ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLIV, s.I.a, Matematică, 1998, f1

# EQUIVARIANT STRONG SHAPE (I)

BY

CORINA MOHORIANU

*Dedicated to Professor Constantin Corduneanu on the occasion of his 70th birthday*

**Abstract:** This is the first part of a serie of articles on equivariant strong shape. The aim of this paper is to prove that equivariant resolutions of $G$-spaces are equivariant strong expansions.

**1. Introduction.** The group $G$ is compact (Hausdorff). We consider the definitions of invariant covering, G-covering and regular $G$-covering for a $G$-space $X$ from [2].

**Definitions.**

1. For a $G$-space $X$ we say that an open covering $\mathcal{U}$ is invariant if $g\mathcal{U} = \mathcal{U}$, for all $g \in G$.
2. We call an invariant covering $\mathcal{U}$ a $G$-covering if it satisfies the following condition: $U \in \mathcal{U}$, $g \in G$, $U \cap gU \neq \emptyset \Longrightarrow U = gU$.
3. We shall call a $G$-covering $\mathcal{U}$ regular if its nerve is a regular $G$-complex, [2, p.134].
4. Let $\mathcal{U}$ be a locally finite invariant covering of $X$, a paracompact $G$-space and let $f = \{f_U \mid U \in \mathcal{U}\}$ be a partition of unity subordinate to $\mathcal{U}$. Then we define a $G$-partition of unity subordinate to this invariant covering $f' = \{f_U \mid U \in \mathcal{U}\}$, $f'_U(x) = \dfrac{1}{|G|} \displaystyle\sum_{g \in G} f_{g^U}(x)$.
5. From [1] a morphism $\underline{p} = (p_\lambda)_{\lambda \in \Lambda} : X \longrightarrow \underline{X} = (X_\lambda, p_\lambda^{\lambda'}, \Lambda)$ of pro$G$Top is called a $G$-resolution of a $G$-space $X$ or an equivariant resolution, provided for every $G$-ANR $P$ and every open invariant covering $\mathcal{V}$ of $P$ the following two conditions are satisfied:

(G-R1) For every $G$- morphism $f : X \longrightarrow P$ there is a $\lambda \in \Lambda$ and a $G$-map $h : X_\lambda \longrightarrow P$ such that $f, h \cdot p_\lambda$ are $\mathcal{V}$-near. We denote this $(f, h \cdot p_\lambda) \leq \mathcal{V}$.

(G-R2) There is an open invariant covering $\mathcal{V}'$ for $P$ with the following property. Whenever $(\forall)\, \lambda \in \Lambda$, $h, h' : X_\lambda \longrightarrow P$, $(h \cdot p_\lambda, h' \cdot p_\lambda) \leq \mathcal{V}'$, $h$ and $h'$ $G$-maps, then there exists a $\lambda' \in \Lambda$, $\lambda' \geq \lambda$, such that $(h \cdot p_\lambda^{\lambda'}, h' \cdot p_\lambda^{\lambda'}) \leq \mathcal{V}$.

If in a $G$-resolution every $X_\lambda$ is a $G$-ANR, then we speak of a $G$-ANR resolution. Every $G$-space admits a $G$-ANR resolution [1]. Generalizing [5,p.276] we have the following definition for a $G$-strong expansion.

**Definition 1.** A morphism $\underline{p} = (p_\lambda) : X \longrightarrow \underline{X} = (X_\lambda, p_\lambda^{\lambda'}, \Lambda)$ of pro$G$Top is an equivariant strong expansion or a $G$-strong expansion provided it satisfies the conditions:

(G-SE1) If $P$ is an $G$-ANR (for metric spaces) and $f : X \longrightarrow P$ is a $G$–mapping, then there exist a $\lambda \in \Lambda$ and a $G$-mapping $h : X_\lambda \longrightarrow P$ such that $h \cdot p_\lambda \approx_G f$ (equivariant homotopic).

(G-SE2) If $P$ is an $G$-ANR,$f_0, f_1 : X_\lambda \longrightarrow P$ are $G$–mappings, $\lambda \in \Lambda$ and $F : X \times I \longrightarrow P$ a $G$-homotopy such that $F : f_0 \cdot p_\lambda \approx_G f_1 \cdot p_\lambda$ then $(\exists)\lambda' \geq \lambda$ and a $G$-homotopy $H : X_{\lambda'} \times I \longrightarrow P$ such that $H_0 = f_0 \cdot p_\lambda^{\lambda'}$, $H_1 = f_1 \cdot p_\lambda^{\lambda'}$, $H(p_{\lambda'} \times 1) \approx_G F$ (rel $X \times \partial I$).

We want to generalize in the equivariant case the following result proved in [6]: "Every resolution of a space is a strong expansion."

