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APPROXIMATING THE LARGER EDDIES IN FLUID MOTION III: THE BOUSINESQ MODEL FOR TURBULENT FLUCTUATIONS[1]

BY

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Dedicated to Professor Constantin Corduneanu on the occasion of his 70th birthday

Abstract. In 1877 Boussinesq (and others) put forward the basic analogy between the mixing effects of turbulent fluctuations and molecular diffusion: $-\nabla \cdot \overline{(u'u')} \sim -\nabla \cdot (\nu_{TURB}(\nabla \bar{u} + \nabla \bar{u}^t))$. This assumption lies at the heart of essentially all turbulence models and subgridscale models. By revisiting the original arguments of Boussinesq, Saint–Venant, Kelvin, Reynolds and others, we give three new approximations for the turbulent viscosity coefficient ν_T in terms of the mean flow based on approximation for the distribution of kinetic energy in u' in terms of the mean flow $\bar{u}$. We prove existence of weak solutions for the resulting system (NSE plus the proposed subgridscale term). Finite difference implementations of the new eddy viscosity/subgrid–scale model are transparent. We show how it can be implemented in finite element procedures and prove its action is no larger than that of the popular Smagorinski–subgrid–scale model.

I. Introduction. Consider the turbulent flow of an incompressible fluid. Beginning with the Navier–Stokes equations:

$$(1.1)\qquad \begin{cases} u_t + \nabla \cdot (uu) - \nu \Delta u + \nabla p = f, \text{ in } \Omega \subset \mathbb{R}^3, \\ \nabla \cdot u = 0, \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \; u(x,0) = u_0(x), \text{ in } \Omega \text{ and } \displaystyle\int_\Omega p(x,t)dx = 0, \end{cases}$$

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the usual approach, developed by R e y n o l d s [29], is to seek not to approximate (u, p) but rather some averages of u and p:

$$\bar{u} := \text{ average of } u, \bar{p} = \text{ average of } p.$$

(Several averaging operators are commonly used; we will be more specific in Section 2.) The fluctuations u' about $\bar{u}$ are defined via $u = \bar{u} + u'$ so

$$u' := u - \bar{u},\ p' = p - \bar{p}.$$

It is interesting to note that, while this decomposition into means and fluctuations was developed by Reynolds, it was advanced much earlier by, for example, da Vinci in 1510 in his description of vortices trailing a blunt body (as translated by P i o m e l l i [28]):

"*Observe the motion of the water surface, which resembles that of hair, that has two motions: One due to the weight of the shaft, the other due to the shape of the curls; thus water has eddying motions, one part of which is due to the principal current, the other to the random and reverse motion.*"

For commonly used averaging operators, averaging and differentiation commute. For these, averaging (1.1) gives

$$\bar{u}_t + \nabla \cdot (\overline{uu}) - \nu \Delta \bar{u} + \nabla \bar{p} = \bar{f},\ \text{and } \nabla \cdot \bar{u} = 0,\ \text{in } \Omega.$$

Since $\overline{uu} \neq \bar{u}\,\bar{u}$, inserting and expanding $\overline{uu} = \overline{(\bar{u} + u')(\bar{u} + u')}$ in the second term above gives

$$\bar{u}_t + \nabla \cdot \overline{(\bar{u}\ \bar{u})} + \nabla \cdot \overline{(\bar{u}u' + u'\bar{u})} - \nu \Delta \bar{u} - \nabla \cdot \tau + \nabla \bar{p} = \bar{f},\ \text{in } \Omega,$$

where τ represents the so called turbulent or Reynold's stresses given by:

$$\tau_{ij} = -\overline{u_i' u_j'}.$$

The cross terms $\overline{\bar{u}u' + u'\bar{u}}$ typically either vanish or are modeled in various ways. We consider herein the modeling of the turbulent stresses.

The fundamental relation still used for modeling turbulent stresses was put forward by B o u s s i n e s q in 1877, [2] stating, in effect, that the interaction of turbulent fluctuations are dissipative in the mean. This "Boussinesq model" is based upon an analogy between the mixing effects of turbulent fluctuations and molecular diffusion and is written as:

$$\tau \sim \nu_T (\nabla \bar{u} + \nabla \bar{u}^t) - \frac{2}{3} k \delta_{ij},\ \nu_T := \text{ turbulent diffusion parameter.} \tag{1.2}$$

Relations like (1.2) have been verified for some simplified settings such as convection of a passive scalar under various assumptions. This work is presented and surveyed well in Part III of M o h a m m a d i and P i r o n n e a u [26]. However, we emphasize that the Boussinesq "model/approximation" (1.2) is not strictly speaking an approximation but rather a physical analogy.

