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SINE THEOREM FOR ROSETTES

BY

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1. Introduction. In the paper [5] the second and third author introduced the isoptics of rosettes. From the other hand the first author gave in [4] a sufficient condition for a Jordan curve to be an isoptic of an oval. In the present paper we extend this result to the class of rosettes. We begin by recalling the basic notations from [5].

Definition 1.1. A plane, closed, positively oriented curve of positive curvature is said to be a rosette.

Fig. 1

Let us present the definition of oriented support function. Consider a rosette $C : Z = Z(s)$ parametrized by arc length and suppose that the index of C is equal to j. Choose a point O as a center of our coordinate system and suppose that the curve C is considered in this system. Let us fix a point $Z_0 = Z(s_0)$ and consider the tangent line at Z_0. We can suppose that Z_0 is chosen in such a way that the tangent line is perpendicular to the x-axis. For an arbitrary point $Z = Z(s)$ we define a vector $e^{it} = \cos t + i \sin t$ as in Fig. 1, where t is an oriented angle between the positive direction it the x-axis and vector e^{it}. Since the curve C has index j, so $t \in [0, 2\pi j]$. Now we define an oriented distance $p(t)$ from origin O of coordinates system to the tangent line to C. Fix a point $Z = Z(s)$. Then we take e^{it} as a normal vector to C at this point (Fig. 1). If the vector e^{it} points to this half-plane which contains O then we put $p(t)$ equal minus ordinary distance between O and tangent line at $Z(s)$. If not we define $p(t)$ as ordinary between O and the tangent line at $Z(s)$. Since the rosette is locally convex, then such obtained function $p(t)$, $t \in [0, 2\pi j]$ is at least at the class C^1 (cf. [6]).

Using $p(t)$ we obtain a new parametrization of C given by

$$z(t) = p(t)e^{it} + \dot{p}(t)ie^{it}, \qquad t \in [0, 2\pi j].$$

Definition 1.2. Above constructed function p we call the oriented support function.

Let us fix a point $z(t_0) \in C$ and consider the tangent line to C at $Z(t)$. Let $z(t'), t' > t$ be the closest point (in the sense parametrization) such that the angle between the tangent lines at $z(t)$ and $z(t')$ is equal to $\pi - \alpha$.

Definition 1.3. The cut locus C_α of the intersection points of the above defined pairs of tangent lines is said to be an α-isoptic of C.

In the same way as in [2], [3] we obtain the following parametrization of C_α

$$z_\alpha(t) = p(t)e^{it} + (-p(t)\cot\alpha + \frac{1}{\sin\alpha}p(t+\alpha))ie^{it}, \quad t \in [0, 2\pi j].$$

2. Generalized isoptics of rosettes. Let us fix a direction k. Since the rosette C is at index j then there exist $2j$ values of parameter t

$$0 \le t_0 < t_1 < \ldots < t_{2j-1} < 2\pi j$$

such that the tangent lines to C at $z(t_0), z(t_1), \ldots, z(t_{2j-1})$ are parallel to k. This observation allows us to define $2j-1$ different types of isoptics (cf. Fig. 2). Consider the tangent line at $z(t)$ and fix an angle α. This determines the direction k subtending a positively oriented angle α with this tangent line. Let $t_0 < t_1 < \ldots < t_{2j-1} < 2\pi j$ be such that the tangent line to C at these points are all parallel to k.

Fig. 2

Definition 2.1. The cut locus of the intersection points of the tangent lines at t and t_k is said to be an isoptic of the type (k, α) and is denoted $C_{k,\alpha}$.

Remark 2.1. Observe that $C_{2j-1,\alpha} = C_{0,\pi-\alpha}$.

Similarly as in the case of $C_{0,\alpha} = C_\alpha$ one can obtain an equation of $C_{k,\alpha}$, $(k = 0, 1, \ldots, 2j-1)$

$$z_{k,\alpha}(t) = p(t)e^{it} + (-p(t)\cot\alpha + (-1)^k \frac{1}{\sin\alpha} p(t + \alpha + k\pi))ie^{it}$$

From the theory of isoptics of ovals ([2]) it is well known that through each point of the exterior of an oval passes a unique α-isoptic. The set of isoptics of a given oval is disjoint family of curves covering the exterior of the oval.

In the case of rosettes the situation is completely different. Isoptics of the rosette (even of the same type) corresponding to different angles can intersect one another. The situation is even more complicated since the different isoptics can overlap on some open sets (in the sense of their relative topology) (cf. Fig. 3).

Fig. 3

In the above figure the arcs $\widehat{AB}$ and $\widehat{CD}$ are semi–circles of common center.

Theorem 2.1. *Sets C_α and C_β are different for each $\alpha \neq \beta$.*

Proof. Suppose that $\alpha < \beta$. Let us consider point $z(t) \in C$ and the half–line tangent to C at this point directed in the same way as $z'(t)$. This half–line contains the points z_1 and z_β of C_α and C_β, respectively.

