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A RADON–NIKODYM THEOREM FOR MULTIMEASURES

BY

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Dedicated to Professor Constantin Corduneanu on the occasion of his 70th birthday

1. Introduction. One of the most interesting problems arising in the theory of integration is that of the existence of a Radon–Nikodym derivative. We have introduced in [6] a set–valued integral for multifunctions with respect to multimeasures. In this paper we obtain a Radon–Nikodym theorem for the integration described in [6]. This is done under conditions of the classical Maynard–type. So, essential in the construction of Radon–Nikodym derivatives is the idea of exhaustion, first identified by M a y n a r d [12], whereby the underlying set is partitioned into subsets having a certain prescribed property.

2. Preliminaries and definitions. Let S be a nonempty set, let $\mathcal{A}$ be an algebra of subsets of S and let X be a real commutative Banach algebra. $\mathcal{P}_{kc}(X) = \mathcal{P}_{kc}$ is the family of all nonempty compact convex subsets of X and D is the usual Hausdorff metric defined on $\mathcal{P}_{kc}$.

We define $\|A\| = D(A, 0)$, $A \in \mathcal{P}_{kc}$, where 0 is used in place of $\{0\} \in \mathcal{P}_{kc}$. If $A, B \subseteq \mathcal{P}_{kc}$, $\lambda \in \mathbb{R}$,

$$(*) \qquad A + B = \{x + y \mid x \in A,\ y \in B\},$$

$AB = \{xy \mid x \in A,\ y \in B\}$, $\lambda A = \{\lambda x \mid x \in A\}$.

Let $\mathcal{K} \subset \mathcal{P}_{kc}$ be a semigroup with identity 0 under the operation $(*)$, satisfying the conditions:

1. AB is convex for every $A, B \in \mathcal{K}$.
2. $A(B + C) = AB + AC$, $\forall A, B, C \in \mathcal{K}$.

A multimeasure $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ is a finitely additive multifunction (namely $\varphi(E_1 \cup E_2) = \varphi(E_1) + \varphi(E_2)$, whenever $E_1, E_2 \in \mathcal{A}$, $E_1 \cap E_2 = \emptyset$).

The total variation $v(\varphi, \cdot)$ of a multimeasure φ is the non–negative (possibly infinite) set function defined on $\mathcal{A}$ as follows:

$$v(\varphi, A) = \sup \sum_{i=1}^{n} \|\varphi(E_i)\|$$

where the supremum is taken over all finite partitions $\{E_i\}_{i=1}^{n}$ of $A \in \mathcal{A}$, with $E_i \in \mathcal{A}$ for all $i = 1, ..., n$.

Let $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ be a multimeasure and let $\mu : \mathcal{A} \to \mathbb{R}_+$ be a finitely additive measure. φ is said to be μ–continuous, written $\varphi \ll \mu$, if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $E \in \mathcal{A}$ and $\mu(E) < \delta$ implies $v(\varphi, E) < \varepsilon$.

Let $F, G : S \to \mathcal{P}_{kc}$ be multifunctions. We denote by $\widetilde{D}(F, G)$ the function defined on S as $(\widetilde{D}(F, G))(s) = D(F(s), G(s))$ and $\|F\| = \widetilde{D}(F, \theta)$, where θ is the multifunction defined by $\theta(s) = 0$, $\forall s \in S$.

Definition 2.1.

a) A family $\mathcal{C} \subset \mathcal{P}_{kc}$ is bounded if there exists $L > 0$ such that $\|B\| \leq L$, $\forall B \in \mathcal{C}$.

b) A multifunction $F : S \to \mathcal{P}_{kc}$ (resp. $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$) is bounded if the family $\{F(s) \mid s \in S\}$ (resp. $\{\varphi(A) \mid A \in \mathcal{A}\}$) is bounded.

Definition 2.2. Let $\mu : \mathcal{A} \to \mathbb{R}$ be a finitely additive measure and let $|\mu|$ be the total variation of μ. A finite or countable family of pairwise disjoint sets $(E_i)_i \subset \mathcal{A}$ will be called a μ–exhaustion of S if for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $|\mu| \left(S - \bigcup_{i=1}^{n_0} E_i \right) < \varepsilon$.