Since (1.2) is the first step and very heart of turbulence models, subgrid–scale models and even the shallow water equations, it is useful first to review the reasoning of Boussinesq behind (1.2) and a few representative attempts to determine $\nu_T = \nu_T(x, t, \bar{u}, \bar{p}, Re, ...)$.

The assumption behind (1.2) is that in the mean the small eddies or turbulent fluctuations are isotropic and collide elastically and exchange momentum like molecules. Eddies do not, of course, interact perfectly elastically, see F r i s c h [10] or C o r r s i n [4]. Furthermore, there are also turbulent flows for which the turbulent fluctuations do not even seem to be isotropic. See Section 9.6 of F r i s c h [10] for more on the physical reasoning underlying (1.2).

There are numerous approaches to calculating the coefficient ν_T. Turbulence models (algebraic, mixing length, one equation, two equations...) typically give formulas of increasing compelxity which are solved approximately to determine ν_T – see, for example, [3], [13], [26], [21] for good surveys of these approaches. Most recently, the trend has been away from these approaches to much simpler subgrid–scale models combined with resolution and adaptivity. The most common such subgrid–scale model is due to S m a g o r i n s k i [31] in which:

$$\nu_T \sim \mu_0 \delta^2 \left|\nabla \bar{u} + \nabla \bar{u}^t\right|, \ \delta := \text{ length scale of resolvable eddies,}$$

so that

$$\nabla \cdot \tau \sim \nabla \cdot (\mu_0 \delta^2 \left|\nabla \bar{u} + \nabla \bar{u}^t\right| (\nabla \bar{u} + \nabla \bar{u}^t)) - \frac{2}{3} \nabla k; \tag{1.3}$$

see also [17], [20], [22], [23], [7], [13] for more on the mathematical foundation of this model. The parameter μ_0 in (1.3) was originally taken a constant; in the dynamic subgridscale model of [17], $\mu_0 = \mu(x, t)$ is determined by a clever extrapolation procedure. The form of (1.3) is consistent with K r a i- c h n a n's [20] extension of Kolmogorov "K-42" theory to 2-d turbulence and, accordingly, (1.3) performs well in 2-d simulations (as reported, e.g., by [26]). The most straightforward ([22], [23]) extension to 3-d is

$$\nu_T \sim \mu_0 \delta^2 \left|\nabla \bar{u} + \nabla \bar{u}^t\right|^{2/3}, \tag{1.4}$$

so that

$$\nabla \cdot \tau \sim \nabla \cdot (\mu_0 \delta^2 \left|\nabla \bar{u} + \nabla \bar{u}^t\right|^{2/3} (\nabla \bar{u} + \nabla \bar{u}^t)) - \frac{2}{3} \nabla k.$$

To explain relations like (1.4), recall that the "K-42" theory of isotropic turbulence developed by Kolmogorov (see F r i s c h [10]) predicts that the smallest length scale of persistent eddies in $O(Re^{-3/4})$. When a formula like (1.4) is used on a computational mesh the smallest resolvable eddy occurs when $|\nabla u + \nabla u^t| \sim O(\delta^{-1})$. Thus the effective local turbulent viscosity in these eddies is $\nu_T \sim \mu_0 \delta^2 \delta^{-2/3} = \mu_0 \delta^{4/3}$. By the K-42 theory eddies below $O((\mu_0 \delta^{4/3})^{3/4}) = O(\delta)$ thus decay exponentially due to the turbulent dissipative (1.4). Thus, the only persistent eddies are precisely those resolvable: $O(\delta)$ or larger.

Again, μ_0 in (1.4) can be regarded as a constant or a distribution to be determined by dynamic extrapolation methods, following [17].

2. A new Boussinesq–type subgridscale model. To motivate the need for another subgrid–scale model, note that the models (1.3) and (1.4) have at least two intuitive shortcomings. First, for flows with linear velocity profiles the formulas (1.3) and (1.4) would still introduce significant amounts of turbulent diffusion even though the flow field is laminar. Second, accepting the reasoning of Boussinesq, the amount of turbulent diffusion by small eddies should depend on the kinetic energy in those small eddies so that $\nu_T = \nu_T \left(\frac{1}{2}\rho|u'|^2\right)$ or $\nu_T = \left(\overline{\frac{1}{2}\rho|u'|^2}\right)$.

This section presents a subgridscale model similar to (1.3), (1.4) which meets these two conditions based on a more complete elaboration of the analogy of Boussinesq. To do so, the average operator must now be specified.

