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SET–CONSTRAINED ALMOST EXTREMALITY VIA NONSMOOTH IMPLICIT FUNCTIONS

BY

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Dedicated to Professor Constantin Corduneanu on the occasion of his 70th birthday

Abstract. A general solvability result (based on maximality principles of Brezis–Browder type) is used to get a nonsmooth implicit function theorem over Banach spaces. The obtained facts are then used to the variational description of the almost/exact solutions to certain set-constrained programming problems.

0. Introduction. Let $(X, \|\cdot\|)$ be a normed space and $F : X \to R$, a function. We consider the set–constrained *optimization problem*

$$(P_{\min}) \qquad \inf\{F(x); x \in M_{TS}\}$$

where the admisible set M_{TS} is of the form

$$M_{TS} = \{x \in M; Tx = 0, Sx \geq 0\}.$$

Here, M is a subset of X, T is a map from X to the normed space Y, and S, a mapping from X to the normed space Z, ordered by the cone Q. By the strong nonlinear character of this problem, we cannot expect, in general, for exact solutions of it; so, the most reasonable device is to look for *approximate solutions* to $(P_{\min})$ which *"almost"* satisfy the necessary first order conditions. Loosely speaking, the main ingredients in solving this problem are

a) the *algebraic one*, based on a variant of the Farkas–Minkowski lemma

b) the *topological* one, consisting in a sort of *implicit function theorem*, to get a path in M_{TS} with a prescribed initial point and initial tangent.

The former of these is to be solved in a standard way,by means of the polar calculus for the dual couple (X, X^*). (This will be the content of Section 4 below). So, the investigations of the present exposition are centered on the latter. These, in the last analysis, follow the lines in T u r i n i c i [20], used for the case of $M = X$; details will be given in Section 5. And, the basic tool in obtaining them is represented by an implicit function theorem (based on Clarke/Bouligand tangent cones) in Section 3. The preliminary material for this is to be found in Sections 1 and 2. Finally, Section 6 treats the exact mimimum case, in the same way as the one developed by M o t r e a–n u [13]. Some further aspects of these facts will be discussed elsewhere.

1. A local solvability result. Let $(X, d, \leq)$ be a quasi–ordered metric space and $(Y, \|\cdot\|)$, a normed space. Take a mapping $T : X \to Y$ and fix some (nonempty) part M of X. Given $x_0 \in M$, denote

(1D1) $M(x_0, \leq) = \{x \in M; x_0 \leq x\}, \quad M(x_0, \geq) = \{x \in M; x_0 \geq x\}$

(the *order intervals* through x_0)

as well as, for each $\rho > 0$,

(1D2) $M(x_0, \rho) = \{x \in M; d(x_0, x) < \rho\}$, $M[x_0, \rho] = \{x \in M; d(x_0, x) \leq \rho\}$ (the *open/closed M-sphere* with center x and radius ρ).

We are interested to determine local (modulo x_0) conditions under which the operator equation

(OE) $Tx = 0$

should have a solution in a certain order sphere

$$M[x_0, \leq, \rho] = M[x_0, \leq] \cap M[x_0, \rho].$$

To formulate the basic statement (referred to as a *local solvability* one for T) we make the hypotheses below about $(X, d, \leq)$:

(1H1) X is $(d, \leq)$-*complete* (each ascending Cauchy sequence converges)

(1H2) $\begin{cases} \leq \text{ is } \textit{self} - \textit{closed} \text{ (the limit of each ascending} \\ \text{sequence is an upper bound of it).} \end{cases}$

And, relative to (T, M), one admits

(1H3) T is *order–continous* ((x_n) ascending and $x_n \to x$ imply $Tx_n \to Tx$).

(1H4) $\begin{cases} M \text{ is } \textit{order} - \textit{closed} \text{ (the limit of each ascending} \\ \text{sequence in M belongs to M).} \end{cases}$

For the specific assumptions to be used, we need some conventions. Call the function $a : [0, \infty[\to [0, \infty[$, *normal*, when

(1D3) $a(0) = 0 < a(t) < t$ and $a^n(t) \to 0$ for each $t > 0$.