Definition 2.3. Let $M : \mathcal{A} \to \mathcal{P}_{kc}$ be a bounded multimeasure, let $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ be a multimeasure with its total variation ν and assume $\nu(S) < \infty$. Let $\mathcal{A}^+ = \{E \in \mathcal{A} \mid \nu(E) > 0\}$ and $\mathcal{A}^2 = \{E \in \mathcal{A} \mid \nu(E) < 2\|\varphi(E)\|\}$. For any $E \in \mathcal{A}$ and $\varepsilon > 0$ we denote:

$$\begin{aligned} E\mathcal{A} &= \{A \in \mathcal{A} \mid A \subseteq E\}, \\ E\mathcal{A}^+ &= \{A \in \mathcal{A}^+ \mid A \subseteq E\}, \\ E\mathcal{A}^2 &= \{A \in \mathcal{A}^2 \mid A \subseteq E\}. \end{aligned}$$

We introduce the following "ranges" of M:

$$A(E\mathcal{A}^2) = \left\{ \frac{M(B)}{\nu(B)};\ B \in E\mathcal{A}^2,\ \nu(B) > 0 \right\}$$

$$A(E, \varepsilon) = \{C \in \mathcal{K};\ D(M(B), C \cdot \nu(B)) \leq \varepsilon \cdot \nu(B),\ \forall B \in E\mathcal{A}\}$$

$$\tilde{A}(E, \varepsilon) = \{C \in \mathcal{K};\ D(M(B), C \cdot \varphi(B)) \leq \varepsilon \cdot \nu(B),\ \forall B \in E\mathcal{A}\}$$

that will be called average range, ε–approximate range and ε–approximate m–range respectively.

Definition 2.4. Let $\mu : \mathcal{A} \to \mathrm{I\!R}$ be a finitely additive measure and let $|\mu|$ be its total variation.

a) A set property P is said to be μ–exhaustive on a set $E \in \mathcal{A}$ if there exists a μ–exhaustion $(E_i)_i \subset E\mathcal{A}$ such that E_i has P for every i.
b) A set property P is null difference if whenever $A, B \in \mathcal{A}^+$, $|\mu|(A\Delta B) = 0$, either A and B both have P or neither does.

Lema 2.5. *Let M, φ and ν be like in Definition* 2.3*. If $M \ll \nu$, then the property $\tilde{A}(E, \varepsilon) \neq \emptyset$ is null difference.*

Proof. The proof is analogous to that of Lemma 3.3 in [11].

Remark 2.6. From Lemma 3.3 [11] and Lemma 2.5 above it easily results that the property $A(E, \varepsilon) \cap \tilde{A}(E, \varepsilon) \neq \emptyset$ is null difference too.

Remark 2.7. Let $\mu : \mathcal{A} \to \mathrm{I\!R}$ be a finitely additive measure. It can prove that if $(S, \mathcal{A}, \mu)$ is complete and P is a null difference set property such that P is μ–exhaustive on S, then it is always possible to obtain a μ–exhaustion of S, $(E_i)_i$, such that each E_i has P and $\bigcup\limits_i E_i = S$.

The following lemma will be needed.

Lema 2.8. *If $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ is a multimeasure, ν is the total variation of φ, $\nu(S) < \infty$ and $E \in \mathcal{A}^+$ then either $E \in \mathcal{A}^2$ or there exists $B \subset E$ such that $B \in \mathcal{A}^2$.*

3. A Radon–Nikodym theorem.

Throughout this section, $\varphi : \mathcal{A} \to \mathcal{K}$ will be a multimeasure, ν will be the total variation of φ and assume $\nu(S) < \infty$.

The theory of integration for multifunctions $F : S \to \mathcal{P}_{kc}$ with respect to a multimeasure φ is that described in [6].

Lema 3.1. *Let $M : \mathcal{A} \to \mathcal{P}_{kc}$ be a bounded multimeasure such that:*

(3.1.1) $\quad M \ll \nu$

(3.1.2) $\quad A(S\mathcal{A}^2)$ *is bounded*

(3.1.3) $\quad$ *for every* $\varepsilon > 0$, *the set property* $A(E, \varepsilon) \cap \tilde{A}(E, \varepsilon) \neq \emptyset$ *is* ν *– exhaustive on every* $E \in \mathcal{A}^+$

Then there exists a φ–integrable multifunction $F : S \to \mathcal{P}_{kc}$ such that $F(S)$ is bounded and $\int_E F\, d\varphi = M(E)$ for all $E \in \mathcal{A}$.