Hereafter, we restrict the averaging operator to be a space filter defined by convolution with a gaussian (see, for example, [1], [6], [15], [11], [17], [18], [27] for more details). Specifically, define

$$g(x) := (\gamma/\pi)^{3/2} \exp(-\gamma(x_1^2 + x_2^2 + x_3^2)) \text{ and } g_\delta(x) := \delta^{-d} g(x/\delta)$$

where $d =$ dimension $(\Omega) = 2$ or 3 and δ can be thought of as the averaging radius. Extending all functions by zero outside $\overline{\Omega}$, define the averages

$$\bar{u}(x,t) := g_\delta * u = \int_{\mathbb{R}^d} g_\delta(x-y) u(y,t) dy,$$

and similar for $\bar{p}, \bar{f}$, and so on.

Averaging of (1.1) by convolution with $g_\delta(\cdot)$ produces the terms which require closure approximations. The cross terms are developed in [15]. Herein, we consider the term describing turbulent fluctuations:

$$-\nabla \cdot (g_\delta * (u'u')). \tag{2.1}$$

We use a closure approximation introduced in [15]. Specifically, since

$$u = \bar{y} + u' \Longrightarrow \bar{u} = g_\delta * \bar{u} + g_\delta * u'.$$

Extending all functions by zero outside, Ω, the Fourier transform of this last equation gives:

$$\hat{\bar{u}} = \hat{g}_\delta \hat{\bar{u}} + \hat{g}_\delta \hat{u}',$$

from which

$$\hat{u}'(k) = ((1/\hat{g}_\delta(k)) - 1)\hat{\bar{u}}(k), \tag{2.2}$$

where $\hat{g}_\delta(k) = \exp\left(-\frac{\delta^2}{4\gamma}(k_1^2 + k_2^2 + k_3^2)\right)$.

The key property of averaging via convolution with $g_\delta(x)$ is that $\bar{u} \in C^\infty$, i.e., high frequencies of $\hat{\bar{u}}$ are attenuated. This a direct consequence of decay of $\hat{g}_\delta(k)$ for large wave numbers. Accordingly, in [15] approximations based upon rational approximation to the exponential are used. The lowest order which preserves some attenuation of high frequencies is given by the (0,1) Padé:

$$\hat{g}_\delta(k) \doteq \frac{1}{1 + \frac{\delta^2}{4\gamma}|k|^2} + O\left(\frac{\delta^4}{16\gamma^2}|k|^4\right).$$

Using the approximation in the formula (2.2) for $\hat{u}'$ gives:

$$\hat{u}'(k) \doteq \frac{\delta^2}{4\gamma}|k|^2 \hat{\bar{u}}(k).$$

After taking the inverse Fourier transform, we arrive at the approximation

$$u' \doteq -\frac{\delta^2}{4\gamma}\Delta\bar{u}(+O(\delta^4) \text{ terms}). \tag{2.3}$$

For more information on these methods see [14] and see [15], [27] for details concerning boundary conditions accompanying space filtered flow models.

The term involving the turbulent fluctuations

$$\nabla \cdot \tau = -\nabla \cdot (\overline{u_i' u_j'})$$

is formally $O(\delta^4)$. However, turbulent fluctuations play an important role in dissipation of energy from large eddies to smaller eddies. Thus, in turbulent flow simulations, moels of the turbulent fluctuations are normally included in the simulation via by some variant of Smagorinski's model as discussed in the introduction.

To model these turbulent fluctuations, reconsider Boussinesq's idea (1.2). Following the analogy between turbulent mixing and molecular diffusion to its logical conclusion, ν_T must be considered as a function of either the local kinetic energy in u' or its local average

$$\nu_T = \nu_T \left(\frac{1}{2}\rho_0 |u'|^2 \right), \quad \text{or } \nu_T = \nu_T \left(\overline{\frac{1}{2}\rho_0 |u'|^2} \right).$$

A classical dimensional argument suggests the correct form of ν_T to be given by the so–called Kolmogorov–Prandtl expression:

$$\nu_T \doteq c_\mu \ell_m \sqrt{k'}, \; k' := \frac{1}{2}\rho|u'|^2, \; \ell_m := \text{"mixing length"}. \tag{2.5}$$

In special filtering the critical length scale is the filter width δ. The kinetic energy in the small eddies can be related back to $\bar{u}$ using the approximation (2.3). Indeed, the approximation (2.3) for u' in (2.5) gives the relation:

$$\nu_T \doteq c_\mu \delta \sqrt{\frac{1}{2}\rho_0 \left(\frac{\delta^2}{4\gamma} \right)^2 |\Delta \bar{u}|^2} = c'_\mu \frac{\delta^3}{\gamma} |\Delta \bar{u}|. \tag{2.6}$$

This yields the subgridscale model for turbulent fluctuations given by

$$-\nabla \cdot \tau \sim -\nabla \cdot \left[c'_\mu \frac{\delta^3}{\gamma} |\Delta \bar{u}| (\nabla \bar{u} + \nabla \bar{u}^t) \right] + \frac{2}{3}\nabla k. \tag{2.7}$$

An important issue arises in how to use (2.7). The model (2.7) can be added to the models used for large eddy motion, such as in [6], [1], [15]. However, there has recently been a trend in the direction of simple fluids models composed of the Navier–Stokes equations augmented by a subgridscale model such as (2.7). These simpler models are paired with highly resolution in simulations by adaptive algorithms (see, e.g., [9] for one example).

