ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLIV, s.I.a, Matematică, 1998, f1

ASYMPTOTIC BEHAVIOR IN VOLTERRA PARTIAL INTEGRODIFFERENCE EQUATION

BY

YOSHIHIRO HAMAYA

Dedicated to Professor Constantin Corduneanu on the occasion of his 70th birthday

Abstract. Volterra partial integrodifference equation, which appears as a model to describe the evolution of a single population, is treated. The asymptotic behavior of solution of the above equation is studied by applying the maximum principle.

1. Introduction. Recently, REDLINGER [6] has studied the asymptotic behavior of the integrodifferential equation with diffusion

$$\frac{\partial u}{\partial t}(t,x)=\Delta u(t,x)+u(t,x)[a-bu(t,x)-\int_0^t K(t-s)u(s,x)ds], t\geq 0, x\in\Omega,$$
$$\frac{\partial u}{\partial n}(t,x)=0, \quad t\geq 0, x\in\partial\Omega,$$

which is a partialization of the equation

$$\frac{du}{dt}(t)=u(t)[a-bu(t)-\int_0^t K(t-s)u(s)ds],\ t\geq 0$$

proposed by VOLTERRA [7] and MILLER [3] as a simple model to describe the evolution of a single population.

In [4], MURAKAMI and YOSHIZAWA have considered the same problem

for the following a system of differential equations

$$\frac{\partial u_i}{\partial t}(t,x) = d_i \Delta u_i(t,x) + u_i(t,x)[a_i(t) - b_i(t)u_i(t,x) - \\ -\sum_{j=1}^{N} c_{ij}(t) \int_0^t K_{ij}(t-s)u_j(s,x)ds],\ t\geq 0,\ x\in\Omega,\ i=1,...,N.$$

For the difference equation, CHENG [1] has researched the stability property and the maximum principle of partial differential equations. Moreover, HAMAYA [2] have obtained the global attractivity for some partial–difference equation by using the results of maximum principle in [1].

In this paper, we shall consider the following Volterra partial integrodifference equation

$$\begin{gathered} \Delta_2 u(m,n) = \Delta_1^2 u(m-1,n) + \\ +u(m,n)\left[a - bu(m,n) - \sum_{s=0}^{n} K(n-s)u(m,s)\right], n=0,1,...,\ m\in\Omega, \\ \Delta_1 u(m,n) = 0,\ n=0,1,...,\ m\in\partial\Omega, \end{gathered} \tag{1}$$

where $\Delta_k,\ k=1,2$, are the partial differences of a a double sequence $\{u(i,j)\}$ are defined by

$$\begin{gathered} \Delta_1 u(i,j) = u(i+1,j) - u(i,j),\ \Delta_2 u(i,j) = u(i,j+1) - u(i,j), \\ \Delta_1^2 u(i,j) = \Delta_{11} u(i,j) = \Delta_1(\Delta_1 u(i,j)), \end{gathered}$$

etc. $\Omega\subset Z^m$ is a bounded domain in the lattice plane with exterior boundary $\partial\Omega$. For the equation (1), we assume that

$$a>0,\ b>0, \tag{2}$$

are positive rate constant and $K: Z^+ \to (-\infty,\infty)$ is a capacity function. Furthermore, we assume that

$$\begin{gathered} K(n) \text{ is nondecreasing on } [0,\infty), \\ \sum_{s=0}^{\infty}|K(s)| = 1 \text{ and } b > \sum_{s=0}^{\infty} K^-(s) =: C, \end{gathered} \tag{3}$$

where $K^-(n) = \max(0, -K(n))$.

For the sake of convenience, we will denote the set $\{a, a+1, ..., b\}$ of consecutive integers by $[a,b]$, and $\{a, a+1, ...,\}$ by $[a,\infty)$, etc. A population size–function $u: \overline{\Omega}\times Z \to R$ is called a solution of (1) if $\Delta_1 u, \Delta_2 u$ and $\Delta_1^2 u$

defined on $\Omega\times(0,\infty)$, $\Delta_1 u$ exists on $\partial\Omega\times(0,\infty)$ and (1) is identically satisfied. From [1], we see that the existence of solutions is guaranteed for (1) whenever the initial function $u(m,0) =: \phi(m,0)$, $m \in \Omega$, with $\phi(\cdot,0) \in \overline{\Omega}$. It is easy to see that (1) has a unique equilibrium, and (2) and (3) imply that there exists $\gamma > 0$, which satisfies the relation

$$\gamma = a\left(b - \sum_{s=0}^{\infty} K^-(s)\right)^{-1}. \tag{4}$$

In addition to (2) and (3), we impose the following assumption, too:

$$a > \gamma \sum_{s=0}^{\infty} K^+(s), \tag{5}$$

where $K^+(n) = \max(0, K(n))$ and γ is the one in (4).

