)|.$$

In the limit, and since $T$ is a linear operator, $0 \geq N_1 v(t_0) + N_2 Tv(t_0) > 0$, which is a contradiction. Hence $v(t) = 0$ for a.e. $t \in I^\circ$. If $t_0 = 0, v'(0) \leq 0$ which implies $x'(0) \leq y'(0)$ and by the property of $L_0(s, t)$,

$$0 = L_0(y(0), y'(0)) = L_0(x(0), x'(0)) \geq L_0(x(0), y'(0)),$$

which implies $y(0) \geq x(0)$, that is, $v(0) \leq 0$ which is a contradiction. If $t_0 = 1, v'(1) \geq 0$ and $x'(1) \geq y'(1)$. Again the property of $L_1(s, t)$ implies, $0 = L_1(y(1), y'(1)) = L_1(x(1), x'(1)) \geq L_1(x(1), y'(t))$, which in turn implies $y(1) \geq x(1)$ or $v(1) \leq 0$ also a contradiction. Hence $v(t) = 0$ for a.e. $t \in I$, that is, for every $\eta \in [\alpha, \beta]$, the $(BVP)_\eta$ has a unique solution $x(t)$.

We now present the result of a monotone iterative scheme for the BVP (1),(2).

**Theorem 3.5.** *Assume that the hypothesis of Lemma 3.4 hold. Then there exist monotone sequences $\alpha_k$ and $\beta_k$ in $W^{2,2}(I)$ which converge uniformly on $I$ to solutions $u$ and $w$ of the $BVP$ (1),(2) and satisfies on $I$ the inequalities,*

$$\alpha(t)\leq\alpha_1(t)\leq\alpha_2(t)\leq\ldots\leq\alpha_k(t)\leq u(t)\leq w(t)\leq\beta_k(t)\leq\ldots\leq\beta_2(t)\leq\beta_1(t)\leq\beta(t).$$

**Proof.** By Lemma 3.4, for every $\eta\in[\alpha,\beta]$, the $(BVP)_\eta$ has a unique solution $x_\eta$. Define the mapping $P:[\alpha,\beta]\to[\alpha,\beta]$ by setting for $\eta\in[\alpha,\beta], P\eta=x_\eta$ where $x_\eta$ is the unique solution of $(BVP)_\eta$. We claim that (i) $\alpha\leq P\alpha$ $(ii) P\beta\leq\beta$ and (iii) $P$ is monotone nondecreasing.
Case (i). Set $P\alpha=\alpha_1$ and let $v(t)=\alpha(t)-\alpha_1(t)$. Suppose $\alpha(t)-\alpha_1(t)>0$ for some $t\in I$, then $v$ has a positive maximum at some $t_0\in I$. If $t_0\in I^\circ$, then $v(t_0)>0, v'(t_0)=0$ and there exists a sequence $\{t_n\}\in I^\circ$ such that $v''(t_n)\leq 0$ for $n=1,2,3,\ldots$ with $\lim\limits_{n\to\infty} t_n=t_0$. For such $t_n$,

$$0\geq v''(t_n)\geq N_1v(t_n)+N_2Tv(t_n)-|R|_{L^\infty(I)}|v'(t_n)|.$$

In the limit, since $T$ is linear and monotone, $0\geq v''(t_0)\geq N_1v(t_0)+N_2Tv(t_0)>0$, which is a contradiction. Hence $v(t)\leq 0$ for all $t\in I^\circ$. If $t_0=0, \alpha'(0)-\alpha'(0)\leq 0$ and $0=L_0(\alpha(0),\alpha'(0))=L_0(\alpha_1(0),\alpha_1'(0))\leq L_0(\alpha_1(0),\alpha'(0))$, which implies $\alpha(0)\leq\alpha_1(0)$ a contradiction. Similarly if $t_0=1$ we arrive at a contradiction. Hence $\alpha(t)-(P\alpha)(t)\leq 0$ on $I$ which proves claim (i).
Case (ii). Using the same type of arguments as in case (i), we obtain $P\beta\leq\beta$ on $I$.
Case (iii). Let $\alpha(t)\leq\eta_1(t)\leq\eta_2(t)\leq\beta(t)$ and suppose $P\eta_1=u_1$ and $P\eta_2=u_2$. If $z=u_1-u_2$ and suppose there exist some $t\in I$ such that $z(t)>0$, then $z$ has a positive maximum at some $t_0\in I$. Clearly, arguments similar to the one above leads to a contradiction and hence $P$ is monotone nondecreasing on $I$.