## 2. An equivariant lemma on homotopy equalizers.

**Lemma 1.** *Let $X$ be a $G$-space, let $P, P'$ be $G$-ANR's, let $f : X \to P'$ and $g_0, g_1 : P' \to P$ be $G$-maps such that $F : g_0 \cdot f \approx_G g_1 \cdot f$ with $F : X \times I \to P$ an equivariant homotopy.*

$$F(x, 0) = g_0 \cdot f(x) \tag{1}$$

$$F(x, 1) = g_1 \cdot f(x) \tag{2}$$

*Then there exists a $G$-ANR, $P''$ and $G$-maps $f' : X \longrightarrow P''$ and $h : P'' \longrightarrow P'$ such that*

$$h \cdot f' = f \tag{3}$$

*and $H : P'' \times I \longrightarrow P$ an equivariant homotopy $H : g_0 \cdot h \approx_G g_1 \cdot h$ such that*

$$H(x, 0) = g_0 \cdot h(x) \tag{4}$$

$$H(x, 1) = g_1 \cdot h(x) \tag{5}$$

$$H \cdot (f' \times 1) = F \tag{6}$$

**Proof.** This lemma strengthens [1, p. 220, Lemma 5] by the additional requirement (6). In the proof we use ideas from the proof given in [1] together the analogous result in ordinary strong shape [6, p. 182].

By (1) and (2) there exists an equivariant homotopy $F : g_0 \cdot f \approx_G g_1 \cdot f$. Consider the space $C(I, P) = P^I$ of all continuous maps endowed with the metric

$$\rho(\varphi, \varphi') = \sup\{d(\varphi(t), \varphi'(t)) \mid t \in I\},\ \varphi, \varphi' \in P^I.$$

This gives the compact–open topology. The action of $G$ on $P$ induces an action of $G$ on $P^I$ given by

$$(g\varphi)(t) = g(\varphi(t)),\ g \in G,\ t \in I,\ \varphi \in P^I. \tag{7}$$

Let $q : X \to P^I$ be the map defined by

$$q(x)(t) = F(x, t), \quad x \in X,\ t \in I. \tag{8}$$

This is a continuous map ([3, XII]). By (7) and (8) we have

$$q(gx)(t) = F(gx, t) = gF(x, t) = g(q(x)(t)) = (gq(x))(t)$$

which means that $q$ is an equivariant map. Also we have

$$q(x)(0) = F(x, 0) = g_0 \cdot f(x),\ q(x)(1) = F(x, 1) = g_1 \cdot f(x). \tag{9}$$

We now define $f' : X \longrightarrow P' \times C(I, P)$ by

$$f'(x) = (f(x), q(x)). \tag{10}$$

Clearly $f'$ is an equivariant map. If we denote by $h : P' \times C(I, P) \longrightarrow P'$ the first projection, i.e., $h(y, \omega) = y$, $y \in P'$, $\omega \in C(I, P)$ then $h$ is also equivariant and (3) holds.

Now we define $P'' \subseteq P' \times C(I, P)$ by

$$P'' = \{(y, \varphi) \in P' \times C(I, P) \varphi(0) = g_0(y), \varphi(1) = g_1(y)\} \tag{11}$$

From (9) and (10) we have $f'(X) \subseteq P''$. So $P''$ is an invariant subset of $P' \times C(I, P)$ because:

$$\begin{aligned}(g\varphi)(0) &= g(\varphi(0)) = g(g_0(y)) = g_0(gy)\\ (g\varphi)(1) &= g(\varphi(1)) = g(g_1(y)) = g_1(gy).\end{aligned}$$

So $g(y,\varphi)=(gy,g\varphi)\in P''$ for all $g\in G$ and $(y,\varphi)\in P''$. Now we want to prove that the restrictions $g_0\cdot h|_{P''}, g_1, g|_{P''}$ are $G$-homotopic. For that we define $H: P''\times I\longrightarrow P$ by

$$H((y,\omega),t)=\omega(t). \tag{12}$$

It is continuous from [13, XII] and is an equivariant homotopy because

$$H(g(y,\omega),t)=H((gy,g\omega),t)=(g\omega)(t)=g\omega(t)=gH((y,\omega),t).$$

Restricting $H$ la $P''$ we obtain an equivariant homotopy from $g_0\cdot h$ to $g_1\cdot h$. So $H$ satisfies (4) and (5) and is an equivariant map. Moreover, by (10) and (12) and by the definition of $q$ we have:

$$H(f'\times 1)(x,t)=H((f(x),q(x)),t)=q(x)(t)=F(x,t). \tag{13}$$

This shows that (6) is also satisfied. In order to complete the proof we will show that $P''$ is a $G$-ANR($\mathcal{M}$) or equivalent a $G$-ANE (see [1, p. 221]). Let $Z$ be a metric $G$-space, let $A\subseteq Z$ be a closed invariant subset of $Z$ and let $\varphi: A\longrightarrow P''$ be an equivariant map. We must find an invariant neighbourhood $V$ of $A$ in $Z$, $A\subseteq V\subseteq Z$ and an equivariant extension $\widetilde{\varphi}: V\longrightarrow P''$ of $\varphi$. Denote by $k: P'\times C(I,P)\longrightarrow C(I,P)$ the second projection. This is clearly an equivariant map. Therefore $k\varphi: A\longrightarrow P''\longrightarrow C(I,P)$ is also an equivariant map and it induces a homotopy $\Phi: A\times I\longrightarrow P$ defined by

$$\Phi(a,t)=(k\varphi(a))(t). \tag{14}$$

The continuity of $\Phi$ follows from [3, XII, Th.3.12].$\Phi$ is a $G$-map because:

$$\begin{gathered}\Phi(g(a,t))=\Phi(ga,t)=(k\cdot\varphi(ga))(t)=(g(k\cdot\varphi(a)))(t)=\\ =g((k\cdot\varphi)(a)(t))=g\Phi(a,t),\ (a,t)\in A\times I,\ g\in G\end{gathered} \tag{15}$$