In what follows, we assume the existence of solutions of (1) for each nonnegative initial function ϕ, and for this unique solution u of (1), we can see that the following relation holds:

$$0 < u(m,n) \le \max\left(\frac{a}{b-C},\ \sup_{m\in\overline{\Omega}} u(m,0)\right),\ n > 0,\ m \in \overline{\Omega}. \tag{6}$$

2. Maximum principles. In this section, we briefly demonstrate the ideas in CHENG's results [1]. So, we omit the proof of theorems.

Let Ω be a finite domain in the lattice plane and $\partial\Omega$ its exterior boundary. We have the following abbreviations:

$$\begin{aligned} (Lu)_{ij} &= (L_1u)_{ij} + (L_2u)_{ij}, && \text{where} \\ (L_1u)_{ij} &= -p_{ij}\Delta_1 u_{i-1,j} + q_{ij}\Delta_1 u_{ij}, && \text{and} \\ (L_2u)_{ij} &= -r_{ij}\Delta_1 u_{i-1,j} + s_{ij}\Delta_1 u_{ij}. \end{aligned}$$

Let $u = \{u(i,j)\} = \{u_{ij}\}$ be a real sequence defined on $\Omega + \partial\Omega$. Then, we have the following Theorems A and B for certain parabolic type partial difference equation.

Theorem A. *Suppose the functions p, q, r are positive on Ω, and suppose u satisfies the inequality*

$$(L_1u)_{ij} - r_{ij}\Delta_2 u_{i,j-1} \le 0,\ (i,j) \in \Omega.$$

Let $M = \max_{(i,j)\in\Omega+\partial\Omega} u_{ij}$. *Then* $u_{ij} < M$ *for all* $(i,j) \in \Omega$ *unless* $u \equiv M$.

We can use another type theorems of maximum principles corresponding to differential equations case [5].

Theorem B. *Suppose the functions* p, q, r *are positive on* Ω, $f_{ij} > 0$ *is bounded on* Ω *and suppose* u *satisfies the inequality*

$$(L_1 u)_{ij} - r_{ij}\Delta_2 u_{i,j-1} \geq f_{ij},\ (i,j) \in \Omega.$$

Let $M = \max_{(i,j)\in\Omega+\partial\Omega} u_{ij}$. *Then* $u_{ij} < M$ *for all* $(i,j) \in \Omega$ *unless* $u \equiv M$.

3. Global attractivity of the equilibrium. The main result in this paper is the following:

Theorem. *Suppose* (2), (3) *and* (5). *Then the solution* $u_{m,n}$ *of* (1) *satisfies*

$$\sup_{m\in\overline{\Omega}} |u_{m,n} - u^*| \longrightarrow 0 \quad as \quad n \longrightarrow \infty,$$

where $u^* = a\left(b + \sum_{s=0}^{\infty} K(s)\right)^{-1}$.

We are now in a position to prove the theorem by maximum principles.

Proof of Theorem. First of all, we establish the following inequality:

$$\frac{a - \gamma\sum_{s=0}^{\infty} K^+(s)}{b} \leq \liminf_{n\to\infty} \min_{m\in\overline{\Omega}} u(m,n) \leq \limsup_{n\to\infty} \max_{m\in\overline{\Omega}} u(m,n) \leq \gamma. \tag{7}$$

This inequality will be proved by employing the argument in [6]. We first consider the solution $p_1(n)$ of the ordinary–type difference equation

$$\Delta p_1 = p_1(a - bp_1 + C\gamma_0),\ n > 0,$$

with $p(0) > \gamma_0$, where $\gamma_0 = \max(a/(b-C), \sup_{m\in\overline{\Omega}} u(m,0))$. We claim that

$$u(m,n) < p_1(n),\ n \geq 0,\ m \in \overline{\Omega}. \tag{8}$$

To establish that the claim by a contradiction, we suppose that (8) is not true. Then it follows from (6) that there exists some $(m_0, n_0) \in \overline{\Omega}\times(0,\infty)$ such that $u(m_0,n_0) = p_1(n_0)$ and $u(m,n) < p_1(n)$ on $\overline{\Omega}\times[0,n_0)$. Set $w(m,n) = u(m,n) - p_1(n)$. Then $w(m_0,n_0) = 0$ and $w(m,n) < 0$ on $\overline{\Omega}\times[0,n_0)$. Moreover, we get

$$\Delta_2 w = \Delta_2 u - \Delta p_1 = \Delta_1 u + (w+p_1)[a - b(w+p_1) - \\ -\sum_{s=0}^{n} K(n-s)u(m,s)] - p_1[a - bp_1 - C\gamma_0] = \Delta_1 w + f_1 w + f_2$$