Define $\{\alpha_k\}$ and $\{\beta_k\}$ as follows :

$$\alpha_0 = \alpha, \alpha_1 = P\alpha = P\alpha_0, \ldots, \alpha_{k+1} = P\alpha_k, k = 0, 1, 2, 3, \ldots$$

$$\beta_0 = \beta, \beta_1 = P\beta = P\beta_0, \ldots, \beta_{k+1} = P\beta_k, k = 0, 1, 2, 3, \ldots$$

By induction $\alpha_k \leq \beta_k$ for all $k$ on $I$, hence

$$\alpha_0 = \alpha \leq \alpha_1 \leq \ldots \leq \alpha_k \leq \beta_k \ldots \beta_1 \leq \beta_0 = \beta \text{ on } I.$$

Theorem 3.3. implies there exists $Q > 0$ such that for all $k$ and $u_k \in \{\alpha_k, \beta_k\}$, $|u_k|_{C'(I)} < Q$. Hence sequences $\{u_k\}$ and $\{u'_k\}$ are uniformly bounded and equicontinuous. Ascoli–Arzela Theorem implies the existence of subsequences $\{u_{k_j}\}$ and $\{u'_{k_j}\}$ which converge in $C(I)$. Let $u = \lim_{k\to\infty} \alpha_k$ and $w = \lim_{k\to\infty} \beta_k$, then $u, w \in C(I)$ and $\alpha'_k, \beta'_k$ converge to $u'$ and $w'$ respectively in $C(I)$. For all $k$ and $u_k \in \{\alpha_k, \beta_k\}$ and $t, t_0 \in I$, we have,

$$\begin{aligned} u_k(t) &= u_k(t_0) + u'_k(t_0)(t - t_0) + \\ &+ \int_{t_0}^{t} (t - s)[f(s, Tu_{k-1}(s), u_{k-1}(s), u'_{k-1}(s)]ds + \\ &+ \int_{t_0}^{t} (t - s)[N_1(u_k - u_{k-1})(s) + N_2(Tu_k - Tu_{k-1})(s)]ds, \end{aligned}$$

and $L_0(u_k(0), u'_k(0)) = 0 = L_1(u_k(1), u'_k(1))$.
Taking limits, $u$ is a solution of the $BVP$ (1),(2). Similarly $w$ is a solution and by definition of the sequences, $\alpha(t) \leq \alpha_1(t) \leq \ldots \leq \alpha_k(t) \leq u(t) \leq w(t) \leq \beta_k(t) \leq \ldots \leq \beta_1(t) \leq \beta(t)$, for all $t \in I$.

**Remark 3.5.** If $I = [a, b]$ the results presented here remain valid. Let $f(t, Tx(t), x(t), x'(t)) = f(t, x(t), x'(t)) + Tx(t)$, where for $t, t_0 \in I, Tx(t) = \int_{t_0}^{t} K(t, s)x(s)ds \quad t_0 \leq s \leq t$ and $|K(t, s)| \leq K_0$. Then we have a special case of the $BVP$ (1),(2) of the form

$$x''(t) = f(t, x(t), x'(t)) + \int_{t_0}^{t} K(t, s)x(s)ds, \quad 0 < t < 1 \tag{8}$$

$$L_\mu(x(\mu), x'(\mu)) = 0, \quad \mu = 0, 1 \tag{9}$$

The case $K(t,s) = 0$ means that the $BVP$ (8),(9) does not include the integral term and the corresponding results on solutions have been studied by Adjei [1,2,3] with stronger conditions on $f$. If in addition $L_\mu$ is linear, then we also have several known results as special cases [4,5,7,11]. It will be of interest to extend the results obtained here to the case where $T$ is a nonlinear Volterra operator of the form $Tx(t) = \int_{t_0}^{t} S(t,s,x(s))ds, \quad t, t_0 \in I,\ t_0 \leq s \leq t$, where $S \in C(I \times I \times C(I, \boldsymbol{R}), \boldsymbol{R})$ and $S(t,s,x)$ is monotone nondecreasing in $x$ for fixed $t$ and $s$. The general problem for nonlinear Volterra operator $T$ remains open.

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