Proof. Since a multifunction which satisfies the conclusion for the completion of $(S, \mathcal{A}, \nu)$ will do so for $(S, \mathcal{A}, \nu)$ as well, it suffices to prove the lemma for complete $(S, \mathcal{A}, \nu)$.

Analogously to what is done in [9] we can obtain a sequence of ν–exhaustions of S, $(E^n_\alpha)_n$, $\alpha \in \mathbb{N}^n$ such that:

(3.1.4) $A(E^n_\alpha, 2^{-n}) \cap \tilde{A}(E^n_\alpha, 2^{-n}) \neq \emptyset$ for every $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$

(3.1.5) $E^n_\alpha = \bigcup\limits_i E^{n+1}_{\alpha,i}$, where $(E^{n+1}_{\alpha,i})_i$ is a $\nu-$exhaustion of E^n_α for every $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$

(3.1.6) for every fixed $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n, S = \bigcup\limits_\alpha E^n_\alpha$ and $(E^n_\alpha)_\alpha$ is a $\nu-$exhaustion of S.

Now, let $C^n_\alpha \in A(E^n_\alpha, 2^{-n}) \cap \tilde{A}(E^n_\alpha, 2^{-n})$ for each n and α and define $F_n = \sum\limits_\alpha C^n_\alpha \cdot 1_{E^n_\alpha}$.

Using Lemma 2.8 it can prove that $\|C^n_\alpha\| \leq 1 + L$ for every $\alpha \in \mathbb{N}^n$, $n \in \mathbb{N}$, where $L = \sup\limits_{B \in A(S\mathcal{A}^2)} \|B\|$. From Proposition 19 [6] it results that F_n is φ–integrable.

For $m, n \in \mathbb{N}$, $m < n$, $\alpha \in \mathbb{N}^m$, $\beta \in \mathbb{N}^{n-m}$, $C^m_\alpha, C^n_{\alpha,\beta} \in A(E^m_{\alpha,\beta}, 2^{-m}) \cap \cap \tilde{A}(E^m_{\alpha,\beta}, 2^{-m})$ and using Lemma 2.8 it follows that $D(C^m_\alpha, C^n_{\alpha,\beta}) \leq 2^{1-m}$.

This implies that

$$\widetilde{D}(F_m, F_n) \leq 2^{1-m}. \tag{1}$$

Hence the sequence $\{F_n(s)\}_n$ is Cauchy in $\mathcal{P}_{kc}$, uniformly with respect to $s \in S$.

Let $F(s) = \lim\limits_{n \to \infty} F_n(s)$, $\forall s \in S$.

From (1) and the definition of F it results that $D(F(s), 0) \leq 1 + L$, $\forall s \in S$. From Proposition 22 [6] it follows that F is φ–integrable and

$$\lim_{n \to \infty} D\left(\int_E F_n \, d\varphi, \int_E F \, d\varphi\right) = 0, \quad \forall E \in \mathcal{A}.$$

Next we prove that

$$D\left(M(E), \int_E F \, d\varphi\right) = 0, \ \forall E \in \mathcal{A}. \tag{2}$$

Let $\varepsilon > 0$ fixed; for every $E \in \mathcal{A}$ and $n \in \mathbb{N}$ fixed, the family $\{E \cap E^n_\alpha\}_{\alpha \in \mathbb{N}^n}$ is a ν–exhaustion of E. Let $q \in \mathbb{N}$ be such that for every $\alpha \in \mathbb{N}^n, \alpha < (q, \ldots, q)$,

$$\nu\left(E - \bigcup_{\alpha < (q,\ldots,q)} (E \cap E^n_\alpha)\right) < \delta$$

where δ is determined from the condition (3.1.1) written for $\frac{\varepsilon}{4}\cdot$