on $\overline{\Omega}\times(0,n_0]$, where

$$f_1 = f_1(m,n) = a - bu(m,n) - p_1(n)b - \sum_{s=0}^{n} k(n-s)u(m,s) \quad \text{and}$$

$$f_2 = f_2(m,n) = p(n)\left[-\sum_{s=0}^{n} k(n-s)u(m,s) - C\gamma_0\right].$$

Clearly, $f_2 \le 0$, and hence $\Delta_1 w - \Delta_2 w + f_1 w \ge 0$ on $\overline{\Omega}\times(0,n_0]$. Since f_1 is bounded on $\overline{\Omega}\times[0,n_0]$, we get a contradiction by the maximum principle (cf. [1]). Indeed, if $m_0 \in \Omega$, then $w(m,n) \equiv 0$ on $\overline{\Omega}\times[0,n_0]$ by the maximum principle, which is a contradiction because of $u(m,0) \le \gamma_0 < p_1(0)$. We thus obtain $m_0 \in \partial\Omega$ and $w(m,n) < 0$ on $\overline{\Omega}\times[0,n_0]$, and hence $\Delta_1 w > 0$ by the maximum principle, again. But this is also a contradiction, because of $\Delta_1 w(m_0,n_0) = \Delta_1 u(m_0,n_0) = 0$.

We next observe that $\lim_{n\to\infty} p_1(n) = (a + C\gamma_0)/b =: \gamma_1$. From (8) it follows that $\limsup_{n\to\infty}(\max_{m\in\overline{\Omega}} u(m,n)) \le \gamma_1$. Let $\varepsilon > 0$. There exists a $n_1 > 0$ such that $\max_{m\in\overline{\Omega}} u(m,n) < \gamma_1 + \varepsilon$, for all $n \ge n_1$, and a $n_2 > n_1$ such that $\sum_{s=n-n_1}^{n} K^-(s) < \varepsilon$, for all $n \ge n_2$. We consider the solution $p_2(n)$ of the ordinary-type difference equation

$$\Delta p_2 = p_2\left[a - bp_2 + (\gamma_1+\varepsilon)\sum_{s=0}^{\infty} K^-(s) + \varepsilon\gamma_0\right], \ n > n_2,$$

with $p_2(n_2) = \gamma_1 + \varepsilon$. Employing the argument for $p_1(n)$, we can prove that $u(m,n) < p_2(n)$, $n \ge n_2$, $m \in \overline{\Omega}$. Since

$$\lim_{n\to\infty} p_2(n) = \left[a + (\gamma_1+\varepsilon)\sum_{s=0}^{\infty} K^-(s) + \varepsilon\gamma_0\right]/b$$

it follows that

$$\limsup_{n\longrightarrow\infty}\ \max_{m\in\overline{\Omega}}\ u(m,n) \leq \frac{a+(\gamma_1+\varepsilon)\sum_{s=0}^{\infty}K^-(s)+\varepsilon\gamma_0}{b},$$

and consequently

$$\limsup_{n\longrightarrow\infty}\ \max_{m\in\overline{\Omega}}\ u(m,n) \leq \gamma_2 =: \frac{a+\gamma_1+\sum_{s=0}^{\infty}K^-(s)}{b},$$

because $\varepsilon > 0$ is arbitrary. Repeating the above reasoning, we get

$$\limsup_{n\longrightarrow\infty}\ \max_{m\in\overline{\Omega}}\ u(m,n) < \gamma_n,$$

where

$$\gamma_n = \frac{a+\gamma_{n-1}\sum_{s=0}^{\infty}K^-(s)}{b},\ n=1,2,...$$

The sequence $\{\gamma_n\}$ is nondecreasing and nonnegative. Hence there exists $\gamma_\infty := \lim_{n\to\infty}\gamma_n$, which satisfies $\gamma_\infty > 0$ and $\gamma_\infty = a/\left(b-\sum_{s=0}^{\infty}K^-(s)\right)$. From (3) it follows that $\gamma_\infty = \gamma$. We thus obtain

$$\limsup_{n\longrightarrow\infty}\ \max_{m\in\overline{\Omega}}\ u(m,n) \leq \gamma,$$

which is a part of the inequality desired. Finally, we will prove the remainder part of the inequality. Let $\varepsilon > 0$. There exists a $n_3 > 0$ such that

$$\max_{m\in\overline{\Omega}}\ u(m,n) < \gamma+\varepsilon, \quad \text{for all } n \geq n_3$$