(Here a^n=the n–th iterate of a, for each n). An immediate example of such function is the *linear* one

(1D4) $a(t) = \alpha t, \quad t \geq 0,$ for some α in $]0, 1[$.

Given this object, we shall say that the couple of functions φ, ψ from $[0, \infty[$ to itself is *compatible* (modulo a) when

(1D5) $\dfrac{\varphi(t)}{t - a(t)} \leq \dfrac{\psi(t) - \psi(s)}{t - s}, \quad t, s > 0, \quad a(t) \leq s < t.$

Note that, in such a case

(1.1) $t \vdash \psi(t)$ is increasing over $]0, \infty[$.

The proof being evident, we omit the details.

Theorem 1.1. *Suppose that a normal function $t \vdash a(t)$ and a couple of compatible* (*modulo a*) *functions b, g may be found so as: for each x in*

$$M_0(g) = M[x_0, \leq, g(\|T(x_0)\|)]$$

with $Tx \neq 0$ there correspond x' in M and ε in $]0, 1]$ with

(1H5) $x \leq x', \ \|Tx' - (1 - \varepsilon)Tx\| \leq \varepsilon a(\|Tx\|), \ d(x, x') \leq \varepsilon b(\|Tx\|).$

Then, the operator equation (OE) *has at least one solution in $M_0(g)$.*

Proof. Suppose by contradiction that

(1H6) $Tx \neq 0$, for each $x \in M_0(g)$ (hence $Tx_0 \neq 0$).

Denote for simplicity

$$M_0^*(g) = \{x \in M_0(g); d(x_0, x) \leq g(\|Tx_0\|) - g(\|Tx\|)\}.$$

Let x be arbitrary fixed in $M_0^*(g)$ and x', its associated by (1H5) point in M. By the imposed conditions, (1.2) $\|Tx'\| \leq (1-\varepsilon)\|Tx\| + \varepsilon a(\|Tx\|)$.

So, if we put

$$t = \|Tx\|, \quad s = (1-\varepsilon)t + \varepsilon a(t),$$

one has

(1.3) $a(t) \leq s < t, \quad \varepsilon = (t-s)/(t-a(t)), \quad \|Tx'\| \leq s$.

This, and the choice of (b,g), yields

(1.4) $d(x,x') \leq b(t)/(t-a(t)) \leq g(t) - g(s)$.

So, by definition, $x \leq x'$ and

(1.5) $d(x_0,x') \leq d(x_0,x) + d(x,x') \leq g(\|Tx_0\|) - g(s) \leq g(\|Tx_0\|)$;

wherefrom $x' \in M_0(g)$.

Hence $0 < \|Tx'\| \leq s$; and this, by g being increasing, yields

(1.6) $x \leq x',\ d(x,x') \leq g(\|Tx\|) - g(\|Tx'\|),\ \|Tx\| \geq \|Tx'\|$.

Note that, $x_0 \leq x'$; and, again by (1.5),

$d(x_0,x') \leq g(\|Tx_0\|) - g(\|Tx'\|)$ (hence $x' \in M_0^*(g)$).

We now introduce an order $\preceq$ over $M_0(g)$ by the convention

(1D6) $x_1 \preceq x_2$ iff (1.6) holds (with $(x,x') = (x_1,x_2)$).

The above developments tell us that,for each x in $M_0^*(g)$, the associated (via (1H5)) point x' belongs to $M_0^*(g)$ and

(1.7) $x \preceq x', \quad x \neq x'$;

hence, no element of $M_0^*(g)$ can be maximal (modulo $\preceq$). On the other hand, by the same way as in T u r i n i c i [20], one may deduce

(1.8) $\begin{cases} \text{each } (\preceq) - \text{ascending sequence in } M_0^*(g) \\ \text{is a Cauchy one, bounded from above (modulo } (\preceq)). \end{cases}$

So, by the author's maximality principle [19], we have that, for the starting point $x_0 \in M_0^*(g)$, there must be a maximal (modulo $(\preceq)$) point $z \in M_0^*(g)$ with $x_0 \preceq z$. The obtained facts are therefore contradictory. Hence (1H6) cannot be true. The proof is complete. **q.e.d.**