By the definition of $P''$ and $\varphi(a)\in P''$ we have

$$\begin{aligned}&\Phi(a,0)=(k\cdot\varphi(a))(0)=g_0\cdot(h\cdot\varphi(a))\ \text{ and}\\ &\Phi(a,1)=(k\cdot\varphi(a))(1)=g_1\cdot(h\cdot\varphi(a)).\end{aligned} \tag{16}$$

This shows that $\Phi$ is a $G$-homotopy, $\Phi: g_0\cdot h\cdot\varphi\approx_G g_1\cdot h\cdot\varphi$. Since $P'$ is a $G$-ANE($\mathcal{M}$) (by hypothesis and from [1, p. 217]) and $h\cdot\varphi: A\longrightarrow P'$ is an equivariant map, there exists an invariant neighbourhood $U$ for $A$ in $Z$, $A\subseteq U\subseteq Z$ and there exists a $G$-map $\Psi: U\longrightarrow P'$ which extends $h\cdot\varphi$. Now

we apply [1, Prop.4, p. 217] to $g_0 \cdot \Psi, g_1 \cdot \Psi : U \longrightarrow P$ and the equivariant homotopy

$$\Phi : g_0 \cdot h \cdot \varphi \approx_G g_1 \cdot h \cdot \varphi \tag{17}$$

and conclude that there exists an invariant neighbourhood $V$ of $A$ in $U$, $A \subseteq V \subseteq U$ and a $G$-homotopy

$$\widetilde{\Phi} : V \times I \longrightarrow P,\ \widetilde{\Phi} : g_0 \cdot \Psi|_W \approx_G g_1 \cdot \Psi|_V \tag{18}$$

which extends $\Phi$. This $G$-homotopy induces a map $\chi : V \longrightarrow P^I$ given by

$$\chi(z)(t) = \widetilde{\Phi}(z,t),\ z \in V,\ t \in I \ \text{ with} \tag{19}$$
$$\chi(z)(0) = \widetilde{\Phi}(z,0) = g_0 \cdot \Psi(z) \tag{20}$$
$$\chi(z)(1) = \widetilde{\Phi}(z,1) = g_1 \cdot \Psi(z). \tag{21}$$

The continuity of $\chi$ follows from [3, XII, Th.3.1.1]. This is an equivariant map because

$$\chi(gz)(t) = \widetilde{\Phi}(gz,t) = g\widetilde{\Phi}(z,t) = g(\chi(z)(t)) = (g\chi(z))(t), \quad g \in G,\ z \in V,\ t \in I, \tag{22}$$

and extends $k \cdot \varphi : A \longrightarrow P'' \longrightarrow C(I,P)$

$$\chi(a)(t) = \widetilde{\Phi}(a,t) = \Phi(a,t) = k \cdot \varphi(a)(t). \tag{23}$$

Consequently, if we define $\widetilde{\varphi} : V \subseteq Z \longrightarrow P' \times C(I,P)$ by

$$\widetilde{\varphi}(z) = (\Psi(z), \chi(z)). \tag{24}$$

Then $\widetilde{\varphi}$ is an equivariant map ($\Psi, \chi$ are $G$-maps) which extends $\varphi$, $\widetilde{\varphi}|_A = \varphi|_A$. To complete the proof it suffices to note that $(h \cdot \widetilde{\varphi}(z), k \cdot \widetilde{\varphi}(z)) \in P''$ for $z \in V$ because of (20) and (21). That is

$$\begin{aligned} k \cdot \widetilde{\varphi}(z)(0) &= \chi(z)(0) = \widetilde{\Phi}(z,0) = g_0 \cdot \Psi(z) = g_0(h \cdot \widetilde{\varphi}(z)) \\ k \cdot \widetilde{\varphi}(z)(1) &= \chi(z)(1) = \widetilde{\Phi}(z,1) = g_1 \cdot \Psi(z) = g_1(h \cdot \widetilde{\varphi}(z)). \end{aligned} \tag{25}$$

So $\widetilde{\varphi}(V) \subseteq P''$ and this completes the proof of this lemma.

## 3. Stacked invariant coverings

**Definition 2.** Let $Z$ be a $G$-space and $\mathcal{W}$ an open invariant covering of $Z$. For each $W \in \mathcal{W}$ let $\mathcal{T}$ be an open covering of the unit interval

$I = [0,1]$ (we may consider $I$ as $G$-space in the trivial way and $\mathcal{T}$ an invariant covering). Then the sets $W \times J$, $W \in \mathcal{W}$, $J \in \mathcal{T}$ form an open invariant covering $\mathcal{S}$, $g(\mathcal{W} \times \mathcal{T}) = g\mathcal{W} \times \mathcal{T} = \mathcal{W} \times \mathcal{T}$. We call $\mathcal{S}$ a stacked invariant covering for $Z \times I$. If we denote $\mathcal{T}_W = \mathcal{T}$ for each $W \in \mathcal{W}$ we have $\mathcal{S} = \{W \times \mathcal{T}\}_{W \in \mathcal{W}}$.