Accordingly, as a first step in the analytical understanding of (2.7) we consider the Navier–Stokes equations supplmented by (2.7):

$$\begin{cases} \nabla \cdot \Omega = 0, \text{ in } \Omega, \\ w_t + \nabla \cdot (ww) + \nabla q - \mu \Delta w - \nabla \cdot \left[c'_\mu \frac{\delta^3}{\gamma} |\Delta w| (\nabla w + \nabla w^t) \right] = \bar{f}, \text{ in } \Omega. \end{cases} \tag{2.8}$$

The subgridscale model (2.7) has the clear features that the turbulent diffusion vanishes for linear mean velocities and the magnitude of the turbulent diffusion is proportional to a consistent approximation of the turbulent kinetic energy.

If ν_T is taken to be a function of the local average of the kinetic energy in the small eddies, the previous derivation is changed only slightly. The resulting subgridscale model (which replaces (2.7) in (2.8)) is given by:

$$-\nabla \cdot \tau \sim -\nabla \cdot \left[c'_\mu \frac{\delta^3}{\gamma} |g_\delta * \Delta \bar{u}| (\nabla \bar{u} + \nabla \bar{u}^t) \right] + \frac{2}{3} \nabla k. \tag{2.9}$$

Interestingly, if we begin with (2.9) and *reverse* the steps in the derivation of (2.7), (2.9) a *third* subgridscale model results. Thee third model reads:

$$-\nabla \cdot \tau \sim -\nabla \cdot \left[C''_\mu \frac{\delta}{\gamma} |\bar{u} - g_\delta * \bar{u}| (\nabla \bar{u} + \nabla \bar{u}^t) \right] + \frac{2}{3} \nabla k. \tag{2.10}$$

Nonlinear diffusion equations whose coefficients depend upon local average of $\bar{u}$ (as in (2.10)) have recently also used for image smoothing, e.g., [25],[5].

3. Implementation issues. the subgrid–scale term (2.7) can, of course, be added to any discretization of any convection dominated problem to enhance numerical stability. (This is the viewpoint of subgrid–scale modelling adopted in, for example, [22], [23].) For this use, (2.7) is computationally attractive with $\delta = h$ (the local meshwidth) since the extra term vanishes when $\Delta \bar{u} = 0$ and is $O(h^3)$ when $|\Delta u| = O(1)$. When $\bar{u}$ fluctuates accross a few mesh points $|\Delta \bar{u}| = O(h^{-2})$ so that in this case (2.6) introduces only $O(h)$ artificial viscosity.

If the convection–dominated problem is discretized by a finite difference methods, the implementation of the model (2.6) is clear – $|\Delta \bar{u}|$ is calculated by the usual five point discrete Laplacian. If a finite element method *using C^1–elements* is used, then the implementation of (2.6) is equally clear. If the most usual case of C^0 elements, e.g., conforming linears on triangles or tetrahedrons, is used then (2.7) must be interpreted correctly as $\Delta \phi^h$ does not exist for such functions ϕ^h.

For specificity let Ω be a polyhedral domain with a finite element mesh $\Pi^h(\Omega)$ constructed which divides Ω into conforming d–simplices. Thus, in 2-d triangles are edge to edge and in 3-d the tetrahedra are face to face. Let X^h denote the usual space of conforming, C^0, piecewise linears

$$X^h := \{ v^h(x) : v^h \in C^0(\Omega) \cap \overset{\circ}{H}{}^1(\Omega) \text{ and } v^h \text{ is affine on each simplex} \}.$$

For such functions $v^h \in X^h$, Δ_h can be correctly interpreted in terms of jumps in $\nabla v^h \cdot \hat{n}$ accross edges ($2d$) or faces ($3d$) of $\Pi^h(\Omega)$.

This interpretation was (to our knowledge) first pointed out and exploited in the work of E r i k s s o n and J o h n s o n [9] (see also [19] for another use of edge jump in a discretization). For K a triangle ($2d$) or tetrahedron ($3d$), following [8], and $v^h \in X^h$ define the piecewise constant, discrete Laplacian as

$$\left\|\Delta_h v^h\right\|_K := \max_{f \subset \partial K \cap \Omega} \left|[\nabla v^h \cdot n_f]_f\right| / \text{diameter}\,(K),$$

where n_f is the exterior unit normal to K on the edge or face f of K, and $[\cdot]_f$ denotes the jump of the indicated (discontinuous) quantity accross f. The function $\left|\Delta_h v^h\right|$ is piecewise constant for a linear finite element space. For higher order spaces Δv^h inside K is added to the above edge jump definition of Δ_h.