Fig. 4

Since C is locally convex, then $|z(t)z_\beta(t)|<|z(t)z_1(t)|$. Let us suppose that $C_\alpha = C_\beta$. Consider the smallest disc K containing the set $C_\alpha = C_\beta$. Evidently $C \subset K$. The boundary ∂K of K has to be tangent to C_β at some point $Z_2(t)$. This means that this point belongs to the half–line $z(t)z_\beta(t)$. But this half–line has to contain a point $z_\alpha(t)$ such that $|z(t)z_\alpha(t)|>|z(t)z_\beta(t)|$. This means that $z_\alpha(t)$ belongs to the exterior of K. This is a contradiction with the choice of the class K.

Let us consider arbitrary isoptics $C_{k,\alpha}, C_{n,\beta}$, $k, n \in \{0, 1, \ldots, 2j-1\}$. Similarly we prove the following

Theorem 2.2. *If $n \neq k$ or $\alpha \neq \beta$, then $C_{k,\alpha} \neq C_{n,\beta}$.*

3. Index of isoptics of rosettes. Let us fix an isoptic $C_{k,\alpha}$ of C.

Fig. 5

In this case the tangent lines determining the corresponding points of isoptic are tangent to C at $z(t)$ and $z(t+\alpha+k\pi)$, respectively.

Let $\lambda_{k,\alpha} = |z(t)z_{k,\alpha}(t)|, \mu_{k,\alpha} = |z(t+\alpha+k\pi)z_{k,\alpha}(t)|$ and $q_{k,\alpha} = z(t) - z(t+\alpha+k\pi)$. Moreover we put

$$b_{k,\alpha} = \{q_{k,\alpha}, e^{it}\},\ B_{k,\alpha} = \{q_{k,\alpha}, ie^{it}\}$$

where $\{a+bi, c+di\} = ad - bc$.

We have

$$\begin{aligned}
\lambda_{k,\alpha} &= -\cot\alpha p(t) - \dot{p}(t) + (-1)^k \frac{1}{\sin\alpha} p(t) \\
\mu_{k,\alpha} &= \cot\alpha p(t+\alpha+k\pi) - \dot{p}(t+\alpha+k\pi) - (-1)^k \frac{1}{\sin\alpha} p(t) \\
b_{k,\alpha} &= -\dot{p}(t) + (-1)^k p(t+\alpha+k\pi)\sin\alpha + (-1)^k \dot{p}(t+\alpha+k\pi)\cos\alpha \\
B_{k,\alpha} &= p(t) - (-1)^k p(t+\alpha+k\pi)\cos\alpha + (-1)^k \dot{p}(t+\alpha+k\pi)\sin\alpha.
\end{aligned}$$

From the above formulas we get

$$\begin{aligned}
b_{k,\alpha} &= \lambda_{k,\alpha} - (-1)^k \cos\alpha \mu_{k,\alpha} \\
B_{k,\alpha} &= -(-1)^k \sin\alpha \mu_{k,\alpha}
\end{aligned}$$

or equivalently

$$\begin{aligned}
\lambda_{k,\alpha} &= b_{k,\alpha} - (-1)^k \cot\alpha B_{k,\alpha} \\
\mu_{k,\alpha} &= -(-1)^k \sin\alpha B_{k,\alpha}.
\end{aligned}$$

Differentiating the formulas for $b_{k,\alpha}$ and $B_{k,\alpha}$ we obtain

$$\begin{aligned}
\dot{b}_{k,\alpha} &= -R + (-1)^k R(t+\alpha+k\pi)\cos\alpha + B_{k,\alpha} \\
\dot{B}_{k,\alpha} &= (-1)^k R(t+\alpha+k\pi)\sin\alpha - b_{k,\alpha},
\end{aligned}$$

where $R = R(t)$ and $R(t+\alpha+k\pi)$ are the radii of curvature and the $\dot{x}$ denotes the differentiation with respect to t.

We have the following formula for the curvature κ of $C_{k,\alpha}$

$$\kappa_{k,\alpha} = \frac{\sin\alpha}{|q_{k,\alpha}|^3}\left(2|q_{k,\alpha}|^2 - \{q_{k,\alpha}, \dot{q}_{k,\alpha}\}\right).$$

Reasoning similarly as in [5] we get

Theorem 3.1. Index $C_{k,\alpha}=$ Index C *for each* k *and* α.

4. Steiner centroid of isoptics of rosettes. In this section we generalize Theorem 4.1 from [2] to the case of rosettes.

Theorem 4.1. *The Steiner centroids of isoptics* $C_{k,\alpha}$ *coincide with the Steiner centroid of the rosette* C.