**Lemma 2.** *Let $Z$ be a normal $G$-space and let $\mathcal{S} = \{W \times \mathcal{T}_W\}_{W \in \mathcal{W}}$ be a stacked invariant covering of $Z \times I$ for $\mathcal{W}$, a locally finite invariant covering of $Z$ and each $\mathcal{T}_W = \mathcal{T}$, for $W \in \mathcal{W}$, is finite. Moreover, for each $W \in \mathcal{W}$ let $\alpha_W$ be a real number, $\alpha_W > 0$. Then there exists a continuous $G$–function $\varphi : Z \longrightarrow I$ with the property that every $z \in Z$ admits a $W \in \mathcal{W}$ such that*

(1) $z \in W$

(2) $0 < \varphi(z) \leq \alpha_W$

**Proof.** Let be $\{\varphi_W\}_{W \in \mathcal{W}}$ a $G$-partition of unity on $Z$ (which exists for $\mathcal{W}$ is a locally finite invariant covering). That is the continuous $G$-maps $\varphi_W : Z \longrightarrow I$ are such that

$$\varphi_W(z) \neq 0 \text{ then } z \in W \tag{3}$$

$$\sum_{W \in \mathcal{W}} \varphi_W(z) = 1 \tag{4}$$

$$\varphi_{gW}(gz) = \varphi_W(z),\ g \in G,\ z \in W \tag{5}$$

Put $\varphi = \sup\{\alpha_W \cdot \varphi_W\}_{W \in \mathcal{W}}$. For each $\varphi_W$ is a $G$-map and $\mathcal{W}$ is an open invariant covering, we have that $\varphi$ is an equivariant map, too:

$$\begin{aligned}\varphi(gx) = \sup\{\alpha_W \cdot \varphi_W(gx)\} = \sup\{\alpha_W \cdot \varphi_{g^{-1}W}(x)\} = \\ = \sup g\{\alpha_{g^{-1}W} \cdot \varphi_{g^{-1}W}(x)\} = g\varphi(x).\end{aligned}$$

By local finitness of $\mathcal{W}$, every point $z_0 \in Z$ admits a neighborhood $U \in \mathcal{V}(z_0)$ such that $\varphi|_U = \sup\{\alpha_W \cdot \varphi_W\}_{W \in \mathcal{W}}$ and $\varphi_W|_U = 0$ except for a finite collection of indexes $\{W_i\}_{i=\overline{1,n}}$, $\varphi_W|_{W_i} \neq 0$.

Then $\varphi|_U = \sup\{\alpha_i \cdot \varphi_{W_i}|_U\}_{i=\overline{1,n}}$. This implies the continuity of $\varphi$ in $z_0 \in Z$. For a given $z \in Z$ one cannot have $\varphi_W(z) = 0$ for all $W \in \mathcal{W}$ because of $\sum_{W \in \mathcal{W}} \varphi_W(z) = 1$. Therefore, $\varphi(z) > 0$ for every $z \in Z$. Every $z \in Z$ admits a $W \in \mathcal{W}$ such that $\varphi(z) = \alpha_W \cdot \varphi_W(z) \leq \alpha_W$. Since $\varphi(z) > 0$ we have $\varphi_W(z) > 0$, and therefore $z \in W$.

**Remark.** From [2, p. 133] if $\mathcal{U}$ is an open covering of $X$ then $g\mathcal{U}$ is also an open covering. $\bigcap_{g\in G} g\mathcal{U}$ is an invariant covering refining $\mathcal{U}$. The $G$-coverings are ordered by existence of equivariant refinements maps [2, p. 135] and form a directed set. This is not true for invariant coverings. For this reason we regard $G$-coverings as the basic notion rather than invariant coverings.

Another result necessary for our scope is the following, presented in [6, p.181] and [7, p.39] classical form and in [1, p.217] the equivariant form.

**Lemma 3.** *Let $P$ be a $G$-ANR and let $\mathcal{U}$ be an open invariant covering of $P$. Then there exists an open invariant covering $\mathcal{V}$ of $P$ such that, whenever for a $G$–space $X$, two $G$-mappings $f_0, f_1 : X \longrightarrow P$ are $\mathcal{V}$–near, $(f_0, f_1) \leq \mathcal{V}$, then there exists an equivariant $\mathcal{U}$-homotopy which connects $f_0, f_1, F : f_0 \approx_G f_1$. Moreover, if for a subset $A \subset X$ (closed and invariant), such that $f_0|_A = f_1|_A$ then $F|_{A\times I}$ is constant.*

**Corollary 1.** *Every $G$-ANR $P$ admits an open invariant covering $\mathcal{V}$ such that any two $G$-maps $f_0, f_1 : X \longrightarrow P$ $\mathcal{V}$-near are homotopic.*

**Proof.** $f_0, f_1 : X \longrightarrow P$ are $\mathcal{V}$-near as in Lemma 3. We can assume $P$ as an invariant closed subset of a normed linear $G$-space. Then we may define, as in the proof of Lemma 3, [1], the linear homotopy $\Phi : X\times I \longrightarrow L$, $\Phi(x,t) = (1-t)\cdot f_0(x) + t\cdot f_1(x)$. This is a $G$-map for $f_0, f_1$ are equivariant.