Let $n_1 \in \mathbb{N}$ such that

$$D\left(\int_E F_n \, d\varphi, \int_E F \, d\varphi\right) < \frac{\varepsilon}{4} \tag{3}$$

$$D\left(\sum_{\alpha<(q,\dots,q)} C_\alpha^n \cdot \varphi(E \cap E_\alpha^n), \int_E F_n \, d\varphi\right) < \frac{\varepsilon}{4} \tag{4}$$

for every $n \geq n_1$. Since $C_\alpha^n \in \tilde{A}(E_\alpha^n, 2^{-n})$,

$$\begin{aligned} D\left(\sum_{\alpha<(q,\dots,q)} M(E \cap E_\alpha^n), \sum_{\alpha<(q,\dots,q)} C_\alpha^n \cdot \varphi(E \cap E_\alpha^n)\right) \leq \\ \leq \sum_{\alpha<(q,\dots,q)} 2^{-n} \cdot \nu(E \cap E_\alpha^n) \leq \frac{1}{2^n} \cdot \nu(E). \end{aligned} \tag{5}$$

Since (3.1.1),

$$D\left(M(E), M\left(\bigcup_\alpha (E \cap E_\alpha^n)\right)\right) < \frac{\varepsilon}{4}\cdot \tag{6}$$

From (3), (4), (5) and (6) it follows that $D\left(M(E), \int_E F \, d\varphi\right) < \varepsilon$, $\forall \varepsilon > 0$. Thus (2) holds and the lemma follows.

Definition 3.2. A set property P is hereditary if $A, B \in \mathcal{A}^+$, $B \subseteq A$ and A has P implies B has P.

Remark 3.3. It is easy to see that the properties $A(E,\varepsilon) \neq \emptyset$, $\tilde{A}(E,\varepsilon) \neq \emptyset$ and $A(E,\varepsilon) \cap \tilde{A}(E,\varepsilon) \neq \emptyset$ are hereditaries.

The following result will be used in the sequel.

Proposition 3.4. (Exhaustion principle) *Let $\mu : \mathcal{A} \to \mathbb{R}$ be a finitely additive measure of bounded variation, let $|\mu|$ be the total variation of μ and let P be a hereditary set property. Then the following two statements are equivalent:*

(e_1) *P is μ–exhaustive on each element of $\mathcal{A}^+$.*

(e_2) *for each $\delta > 0$, there exists $C \in \mathcal{A}^+$ and $\alpha \in (0,1)$ such that:*

- (e_2') $|\mu|(S - C) < \delta$.
- (e_2'') *for all $E \in C\mathcal{A}^+$, there exists $K \in E\mathcal{A}^+$ such that $|\mu|(K) > \alpha|\mu|(E)$ and K has P.*

Proof. See Proposition 3.2 [9].

Theorem 3.5. (Radon–Nikodym) *Suppose* $M : \mathcal{A} \to \mathcal{P}_{kc}$ *is a bounded multimeasure (let* $0 < L = \sup_{A\in\mathcal{A}} \|M(A)\| < +\infty$*),* $\varphi : \mathcal{A} \to \mathcal{K}$ *is a multimeasure,* ν *is the total variation of* φ *and* $\nu(S) < \infty$ *such that:*

(3.5.1) $M \ll \nu$.

(3.5.2) *for all positive numbers* ε *and* δ, *there exists* $C \in \mathcal{A}^+$ *and* $\alpha \in (0,1)$ *satisfying*

(3.5.2.a) $\nu(S - C) < \delta$

(3.5.2.b) $A(C\mathcal{A}^2)$ *is bounded*

(3.5.2.c) *for each* $E \in C\mathcal{A}^+$, *there exists* $K \in E\mathcal{A}^+$ *such that* $\nu(K) > \alpha\nu(E)$ *and* $A(K,\varepsilon) \cap \tilde{A}(K,\varepsilon) \neq \emptyset$.

Then there exists a φ*–integrable multifunction* $F : S \to \mathcal{P}_{kc}$ *such that:*

(3.5.3) *there is a* $\nu -$ *exhaustion* $(E_i)_i$ of S *such that for every* i, $F : E_i \to \mathcal{P}_{kc}$ *is the limit (uniformly with respect to* $s \in E_i$*) of* $\varphi -$ *integrable multifunctions*

(3.5.4) *for each* i, $\exists r_i > 0$ *such that* $\|F(s)\| \leq r_i$ *for all* $s \in E_i$

(3.5.5) $\int_E F\, d\varphi = M(E)$ *for all* $E \in \mathcal{A}$.