For locally quasi–uniform meshes $\left|\Delta_h v^h\right|$ satisfies the following inverse estimates, [8] for all $v^h \in X^h$,

$$\left\|h \Delta_h v^h\right\|_{L^2(\Omega)} \leq C \left\|\nabla v^h\right\|_{L^2(\Omega)},$$

and

$$\left\|h_K \Delta_h v^h\right\|_{L^2(K)} \leq C \left\|\nabla v^h\right\|_{L^2(\omega(K))}, \tag{3.1}$$

where $\omega(K)$ is the union of all the elements sharing a common edge/face with K in $2d/3d$.

The next proposition shows that the effects of the SGS model (2.6) are, *at their largest,* comparable to those of the Smagorinski model (1.3) provided $\delta \sim h_K$.

Proposition 3.1. *Suppose the inverse estimates* (3.1) *hold. Then, for all* $u^h \in X^h$

$$\int_\Omega \delta^3 \left|\Delta^h u^h\right| \nabla u^h \cdot \nabla u^h \leq C \int_\Omega (\delta^3 / h_K) \left|\nabla u^h\right| \nabla u^h \cdot \nabla u^h \, dx,$$

where C depends on the smallest angle in the triangulation.

Proof. Consider one triangle K. Since both ∇u^h and $\Delta^h u^h$ are constant on K

$$\int_K \delta^3 \left|\Delta^h u^h\right| \nabla u^h \cdot \nabla u^h \, dx = \delta^3 \left|\Delta^h u^h\right| \left\|\nabla u^h\right\|^2_{L^2(K)} \leq$$

$$\text{(using(3.1))} \ \leq C \delta^3 h_k^{-1} \left[\sum_{K' \subset \omega(K)} \left\|\nabla u^h\right\|^2_{L^2(K')}\right]^{1/2} \left\|\nabla u^h\right\|^2_{L^2(K)}$$

$$C \delta^3 h_k^{-1} \left(\sum_{K' \subset \omega(K)} \left\|\nabla u^h\right\|^2_{L^2(K)}\right)^{1/2} \left(\sum_{K' \subset \omega(K)} \left\|\nabla u^h\right\|^2_{L^2(K)}\right).$$

Now $\sum_{K' \subset \omega(K)}$ is a sum over a fixed number of terms bounded by $C(\theta_{\min})$. Thus, by norm equivalence the ℓ_2 and ℓ_3 norms are equivalent with constant only depending on the minimum angle $\theta_{\min}$ in the mesh. Thus,

$$\leq C(\theta_{\min})\delta^3 h_K^{-1} \left[\sum_{K' \subset \omega(K)} \|\nabla^h\|_{L^2(K')} \right]^{1/3} \left[\sum_{K' \subset \omega(K)} \|\nabla u^h\|^3_{L^2(K')} \right]^{2/3}.$$

Since $\nabla u^h_{|_{K'}}$ is a constant vector all $L^p(K')$ norms are equivalent. Thus,

$$\int_K \delta^3 |\Delta^h u^h| \cdot |\nabla u^h|^2 \leq C(\theta_{\min})\delta^3 h_K^{-1} \{ \sum_{K' \subset \omega(K)} \|\Delta u^h\|^3_{L^3(K')} \}.$$

Summing this inequality over $K \in \Pi^h(\Omega)$ and again using that $\Pi^h(\Omega)$ satisfies an angle condition gives the result.

This proposition thus establishes that the SGS model (2.6) is more selective than the good and accepted Smagorinski SGS model in that the former vanishes in cases where the latter is significant while (in the sense of the quadratic form) the largest action of the former is bounded by the latter.

4. Existence of solutions. This section considers the question of existence of weak solutions ti the system (2.8) with the modification (2.9). Thus, we seek (w, q) satisfying:

$$\begin{cases} w_t + \nabla \cdot (ww) + \nabla q - \mu \Delta w \\ -\nabla \cdot (\delta^3 |g_\delta * \Delta w| (\nabla w + \nabla w^t)) = \bar{f}, \text{ in } \Omega, t > 0 \\ \nabla \cdot w = 0, \text{ in } \Omega, t > 0, \\ w(x,0) = g_\delta * u_0(x), \text{ in } \Omega, w(x,t) = 0 \text{ for } x \in \partial\Omega. \end{cases} \tag{4.1}$$

The Dirichlet boundary condition we take in (4.1) provides a convenient condition on Γ. It is known, however, that for modeling accuracy near Γ it should be replaced by a slip with friction type condition with recovers no–slip as $\delta \to 0$, [15], [31].