**Corollary 2.** *If $X$ is a metrizable $G$-space, $A \subseteq X$ a closed and invariant subset of $X$, $P$ a $G$-ANR, $f_0, f_1 : X \longrightarrow P$ equivariant maps and $F : f_0|_A \approx_G f_1|_A$ a equivariant homotopy, then there exists an invariant neighborhood $V$ of $A$ in $X$ and there exists an equivariant homotopy $\widetilde{F} : V\times I \longrightarrow P$ from $f_0|_V$ to $f_1|_V$, which extends $F$.*

## 4. Main equivariant lemma.

**Lemma 4.** *Let $X$ be a $G$-space and $\underline{p} = (p_\lambda) : X \longrightarrow \underline{X} = (X_\lambda, p_\lambda^{\lambda'}, \Lambda)$ a $G$–resolution of $X$. Let $\lambda \in \Lambda$, $P$ a $G$-ANR, $f_0, f_1 : X_\lambda \to P$ and $F : X\times I \longrightarrow P$ $G$-maps ($F : f_0 \cdot p_\lambda \approx_G f_1 \cdot p_\lambda$) as in* (G-SE2). *Then for every open $G$-covering $\mathcal{U}$ of $P$ there exists $\lambda' \geq \lambda$ and a $G$-homotopy $\widetilde{H} : X_{\lambda'}\times I \longrightarrow P$ such that*

$$\widetilde{H}_0(y,0) = f_0 \cdot p_\lambda^{\lambda'}(y) \tag{1}$$

$$\widetilde{H}_0(y,1) = f_1 \cdot p_\lambda^{\lambda'}(y) \tag{2}$$

$$(\widetilde{H}(p_{\lambda'}\times 1), F) \leq \mathcal{U}. \tag{3}$$

**Proof.** Let $\underline{p} = (p_\lambda)$ be a $G$-resolution and $\lambda, P, f_0, f_1, F$ as in (G-SE2). Let $\mathcal{U}$ be an open $G$-covering of P and $\mathcal{U}'$ an open invariant star refinement of $\mathcal{U}$, [2, p. 135]. For P is a $G$-ANR($\mathcal{M}$) we have that if $\mathcal{U}$ is a $G$-covering then $\mathcal{U}'$ is a $G$-covering. Then we choose an open $G$-covering $\mathcal{V}$ of P such that the assertion of Lemma 3 holds for $\mathcal{U}'$ and $\mathcal{V}$. We also assume that $\mathcal{V}$ is a $G$-covering star refinement of $\mathcal{U}'$. Next we choose $\mathcal{V}'$ a $G$-covering, star refinement of $\mathcal{V}$ such that (G-R2) holds for $P, \mathcal{V}, \mathcal{V}'$. Put $P' = P \times P$ and denote by $g_0, g_1 : P' \longrightarrow P$ the first and the second projections. If we define the action of $G$ on $P'$ by $g(p_1, p_2) = (gp_1, gp_2)$, $g \in G$, $p_1, p_2 \in P$ we obtain $g_0, g_1$ equivariant maps. Then we define $f : X \longrightarrow P'$ to be the only map for which $f = (g_0 \cdot f, g_1 \cdot f)$, $g_0 \cdot f = f_0 \cdot p_\lambda$, $g_1 \cdot f = f_1 \cdot p_\lambda$. For $(p_\lambda)$ are $G$-maps and $f_0, f_1$ are equivariant maps we have f is a $G$-map.

Now we apply the equivariant Lemma 1 on homotopy equalizers to $X, P, P', f, g_0, g_1, F$ and we conclude that there exist an $G$-ANR $P''$, $G$-maps $f' : X \longrightarrow P''$, $h : P'' \to P'$ and a $G$-homotopy $H : P'' \times I \longrightarrow P$ such that

$$h \cdot f' = f \tag{4}$$

$$H(x, 0) = g_0 \cdot h(x),\ H(x, 1) = g_1 \cdot h \tag{5}$$

$$H \cdot (f' \times 1) = F \tag{6}$$

From [2, p. 141] , $H^{-1}(\mathcal{V}')$ is open $G$-covering of $P'' \times I$ because H is $G$-map and $\mathcal{V}'$ is $G$-covering. We choose a refinement $\mathcal{S}$ which is $G$-covering and stacked covering of $P'' \times I$. $\mathcal{S}$ is given by a locally finite open $G$-covering $\mathcal{W}$ of $P''$ and by finite open coverings $\mathcal{T} = \mathcal{T}_W$ of $I$, one for each $W \in \mathcal{W}$. Now we apply (G-R1) to $f' : X \to P''$ and $\mathcal{W}$. There exists $\lambda'' \geq \lambda$ and a $G$-mapping $k : X_{\lambda''} \longrightarrow P''$ such that

$$(f', k \cdot p_{\lambda''}) \leq \mathcal{W}. \tag{7}$$

For any $W \in \mathcal{W}$, $W \times 0 \subseteq W \times J$, $J \in \mathcal{T}_W$ and $W \times J$ contained in $H^{-1}(V')$, $V' \in \mathcal{V}'$. By the condition (5) we have $g_0 \cdot h(W) = H(W, 0) \subseteq H(W, J) \subseteq V'$, i.e. $g_0 \cdot h(\mathcal{W}) \geq \mathcal{V}'$.