Proof. We have to prove that

$$\int_0^{2\pi j} z(t)dt = \int_0^{2\pi j} z_{k,\alpha}(t)dt.$$

Observe that

$$\int_0^{2\pi j} z(t)dt = 2\int_0^{2\pi j} p(t)e^{it}dt.$$

We have

$$\int_0^{2\pi j} z_{k,\alpha}(t)dt =$$

$$= \int_0^{2\pi j} p(t)e^{it}dt + \frac{i}{\sin\alpha}\int_0^{2\pi j}(-\cos\alpha p(t) + (-1)^k p(t+\alpha+k\pi)e^{it})dt.$$

Let us consider the second integral. Substituting $\tau = t+\alpha+k\pi$ we get

$$\int_0^{2\pi j}(-\cos\alpha p(t) + (-1)^k p(t+\alpha+k\pi)e^{it})dt =$$

$$-\cos\alpha\int_0^{2\pi j} p(t)e^{it} + (-1)^k\int_{\alpha+k\pi}^{2\pi j+k\pi} p(\tau)e^{(\tau-\alpha-k\pi)i}d\tau =$$

$$= -\cos\alpha\int_0^{2\pi j} p(t)e^{it}dt + \int_0^{2\pi j} p(t)e^{it(\cos\alpha - i\sin\alpha)dt)} =$$

$$= -i\sin\alpha\int_0^{2\pi j} p(t)e^{it}dt.$$

Hence

$$\int_0^{2\pi j} z(t)dt = 2\int_0^{2\pi j} p(t)e^{it}dt.$$

5. The sine theorem for isoptics of rosettes. The sine theorem for the first time appeared in [1] in the case ovals. Then this theorem was extended to closed strictly convex curves in [2]. Finally in paper [5] the sine theorem was proved in the case of isoptic $C_1 = C_{0,\alpha}$ of an arbitrary rosette. In this section we shall prove this result for an arbitrary isoptic $C_{k,\alpha}$ and we shall present the inverse theorem which generalizes the paper of G ó ź d ź [4].

Fig. 6

Theorem 5.1. *In the notations of Fig.* 6 *we have*

$$\frac{\lambda_{k,\alpha}}{\sin\gamma} = \frac{\mu_{k,\alpha}}{\sin\delta} = \frac{|q_{k,\alpha}|}{\sin\alpha}$$

Proof. Using the formulas from section 3 and reasoning similarly as in [2] we obtain after some calculations the desired formula.

Let us consider the inverse problem. We shall prove the following

Theorem 5.2. *Let $C : z(t) = p(t)e^{it} + \dot{p}(t)ie^{it}$ be a given rosette of index j and let Γ be a closed curve. Suppose that there exists a differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ such that, the tangent lines to C at $z(t)$ and $z(\varphi(t))$ intersect for each t of some point Γ, moreover we assume that at these points the sine formula holds. Then Γ is an isoptic of C for some k and α.*

Proof. The function $\varphi(t)$ defines the function $\alpha(t)$, $0 < \alpha(t) < \pi$ reasoning the oriented angle between the tangent lines to C at $z(t)$ and $z(\varphi(t))$. It is easy to observe that $\varphi(t) = t + \alpha(t) + k\pi$ for some uniquely determined integer k, where $k \in \{0, 1, \ldots, 2j-1\}$. Note that depending from the situation the vector $ie^{i(t+\alpha(t))}$ points the intersection point of suitable tangent lines or is oppositively directed. It is easy to see that in our case we can obtain the following parametrization of Γ

$$z_\Gamma(t) = p(t)e^{it} + (-\cot\alpha(t)p(t) + \frac{1}{\sin\alpha(t)}p(t+\alpha(t))ie^{it}.$$

In the notations similar to those of section 3 we obtain

$$\lambda_\Gamma = -\cot\alpha(t)p(t) - \dot{p}(t) + \frac{1}{\sin\alpha(t)}p(t+\alpha(t))$$

$$\mu_\Gamma = -\dot{p}(t+\alpha(t)) + \cot\alpha(t)p(t+\alpha(t)) - \frac{1}{\sin\alpha(t)}p(t).$$

Observe that λ_Γ is equal to the length of the segment joining $z(t)$ and $z_\Gamma(t)$ and μ_Γ is the length of the segment $z(t+\alpha(t))z_\Gamma(t)$ for k odd and minus the length for k even. From the assumptions of our theorem we know that

$$\frac{\lambda_\Gamma}{\{ie^{it}, z'_\Gamma(t)\}} = \frac{(-1)^{k+1}\mu_\Gamma}{\{z'_\Gamma(t), (-1)^k ie^{i(t+\alpha(t))}\}}.$$

Taking into account the formulas for $\lambda_\Gamma, \mu_\Gamma$ we can show after some cumbersome calculations that the above condition is equivalent to

$$\lambda_\Gamma \cdot \mu_\Gamma \cdot \dot{\alpha}(t) = 0 \quad \text{for each } t.$$

Hence $\dot{\alpha}(t) = 0$, thus $\alpha = const$.

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