Theorem 4.1. *Let $T > 0$ and Ω a bounded domain in $\mathbb{R}^n$. Then, for any given*

$$u_0 \in L^2(\Omega), \ f \in L^2(\Omega(0,T)),$$

there exists at least one weak solution to (4.1) in $\Omega \times (0,T)$.

Remark. The model (2.7) without regularization is more difficult due to the unbounded coefficient $|\delta u|$ in (2.7). Appropriate mathematical tools for such problems are in their early stages of development, see, e.g. [16].

Proof. We shall use the Galerkin method.

Let $\mathcal{D}(\Omega) := \{\psi \in C_0^\infty(\Omega) : \nabla\cdot\psi = 0 \text{ in } \Omega\}$, $H(\Omega)$ the completion of $\mathcal{D}(\Omega)$ in $L^2(\Omega)$, $H^1(\Omega)$ the completion of $\mathcal{D}(\Omega)$ in $W^{1,2}(\Omega)$ and $\{\psi_r\} \subset \mathcal{D}(\Omega)$ be the orthogonal basis of $H(\Omega)$ given in Lemma 2.3 [11]. We shall look for apporximating solutions v_k of the form:

$$v_k(x,t) = \sum_{r=1}^{k} c_{kr}(t)\psi_r(x),\ k \in \mathbb{N}, \tag{4.2}$$

where the coefficients c_{kr} are required to satisfy the following system of ordinary differential equations

$$\begin{gathered}\frac{dc_{kr}}{dt} + \sum_{i=1}^{k}\mu(\nabla\psi_i, \nabla\psi_r)c_{ki} + \sum_{i,s=1}^{k}(\psi_i\nabla\psi_s, \psi_r)c_{ki}c_{ks} + \gamma^3\sum_{i=1}^{k}c_{ki} \\ \left(\left|\sum_{j=1}^{k}c_{kj}(g_\delta * \Delta\psi_r)\right|(\nabla\phi_i + \nabla\phi_i^t), \nabla\psi_r + \nabla\psi_r^t)\right) = (\bar{f}, \psi - r), \\ r = 1, ..., k\end{gathered} \tag{4.3}$$

with the initial condition

$$c_{kr}(0) = (g_\delta * v_0, \psi_r) \tag{4.4}$$

Multiplying (4.3) by c_{kr}, and summing over r, we get:

$$\begin{gathered}\|v_k(t)\|_2^2 + 2\mu\int_0^t \|\nabla v_k(\zeta)\|_2^2 d\zeta + \\ +2\delta^3\int_0^t |g_\delta * \nabla v_k(\zeta)|\|\nabla v_k(\zeta) + \nabla v_k t(\zeta)\|_2^2 d\zeta = \\ = 2\int_0^t (v_k(\zeta), \bar{f}(\zeta))d\zeta + \|v_k(0)\|_2^2,\ \forall t \in [0,T).\end{gathered} \tag{4.5}$$

Using the Schwarz inequality, Korn's inequality, and Gronwall's lemma, we get:

$$\|v_k(t)\|_2^2 + \int_0^t \|\nabla v_k(\zeta)\|_2^2 d\zeta \le M, \quad \forall t \in [0,T), \tag{4.6}$$

with M independent of t and k. Thus,

$$|c_{kr}(t)| \leq M^{1/2}, \quad \forall r = 1, ..., k. \tag{4.7}$$

From the elementary theory of partial differential equations, (4.7) and (4.3) admits a unique solution $c_{kr} \in W^{1/2}(0, T)$ for all $k \in \mathbb{N}$.

Using the same approach as the one in [11], from these à priori bounds we get the existence of $v \in L^2(0, T, H^1(\Omega))$ such that

$$\lim_{k\to\infty} (v_k(t) - v(t), w) = 0 \text{ uniformly in } t \in [0, T], \ \forall w \in L^2(\Omega) \tag{4.8}$$

$$\lim_{k\to\infty} \int_0^T (\partial_i(v_k - v), w) d\zeta = 0, \forall w \in L^2(\Omega \times [0, T]), \ i = 1, ..., k. \tag{4.9}$$

Now we shall prove the strong convergence of $\{g_\delta * \nabla v_k\}$ to $g_\delta * \Delta v$ in $L^2(w \times [0, T])$ for all $w \subset \Omega$. To show this, we need the following Friedrich's inequality (see, e.g., [12] Lemma II.4.2): Let C be a cube in $\mathbb{R}^n$, then for any $\eta > 0$, there exists $K(\eta, C)$ functions $\psi_i \in L^\infty(C)$, $i = 1, ..., K$, such that

$$\int_0^T \|w(t)\|^2_{2,C} dt \leq \sum_{i=1}^k \int_0^T (w(t), \varphi_i)^2_C dt + \eta \int_0^t \|\nabla w(t)\|^2_{2,C} dt. \tag{4.10}$$