Consequently, (7) implies $(g_0 \cdot h \cdot f', g_0 \cdot h \cdot k \cdot p_{\lambda''}) \leq \mathcal{V}'$. From (4) $h \cdot f' = f$ and $g_0 \cdot f = f_0 \cdot p_\lambda$, $g_1 \cdot f = f_1 \cdot p_\lambda$ we have

$$g_0 \cdot (h \cdot f') = g_0 \cdot f = f_0 \cdot p_\lambda = f_0 \cdot p_\lambda^{\lambda''} \cdot p_{\lambda''}, \quad \lambda'' \geq \lambda. \tag{8}$$

So (7) becomes

$$(f_0 \cdot p_\lambda^{\lambda''} \cdot p_{\lambda''}, g_0 \cdot h \cdot k \cdot p_{\lambda''}) \leq \mathcal{V}'. \tag{9}$$

Analogously we obtain

$$(f_1 \cdot p_\lambda^{\lambda''} \cdot p_{\lambda''}, g_1 \cdot h \cdot k \cdot p_{\lambda''}) \leq \mathcal{V}'. \tag{10}$$

By the choise of $\mathcal{V}$ in (G-R2) there is a $\lambda' \geq \lambda''$ such that

$$(f_0 \cdot p_\lambda^{\lambda'}, g_0 \cdot h \cdot k \cdot p_{\lambda''}^{\lambda'}) \leq \mathcal{V} \tag{11}$$

$$(f_1 \cdot p_\lambda^{\lambda'}, g_1 \cdot h \cdot k \cdot p_{\lambda''}^{\lambda'}) \leq \mathcal{V}. \tag{12}$$

By the choice of $\mathcal{V}$ as $G$-covering of P we conclude that there exist $\mathcal{U}'$-$G$-homotopies $K, L : X_{\lambda'} \times I \longrightarrow P$ such that

$$K : f_0 \cdot p_\lambda^{\lambda'} \approx_G g_0 \cdot h \cdot k \cdot p_{\lambda''}^{\lambda'} \tag{13}$$

$$L : f_1 \cdot p_\lambda^{\lambda'} \approx_G g_1 \cdot h \cdot k \cdot p_{\lambda''}^{\lambda'} \tag{14}$$

Furthermore, by the fact that $\mathcal{S} = \{W \times \mathcal{T}_W\}_{W \in \mathcal{W}}$ is a stacked invariant cover and by (7), for any $t \in I$ both points $(f'(x), t), (k \cdot p_{\lambda''}(x), t)$ belong to some member of $\mathcal{S}$, and therefore also to $H^{-1}(V')$, for some $V' \in \mathcal{V}'$. Consequently,

$$(H \cdot (f' \times 1), H \cdot (k \cdot p_{\lambda''} \times 1)) \leq \mathcal{V}' \tag{15}$$

and so $(H \cdot (f' \times 1), H \cdot (k \cdot p_{\lambda''} \times 1)) \leq \mathcal{V}$.

We will now use the $G$-maps $K, L, H \cdot (k \cdot p_{\lambda''}^{\lambda'} \times 1)$ to construct the desired homotopy $\widetilde{H} : X_{\lambda'} \times I \longrightarrow P$. For every $W \in \mathcal{W}$ we choose a number $\alpha_W \in \left(0, \frac{1}{3}\right]$ which is smaller than the Lebesgue number of the covering $\mathcal{T}_W$ of the compact $I$. If $t, t' \in I$, $|t - t'| \leq \alpha_W$ then there is a $J \in \sqcup_{\mathcal{W}}$, $t, t' \in J$. Consequently, if $z \in W$ and $|t - t'| \leq \alpha_W$ then $(z, t), (z, t') \in W \times J$. Therefore both $H(z, t), H(z, t') \in V'$, $V' \in \mathcal{V}'$. Now we apply Lemma 2 and obtain a continuous equivariant function $\varphi : P'' \longrightarrow I$ such that for every $z \in P''$ there is a $W \in \mathcal{W}$ with

$$z \in W \tag{16}$$

$$0 < \varphi(z) \leq \alpha_W \leq \frac{1}{3} \tag{17}$$

Note that $0 < \varphi(z) \leq 1 - \varphi(z) < 1$, $(\forall)\, z \in P''$. Now we define $\widetilde{H} : X_{\lambda'} \times I \longrightarrow P$ for each $y \in X_{\lambda'}$, $t \in I$ by

$$\widetilde{H}(y,t) = \begin{cases} K\left(y, \dfrac{t}{\varphi(z)}\right), & t \in [0, \varphi(z)] \\ H\left(z, \dfrac{t-\varphi(z)}{1-2\varphi(z)}\right), & t \in [\varphi(z), 1-\varphi(z)] \\ L\left(y, \dfrac{1-t}{\varphi(z)}\right), & t \in [1-\varphi(z), 1] \end{cases} \tag{18}$$

where

$$z = k \cdot p_{\lambda''}^{\lambda'}(y). \tag{19}$$

$\widetilde{H}$ is well defined, continuous and equivariant ($K, H, L$ are $G$-maps). It remains to prove $(\widetilde{H}(p_{\lambda'} \times 1), F) \leq \mathcal{U}$, that is $(\forall)\,(x,t) \in X \times I$, $(\exists)\, U \in \mathcal{U}$ such that

$$\widetilde{H}(p_{\lambda'} \times 1)(x,t), F(x,t) \in U. \tag{20}$$

Put $y = p_{\lambda'}(x)$, $z = k \cdot p_{\lambda''}^{\lambda'}(y) = k \cdot p_{\lambda''}^{\lambda'} \cdot p_{\lambda'}(x) = k \cdot p_{\lambda''}(x)$, $z \in P''$, $y \in X_{\lambda'}, x \in X$.

