ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLIV, s.I.a, Matematică, 1998, f1

AN EXISTENCE THEOREM ON NONLINEAR VARIATIONAL INEQUALITIES

BY

RAM U. VERMA

Abstract. The solvability of a class of strongly monotone nonlinear variational inequality problems in a Banach space setting is presented.

1. Introduction. Let K be a subset of a real reflexive Banach space X and let the mapping $T : K \to X^*$ be strongly monotone. Let (w, u) denote the duality pairing between the elements w in X^* and u in X and $\|u\|$ denote the norm on X. Let H and K be nonempty closed subsets of the Banach space X and, let $d_H K$ denote the set of z in K such that the intersection of $U(z)$ and $(H - K)$ is non–void for every neighborhood $U(z)$ of z and $i_H K$ denote the set of z in K such that the intersection of $U(z)$ and $(H - K)$ is void for some neighborhood $U(z)$ of z.

Hirano and Takahashi [3] using a result of Barbu and Precupanu [1] established the following existence theorem on the solvability of the nonlinear variational inequalities.

Theorem 1.1. *Let C be a nonempty closed convex subset of a reflexive Banach X and $T : C \to X^*$ be hemicontinuous and monotone. Then the following inequalities are equivalent:*

(i) There exists an element x_0 in C such that

$$(Tx_0, y - x_0) \geq 0 \quad \textit{for all } y \textit{ in } C.$$

This research is announced in the Mathematical Sciences Research Hot–Line 1 (1) (1997), to appear.

(ii) *There exists a bounded closed convex subset K of C such that for each z in $d_C K$, there is an element y in $i_C K$ such that*

$$(Tz, y - z) \leq 0.$$

Here our aim is to generalize this theorem to the case of strongly monotone operators in a similar setting.

An operator $T : C \to X^*$ is strongly monotone if for all u, v in C there exists a constant $r > 0$ such that

$$(Tu - Tv, u - v) \geq r\|u - v\|^2.$$

The operator T is called monotone if for all u, v in C,

$$(Tu - Tv, u - v) \geq 0.$$

T is called hemicontinuous if T is continuous from line segments in C to weak topology of X^*.

2. The main result. Now we consider our main result on the existence of solutions of a class of nonlinear variational inequalities.

Theorem 2.1. *Let X be a real reflexive Banach space and C be a nonempty closed convex subset of X. If $T : C \to X^*$ is hemicontinuous and strongly monotone then the following conditions are equivalent:*

(1) *There exists an element x_0 in C such that*

$$(Tx_0, y - x_0) \geq 0 \quad \textit{for all } y \textit{ in } C.$$

(2) *There exists a bounded closed convex subset K of C such that for each z in $d_C K$, there is an element y in $i_C K$, which satisfies*

$$(Tz, y - z) \leq -r\|y - z\|^2 \quad \textit{for } r > 0.$$

Proof. (1) $\Rightarrow$ (2) : This part of the proof follows from the strong monotonicity of the operator T.

(2) $\Rightarrow$ (1) : Since under the assumptions of the theorem K is weakly compact convex, there by [1] exists an element x_0 in K such that

$$(Tx_0, x - x_0) \geq 0 \quad \text{for all } x \text{ in } K.$$

If x_0 is in $i_C K$ then for each y in C one can choose $t > 0$ sufficiently small that $x = ty + (1-t)x_0$ belongs to K, and as a result, we find that

$$(Tx_0, y - x_0) \geq 0 \quad \text{for all } y \text{ in } C$$

If x_0 belongs to $d_C K$, then by the hypothesis there exists z_0 in $i_C K$ such that

$$(Tx_0, z_0 - x_0) \leq -r\|z_0 - x_0\|^2.$$

Since $(Tx_0, x - x_0) \geq 0 \quad \text{for all in } K$, this implies that

$$(Tx_0, x - z_0) \geq r\|z_0 - x_0\|^2 \quad \text{for all } x \text{ in } C.$$

As z_0 is in $i_C K$, for each y in C, there exists a constant $t > 0$ such that $x = ty + (1-t)z_0$ belongs to K. It follows that

$$(Tx_0, y - z_0) \geq r\|z_0 - x_0\|^2 \quad \text{for all } y \text{ in } C.$$

Next since $(Tx_0, z_0 - x_0) \geq -r\|z_0 - x_0\|^2$, this implies that

$$(Tx_0, y - x_0) = (Tx_0, y - z_0) + (Tx_0, z_0 - x_0) \geq 0.$$

REFERENCES

1. B a r b u, V. and P r e c u p a n u, Th.– *Convexity and Optimization in Banach Spaces*, Editura Academiei R.S.R., Bucharest, 1975.
2. T a k a h a s h i, W.– *Nonlinear variational inequalities and fixed point theorems*, J.Math.Soc.Japan 28(1976), 168–181.
3. H i r a n o, N. and T a k a h a s h i, W.– *Existence theorems on unbounded sets in Banach spaces*, Proc. Amer. Math. Soc. 80 (1980), 647–650.
4. V e r m a, R.U.– *Iterative algorithms for variational inequalities and associated nonlinear equations involving relaxed Lipschitz operators*, Applied Math. Lett. 9(4) (1996), 61–63.
5. V e r m a, R.U.– *Generalized variational inequalities involving multivalued relaxed monotone operators*, Appl. Math. Lett. (accepted).
6. V e r m a, R.U.– *Nonlinear variational and constrained hemivariational inequalities involving relaxed operators*, Z.Angew. Math. Mech. (accepted).

7. V e r m a, R.U.– *Nonlinear variational inequalities involving relaxed monotone operators*, Comm. Appl. Anal. (accepted).
8. Z e i d l e r, E.– *Nonlinear Functional Analysis and its Applications IV*, Springer–Verlag, New York, 1988.

Received: 12.XI.1996

International Publications
12046 Coed Drive
Orlando, FL 32826
USA