Aplying the above inequality with $w := g_\delta * \Delta v_k - g_\delta * \Delta w_k$, we get

$$\begin{aligned}
\int_0^t \|g_\delta + \Delta v_k - g_\delta * \Delta v\|^2_{2,C} dt &\leq \sum_{i=1}^k \int_0^T (g_\delta * \Delta v_k - g_\delta * \Delta v, \varphi_i)^2_C dt + \\
+\eta \int_0^T \|\nabla(g_\delta * \Delta v_k - g_\delta * \Delta v)\|^2_{2,C} dt &= -\sum_{i=1}^k \int_0^T (\nabla v_k - \nabla v, g_\delta * \nabla \varphi_i)^2_C dt + \\
&+\eta \int_0^T \|\nabla(g_\delta * \Delta v_k - g_\delta * \Delta v)\|^2_{2,C} dt.
\end{aligned}$$

Using (4.9) and the fact that

$$\|\nabla(g_\delta * \Delta w_k - g_\delta * \Delta v)\|^2_{2,C} \leq C(g, \delta) \|v_k - v\|^2_{2,C}$$

we get

$$\lim_{k\to\infty} \int_0^T \|g_\delta * \Delta v_k - g_\delta * \Delta v\|^2_{2,C} = 0. \tag{4.11}$$

Applying (4.10) with $w := v_k - v$, and using (4.9), we get

$$\lim_{k\to\infty}\int_0^T \|v_k - v\|_{2,C}^2 dt = 0. \tag{4.12}$$

Now we shall prove that v is a weak solution of (4.1). Integrating (4.3) from 0 to $t \le T$, we get:

$$\begin{aligned}\int_0^t -\mu(\nabla v_k, \nabla\psi_r) - (v_k \cdot \nabla v_k, \psi_r)d\zeta = -\int_0^t (\bar{f}.\psi_r)d\zeta + \\ +\delta^3\int_0^t (|g_\delta * \Delta v_k|(\nabla v_k + \nabla v_k^t), \nabla\psi_r + \nabla\psi_r^t) + (v_k(t), \psi_r) - (v_k(0), \psi_r).\end{aligned} \tag{4.13}$$

From (4.8) and (4.9) we get

$$\lim_{k\to\infty}(v_k(t) - v(t), \psi_r) = 0, \ \lim_{k\to\infty}\int_0^t (\nabla v_k(\zeta) - \nabla v(\zeta), \nabla\psi_r)d\zeta = 0. \tag{4.12}$$

Let C be a cube containing the support of ψ_r. Then:

$$\begin{aligned}\left|\int_0^t (v_k\cdot\nabla v_k, \psi_r) - (v\cdot\nabla v, \psi_r)d\zeta\right| \le \\ \le \left|\int_0^t ((v_k - v)\cdot\nabla v_k, \psi_r)_C d\zeta)\right| + \left|\int_0^t (v\cdot\nabla(v_k - v), \psi_r)_C d\zeta\right|.\end{aligned} \tag{4.15}$$

Setting $S := \max_{x\in C}|\psi_r(x)|$, and using (4.6), we also have:

$$\begin{aligned}\left|\int_0^t ((v_k - v)\cdot\nabla v_k, \psi_r)_C d\zeta\right| \le \\ \le S\left(\int_0^t \|v_k - v\|_{2,C}^2 d\zeta\right)^{1/2}\left(\int_0^t \|\nabla v_k\|_{2,C}^2 d\zeta\right)^{1/2} \le \\ \le SM^{1/2}\left(\int_0^t \|v_k - v\|_{2,C}^2 d\zeta\right)^{1/2}.\end{aligned}$$

Thus, using (4.12), we get:

$$\lim_{k\to\infty}\left|\int_0^t ((v_k - v)\cdot\nabla v_k, \psi_r)_C d\zeta\right| = 0. \tag{4.16}$$

We also have:

$$\left|\int_0^t (v\cdot\nabla(v_k-v),\psi_r)_C d\zeta\right| \le \sum_{i=1}^{n}\left|\int_0^t (\partial_i(v_k-v),v_i\psi_r)_C d\zeta\right|$$

and since $v_i\psi_r \in L^2(\Omega\times[0,T])$, (4.9) implies:

$$\lim_{k\to\infty}\left|\int_0^t (v\cdot\nabla(v_k-v),\psi_r)_C d\zeta\right| = 0. \tag{4.17}$$