**Case 1:** $t \in [\varphi(z), 1 - \varphi(z)]$

Then by the definition of $\widetilde{H}$ and by $H \cdot (f' \times 1) = F$ we have

$$\widetilde{H}(y,t) = \widetilde{H}(p_{\lambda'}(x), t) = H\left(z, \frac{t-\varphi(z)}{1-2\varphi(z)}\right),\ z = k \cdot p_{\lambda''}^{\lambda'}(y) \tag{21}$$

$$H \cdot (f'(x), t) = F(x,t). \tag{22}$$

By the definition of $\mathcal{W}$ there is $W \in \mathcal{W}$ such that

$$f'(x),\ z = k \cdot p_{\lambda''}(x) \in W. \tag{23}$$

From the definition of equivariant stacked covering $\mathcal{S} = \{W \times \mathcal{T}_W\}_{W \in \mathcal{W}}$ there is a $W' \in \mathcal{W}$ such that $z \in W'$, $0 < \varphi(z) \leq \alpha_{W'} \leq \dfrac{1}{3}\cdot$ Also note that $t \in [\varphi(z), 1 - \varphi(z)]$ implies

$$\left|\frac{1-2t}{1-2\varphi(z)}\right| \leq 1 \tag{24}$$

and therefore

$$|t - \frac{t-\varphi(z)}{1-2\varphi(z)}| \leq \varphi(z)|\frac{1-2t}{1-2\varphi(z)}| \leq \varphi(z) \leq \alpha_{W'} \tag{25}$$

By the choice of the numbers $\alpha_{W'}$, $z \in W'$ we have the existence of some $V_1' \in \mathcal{V}'$ for which

$$H(z,t), H\left(z, \frac{t-\varphi(z)}{1-2\varphi(z)}\right) \in V_1'. \tag{26}$$

On the other hand, by the fact that $f'(x),\ z \in W$ we have $(f'(x),t),(z,t) \in W \times J$, $W \in J \in \mathcal{S}$ and therefore there is a $V_2' \in \mathcal{V}'$ such that

$$H(f'(x),t), H(z,t) \in V_2'. \tag{27}$$

Since $\mathcal{V}'$ is a $G$-star refinement of $\mathcal{V}$ we have that there is a $V \in \mathcal{V}$ such that $V_1' \cup V_2' \subset V$ which contains the points $H\left(z, \frac{t-\varphi(z)}{1-2\varphi(z)}\right), H(f'(x),t) \in V$. Since $\mathcal{V}$ refines $\mathcal{U}$ we have $F(t,x), \widetilde{H}(p_{\lambda'}(x),t) \in U$, $U \in \mathcal{U}$. This establishes (20).

**Case 2.** $t \in [0, \varphi(z)]$

Choose $W \in \mathcal{W}$ so that $z \in W$ and $0 < \varphi(z) \leq \alpha_W \leq \frac{1}{3}$.

From $|t-0| \leq \varphi(z) \leq \alpha_W$ there is a $J \in \mathcal{T}_W$ such that $0, t \in J$. Consequently there is $V' \in \mathcal{V}'$ such that

$$H(z,0), H(z,t) \in V'. \tag{28}$$

On the other hand, by (7) there is $W' \in \mathcal{W}$ which contains the points $f'(x), z = k \cdot p_{\lambda''}(x) \in W'$. Therefore, for any $t \in I$, $(f'(x),t),(z,t) \in W' \times J_t$, $J_t \in \mathcal{T}_{W'}$. Consequently, there is a $V_t' \in \mathcal{V}'$ such that

$$F(x,t) = H(f'(x),t),\ H(z,t) \in \mathcal{V}_t'. \tag{29}$$

In particular, for $t = 0$, we have $V_0' \in \mathcal{V}'$ and

$$F(x,0) = H(f'(x),0), H(z,0) \in V_0' \tag{30}$$

Since $\mathcal{V}'$ is a $G$-star refinement of $\mathcal{V}$ we conclude that there is a $V \in \mathcal{V}$, $V' \cup V_t' \cup V_0' \subset V$ such that

$$F(x,0), F(x,t) \in V. \tag{31}$$

Since K is $\mathcal{U}'$-equivariant homotopy, there is a $U' \in \mathcal{U}'$ which contains both points $K(y,0), K\left(y, \frac{t}{\varphi(z)}\right) \in U'$. By the definition of K we have

$$K(y,0) = f_0 \cdot p_\lambda^{\lambda'}(y) = f_0 \cdot p_\lambda^{\lambda'} \cdot p_{\lambda'}(x) = f_0 \cdot p_\lambda(x) = g_0 \cdot f(x), y = p_{\lambda'}(x)$$
$$F(x,0) = H(f'(x),0) = g_0 \cdot h \cdot f'(x) = g_0 \cdot f(x)$$

So

$$K(y,0) = F(x,0). \tag{32}$$

Furthermore

$$K\left(y, \frac{t}{\varphi(z)}\right) = \widetilde{H}(y,t) = \widetilde{H}(p_{\lambda'}(x),t). \tag{33}$$

So we have

$$\widetilde{H}(p_{\lambda'}(x),t), F(x,0) \in U',\ U' \in \mathcal{U}',\ t \in [0,\varphi(z)]. \tag{34}$$

Since $\mathcal{V}$ refines $\mathcal{U}'$ and $\mathcal{U}'$ is a $G$-star refinement of $\mathcal{U}$ we have that $(\exists)\, U \in \mathcal{U}$, $V \cup U' \subset U$ such that $H(p_{\lambda'}(x),t), F(x,t) \in U$.