and a $n_4 > n_3$ such that

$$\sum_{s=n-n_3}^{n} K^+(s) < \varepsilon, \quad \text{for all } n \geq n_4.$$

By (6), there exists a $\delta > 0$ such that $\min_{m\in\overline{\Omega}} u(m, n_4) \geq 2\delta$. We consider the solution $q(n)$ of the ordinary–type difference equation

$$\Delta q + q\left[a - bq - (\gamma+\varepsilon)\sum_{s=0}^{\infty} K^+(s) - \varepsilon\gamma_0\right],\ n > n_4,$$

with $q(n_4) = \delta$. Employing almost the same arguments as for $p_1(n)$, we can prove that

$$u(m, n) > q(n),\ n \geq n_4,\ m \in \overline{\Omega}.$$

In virtue of (5), ε can be taken so small that $a > (\gamma+\varepsilon)\sum_{s=0}^{\infty} K^+(s) + \varepsilon\gamma_0$. Then

$$\lim_{n\to\infty} q(n) = \left[a - (\gamma+\varepsilon)\sum_{s=0}^{\infty} K^+(s) - \varepsilon\gamma_0\right]/b,$$

and hence

$$\liminf_{n\longrightarrow\infty} \min_{m\in\overline{\Omega}} u(m, n) \geq \left[a - (\gamma+\varepsilon)\sum_{s=0}^{\infty} K^+(s) - \varepsilon\gamma_0\right]/b.$$

Letting $\varepsilon \to 0$ in the above, we get the remainder part of the inequality desired. To prove our theorem, we now use an iterative argument starting with (7). Since $\mu_1 := \gamma\left(b - \sum_{s=0}^{\infty} K^+(s) - \sum_{s=0}^{\infty} K^-(s)\right)/b = \left(a - \gamma\sum_{s=0}^{\infty} K^+(s)\right)/b$ and $\nu_1 := \gamma$, we can show that for choosing $\varepsilon \in (0, \mu_1/2)$ there exists a $n_5 > n_3$ such that

$$\mu_1 - \varepsilon \leq \liminf_{n\longrightarrow\infty} \min_{m\in\overline{\Omega}} u(m, n) \leq \limsup_{n\longrightarrow\infty} \max_{m\in\overline{\Omega}} u(m, n) \leq \nu_1 + \varepsilon$$

for all $n \geq n_5 - 1$.

Repeating the above argument, we have two sequences $\{\mu_n\}, \{\nu_n\}$ defined by

$$\mu_{n+1} = \left(a + \mu_n\sum_{s=0}^{\infty} K^-(s) - \nu_n\sum_{s=0}^{\infty} K^+(s)\right)/b \text{ for } n \in N, \tag{9}$$

and by

$$\nu_{n+1} = \left(a - \mu_n\sum_{s=0}^{\infty} K^+(s) + \nu_n\sum_{s=0}^{\infty} K^-(s)\right)/b \text{ for } n \in N. \tag{10}$$

It is easy to check that the sequences $\{\mu_n\}$ and $\{\nu_n\}$ defined by (9) and (10) satisfy the relation

$$0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \cdots \leq \nu_n \leq \cdots \leq \nu_2 \leq \nu_1 = \gamma$$

for all $n \in N$. Then there exist numbers μ and ν such that $\lim_{n\to\infty} \mu_n = \mu$ and $\lim_{n\to\infty} \nu_n = \nu$. It is obvious that $0 < \mu \leq \nu < \gamma$. To establish the theorem, it suffices to certify by (9) and (10) that

$$\mu = \nu = \frac{a}{b + \sum_{s=0}^{\infty} K^+(s) - \sum_{s=0}^{\infty} K^-(s)}.$$

This completes the proof.

REFERENCES

1. C h e n g, S.S. – *Stability of partial difference equations: An introduction*, to appear.
2. H a m a y a, Y. – *Global attractivity in a partial integrodifference equation*, to appear.
3. M i l l e r, R.K. – *On Volterra's population equation*, SIAM J. Appl. Math. 14 (1966), 446–452.
4. M u r a k a m i, S. and Y o s h i z a w a, T. – *Asymptotic behavior in a system of Volterra integrodifferential equations with diffusion*, Dynamical Systems and Applications, 3 (1994), 175–188.
5. P r o t t e r, M.H. and W e i n b e r g e r, H.F. – *Maximum principles in differential equations*, Springer–Verlag New York Inc., 1984.
6. R e d l i n g e r, R. – *On Volterra's population equation with diffusion,* SIAM J. Math. Anal. 16 (1985), 135–142.
7. V o l t e r r a, V. – *Leçons sur la théorie mathématique de la lutte pour la vie*, Gauthier–Villars, Paris, 1931.

Department of Mathematical Information Science
Okayama University of Science
1-1 Ridai-cho, Okayama 700
JAPAN