Relations (4.15)–(4.17) yield:

$$\lim_{k\to\infty}\left|\int_0^t (v_k\cdot\nabla v_k - v\cdot\nabla v,\psi_r) d\zeta\right| = 0. \tag{4.18}$$

Also,

$$\begin{aligned}
&\int_0^t (|g_\delta * \Delta v_k|(\nabla v_k+\nabla v_k^t)-|g_\delta*\Delta v|(\nabla v_\Delta+\nabla v^t),\nabla\psi_r+\nabla\psi_r^t)d\zeta\le\\
&\le\left|\int_0^t (|g_\delta*\Delta v|(\nabla(v_k-v)+\nabla(v_k-v)^t),\nabla\psi_r+\nabla\psi_r^t)d\zeta\right|+\\
&+\left|\int_0^t (|g_\delta*\Delta v_k-g_\delta*\Delta v|)(\nabla v_k,+\nabla v_k^t),\nabla\psi_r+\nabla v_k^t)\nabla\psi_r^t)d\zeta\right|.
\end{aligned}\tag{4.19}$$

We have:

$$\begin{aligned}
&\left|\int_0^t (|g_\delta+\Delta v|(\nabla(v_k-v)+\nabla(v_k-v)^t),\nabla\psi_r+\nabla\psi_r^t)d\zeta\right|\le\\
&\le\sum_{i=1}^{n}\left|\int_0^t (\partial_i(v_k-v),|g_\delta*\Delta v|\nabla\psi_r)d\zeta\right|
\end{aligned}$$

and since $|g_\delta*\Delta v|\nabla\psi_r \in L^2(\Omega\times[0,T])$, (4.9) implies:

$$\lim_{k\to\infty}\left|\int_0^t (|g_\delta*\Delta v|(\nabla(v_k-v)+\nabla(v_k-v)^t),\nabla\psi_r+\nabla\psi_r^t)d\zeta\right| = 0. \tag{4.20}$$

On the other hand, setting $\widetilde{S} := \max_{x\in C}|\nabla\psi_r(x)|$, and using (4.6), we get:

$$\begin{aligned}
&\left|\int_0^t (|g_\delta*\Delta v_k-g_\delta*\Delta v|)(\nabla v_k,+\nabla v_k^t),\nabla\psi_r+\nabla\psi_r^t)d\zeta\right|\le\\
&\le S\left(\int_0^t \|g_\delta*\Delta v_k-g_\delta*\Delta v\|_{2,C}^2 d\zeta\right)^{1/2}\left(\int_0^t \|\nabla v_k\|_{2,C}^2 d\zeta\right)^{1/2}\le\\
&\le \widetilde{S}M^{1/2}\left(\int_0^t \|g_\delta*\Delta v_k-g_\delta*\Delta v\|_{2,C}^2 d\zeta\right)^{1/2}.
\end{aligned}$$

Thus, using (4.12), we get:

$$\lim_{k\to\infty}\left|\int_0^t (|g_\delta * \Delta v_k - g_\delta * \Delta v|)(\nabla v_k + \nabla v_k^t), \nabla\psi_r + \nabla\psi_r^t)d\zeta\right| = 0. \tag{4.21}$$

Relations (4.19)–(4.21) yield:

$$\lim_{k\to\infty}\left|\int_0^t (|g_\delta * \Delta v_k|)(\nabla v_k + \nabla v_k^t - |g_\delta * \Delta v|(\nabla v + \nabla v^t), \nabla\psi_r + \nabla\psi_r^t)d\zeta\right| = 0. \tag{4.22}$$

Therefore, taking the limit over $k\longrightarrow\infty$ in (4.13), and using (4.14), (4.18) and (4.22), we get:

$$\begin{aligned}\int_0^t \{-\mu(\nabla v, \nabla v_r) - (v\cdot\nabla v, \psi_r\} d\zeta = \\ = -\int_0^t (\bar{f}, \psi_r)d\zeta + (v(t), \psi_r) - \\ -(v(0), \psi_r) + \delta^3\int_0^t (|g_\delta * \Delta v|(\nabla v + \nabla v^t), \nabla\psi_r + \nabla\psi_r^t).\end{aligned} \tag{4.23}$$

However, from Lemma 2.3 [11], we know that every function $\psi \in \mathcal{D}(\Omega)$ can be uniformly approximated in $C^2(\overline{\Omega})$ by functions of the form

$$\psi_N(x) = \sum_{r=1}^{N} \gamma_r\psi_r(x), \quad N \in \mathbb{N},\ \gamma_r \in \mathbb{R}.$$

So, writing (4.21) with ψ_N instead of ψ_r, and passing to the limit as $N \to \infty$, we get the validity of (4.21) for all $\psi \in \mathcal{D}(\Omega)$.

Thus, v is a weak solution of (4.1).

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