Proof. Let $(\delta_n)_n$ be a sequence of positive numbers decreasing to zero. Since (3.5.2), for each n there exists $C_n \in \mathcal{A}^+$ such that $\nu(S - C_n) < \delta_n$ and $A(C_n\mathcal{A}^2)$ is bounded. We take $E_1 = C_1$, $E_n = C_n - \bigcup_{i=1}^{n-1} C_i$ for $n \geq 2$, possibly eliminating those sets E_i of zero ν–measure. The sequence thus obtained is a ν–exhaustion of S and $A(E_n\mathcal{A}^2)$ is bounded for each n.

By Lemma 3.1 applied to each E_n one obtains a bounded φ–integrable multifunction F_n which is the limit (uniformly with respect to $s \in S$) of φ–integrable multifunctions such that F_n vanishes outside E_n and such that

$$M(E_n \cap H) = \int_H F_n\, d\varphi, \ \forall H \in \mathcal{A},\ H \subseteq S.$$

Since the support of F_n is E_n and the sets $(E_n)_n$ are pairwise disjoint by construction, it is possible to define the formal series $F(s) = \sum_{n=1}^{\infty} F_n(s)$.

Next, we prove that F is φ–integrable.

Let $B_\varepsilon^n = \left\{ s \in S;\ D\left(F(s), \sum_{k=1}^{n} F_k(s) \right) > \varepsilon \right\} \in \mathcal{A}$. It follows that

$$B_\varepsilon^n \subset S - \bigcup_{k=1}^{n} E_k \quad \text{and} \quad \nu(B_\varepsilon^n) \underset{n\to\infty}{\longrightarrow} 0$$

(namely $\sum_{k=1}^{n} F_k \xrightarrow{\nu} F$).

Let $\Gamma_n(E) = \int_E \left(\sum_{k=1}^{n} F_k\right) d\varphi$, $\forall E \in \mathcal{A}$, $n \in \mathbb{N}$. We now prove that $\Gamma_n \ll \nu$, uniformly with respect to $n \in \mathbb{N}$.

From Theorem 15 [6], we have $v(\Gamma_n, E) = \int_E \left\| \sum_{k=1}^{n} F_k \right\| d\nu$, so it suffices to show that $v(\Gamma_n, \cdot) \ll \nu$, uniformly with respect to $n \in \mathbb{N}$.

In fact, $\int_E \left\| \sum_{k=1}^{n} F_k \right\| d\nu \le \sum_{k=1}^{n} \int_E \|F_k\| d\nu = \sum_{k=1}^{n} v(M, E \cap E_k) \le v(M, E)$.

Therefore, by Theorem 20 [6], it follows that F is φ–integrable and

$$\lim_{n\to\infty} D\left(\int_E \left(\sum_{k=1}^{n} F_k\right) d\varphi, \int_E F\, d\varphi\right) = 0 \qquad \text{namely}$$

$$\lim_{n\to\infty} D\left(M\left(E \cap \left(\bigcup_{k=1}^{n} E_k\right)\right), \int_E F\, d\varphi\right) = 0.$$

Therefore it is enough to show that

$$\lim_{n\to\infty} D\left(M(E), M\left(E \cap \left(\bigcup_{k=1}^{n} E_k\right)\right)\right) = 0.$$

$$D\left(M(E), M\left(E \cap \left(\bigcup_{k=1}^{n} E_k\right)\right)\right) =$$

$$= D\left(M\left(E - \bigcup_{k=1}^{n} (E \cap E_k)\right) + M\left(E \cap \left(\bigcup_{k=1}^{n} E_k\right)\right), M\left(E \cap \left(\bigcup_{k=1}^{n} E_k\right)\right)\right) =$$

$$= D\left(M\left(E - \bigcup_{k=1}^{n} (E \cap E_k)\right), 0\right) = \left\| M\left(E - \bigcup_{k=1}^{n} (E \cap E_k)\right) \right\|$$

and the assertion follows from (3.5.1) and the definition of $(E_n)_n$.

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