The condition (20) is satisfied in this case.

**Case 3.** $t \in [1-\varphi(z), 1]$

This case is symmetric to the case 2 with L instead of K and $f_1, g_1$ instead of $f_0, g_0$. This completes the proof of Lemma 4.

**5. Conclusions.** Now we can proof the main result:

**Theorem 1.** *Every equivariant resolution is an equivariant strong expansion.*

**Proof.** We will now derive this theorem from Lemmas 3 and 4.

Proof of (G-SE1): Take for $\mathcal{U}$ any $G$-covering of P and choose $\mathcal{V}$ a $G$-open covering of P as in Lemma 3. By (G-R1) applied to $\mathcal{V}$ there is a $\lambda \in \Lambda$ and a $G$-mapping $h : X_\lambda \longrightarrow P$ such that $(h \cdot p_\lambda, f) \leq \mathcal{V}$. Then by the choice of $\mathcal{V}$ we have there is a $\mathcal{U}$-equivariant homotopy

$$h \cdot p_\lambda \approx_G f \tag{1}$$

Proof of (G-SE2): Let $\lambda, P, f_0, f_1, F$ be as in (G-SE2). Take any open $G$-covering $\mathcal{U}$ of P and choose $\mathcal{V}$ according to Lemma 3. Now apply Lemma 4 to the $G$-covering $\mathcal{V}$. We obtain a $\lambda' \geq \lambda$ and a $G$-homotopy

$$H : X_{\lambda'} \times I \longrightarrow P \tag{2}$$

such that

$$H : f_0 \cdot p_\lambda^{\lambda'} \approx_G f_1 \cdot p_\lambda^{\lambda'} \tag{3}$$

$$(F, H(p_{\lambda'} \times 1)) \leq \mathcal{V}. \tag{4}$$

Put $Z = X \times I$ a $G$-space, $A = X \times \partial I$ a closed and $G$-invariant subspace in $Z$, $h_0 = F$, $h_1 = H \cdot (p_{\lambda'} \times 1)$, and observe that the $G$-maps $h_0, h_1$ are equal on $A$,

$$h_0|_A = h_1|_A. \tag{5}$$

This is because

$$\begin{aligned} h_0(x,0) = F(x,0) = f_0 \cdot p_\lambda(x) = f_0 \cdot p_\lambda^{\lambda'} \cdot p_{\lambda'}(x) = \\ = H(p_{\lambda'}(x), 0) = h_1(x,0) \end{aligned} \tag{6}$$

$$\begin{aligned} h_0(x,1) = F(x,1) = f_1 \cdot p_\lambda(x) = f_1 \cdot p_\lambda^{\lambda'} \cdot p_{\lambda'}(x) = \\ = H(p_{\lambda'}(x), 1) = h_1(x,1) \end{aligned} \tag{7}$$

From (4) we also have $(h_0, h_1) \leq \mathcal{V}$. Consequently, by the choice of $\mathcal{V}$ in Lemma 3 there exists a $\mathcal{U}$-$G$-homotopy which connects $h_0, h_1$ constant on $A \times I$.

**Remark.** As in the ordinary case we can proof that every $G$-strong expansion is a $G$-expansion. So we obtain the following result:

"Every $G$-resolution is a $G$-expansion."

From [2, p. 135] we have that for every paracompact $G$-space X, with G a finite group, the locally finite regular $G$-coverings of X are cofinal in the set of all coverings of X. From [1, p. 218] and [4, p. 374] we have that every $G$-space X admits a $G$-resolution. So we have the following result:

**Corollary 3.** *Every $G$-space $X$ admits a cofinite strong $G$-ANR expansion, i.e. a strong equivariant expansion*

$$\underline{p} = (p_\lambda) : X \longrightarrow \underline{X} = (X_\lambda, p_\lambda^{\lambda'}, \Lambda)$$

*where all the $X_\lambda$'s are $G$-ANR's, $\underline{p} = (p_\lambda)$ is a morphism in $pro(G - Top)$ and $\Lambda$ is cofinite.*

REFERENCES

1. A n t o n i a n, S.A, M a r d e š i ć, S., – *Equivariant shape*, Fund. Math. 127 (1987).
2. B r e d o n, G.E, – *Introduction to compact Transformation groups,* Academic Press 1972.
3. D u g u n d j i, J., – *Topology*, Allyn & Bacon, Boston 1968.
4. L i s i c a , J u T., M a r d e š i ć, S., – *Coherent prohomotopy and strong shape theory,* Glasnik Mat. Vol 19(39) (1984), 335–399.
5. M a r d e š i ć, S., – *Strong expansions and strong shape theory*, Top. & Appl. 38 (1991), 275-291.
6. M a r d e š i ć, S., – *Resolutions of spaces are strong expansions,* Publ. De l'Inst. Math., Nouv. Serie 49(63), 1991, 179–188.
7. M a r d e š i ć, S., S e g a l, J., – *Shape Theory, An inverse system approach,* North Holland, Amsterdam (1982).

*Received: 5.XI.1997*

*Faculty of Mathematics*
*"Al.I. Cuza" University*
*6600 Iaşi*
*ROMANIA*