Some remarks are in order. The second part of the normality property (1D3) is directly obtainable from

(1H7) $(0 \leq) \limsup_{s \to t+} a(s) < t, \quad t > 0$;

cf. M a t k o w s k i [12]. Further, given such an object, assume that the function $t \vdash b(t)$ from $[0, \infty[$ to itself satisfies

(1H8) $t \vdash \dfrac{b(t)}{t - a(t)}$ is decreasing and $G(t) = \displaystyle\int_0^t \frac{b(s)}{s - a(s)} ds < \infty, \quad t > 0.$

Then, (b, G) is compatible (modulo a). In particular, when a is linear (cf. (1D4)), and

(1H9) $(\leq) = X^2$ (the *trivial* quasi–order on X),

Theorem 1.1 above is nothing but the solvability result in Cramer and Ray [6]. For other aspects, we refer to Altman [1, ch. 5, Sect.2].

2. Conical surjective operators. The information contained in Theorem 1.1 is particularly relevant in the normed case. So, assume the metric d over X is generated by a *norm* $\|\cdot\|$ and the quasi-order $(\leq)$ on X is deductible – in the standard way – from a *cone* K; i.e., a subset of X with

(2D1) $K + K \subseteq K, \quad \lambda K \subseteq K, \quad \lambda \geq 0.$

These are asssumed to satisfy

(2H1) X is complete and K is closed.

(Note that,in such a case, (1H1)+(1H2) are necessarily holding). Also, let $(Y, \|\cdot\|)$ be a normed space. Take a map $T : X \to Y$ and a subset M of X with

(2H2) T is continuous (on X) and M is closed.

(As before, (1H3)+(1H4) are true in this case). Finally, we need the combined assumption

(2H3) $\begin{cases} M \text{ is } \textit{locally } K-\textit{invariant} \text{ (for each } x \in M, h \in K, \text{ the subset} \\ M(x, h) = \{\tau \in]0, 1]; x + \tau h \in M\} \text{ has 0 as accumulation point).} \end{cases}$

For the applications to be established, it will sufice giving the normal function $t \vdash a(t)$ the simplest (linear) form; and the compatible (modulo a) couple of functions (b, g) be taken in a similar way:

(2D2) $b(t) = \beta t$ (hence $g(t) = \dfrac{\beta}{1 - \alpha} t$), $t \geq 0$, for some $\beta > 0$.

Now, a natural choice for the point x' appearing in (1H5) is (under (2H3))

(2D3) $x' = x + \varepsilon h, \;$ for some $h \in K$.

This immediately gives the following local statement (under the precised framework).

Theorem 2.1. *Suppose that the point $z \in M$ is such that, a couple of constants $\alpha \in]0,1]$ and $\beta > 0$ may be found so as: for each x in $M[z; K, (\beta/(1-\alpha))\|Tz\|\,] = (z+K) \cap M[z, (\beta/(1-\alpha))\|Tz\|\,]$, the conditions*

$$\text{(2H4)} \quad \|\varepsilon^{-1}(T(x+\varepsilon h) - Tx)) + Tx\| \leq \alpha \|Tx\|, \ \|h\| \leq \beta \|Tx\|$$

are satisfied, for some $h \in K$ and some ε in $M(x,h)$. Then, Tx=0, for at least one x in $M[z; K, \dfrac{\beta}{1-\alpha}\|Tz\|\ ]$.

Now, the most appropriate setting for the discussion of (2H4) is the differential one. This will necessitate some conventions. Let $L(X,Y)$ stand for the class of all bounded linear operators from X to Y, endowed with the usual norm

$$\|A\| = \sup\{\|Ax\|; \|x\| \leq 1\}, \quad A \in L(X,Y).$$

Given such an object, A, we term it *K–surjective* in case $A(K) = Y$. Denote

$$\text{(2D4)} \quad L^s(X,Y;K) = \{A \in L(X,Y); A \text{ is } K-\text{surjective}\}.$$

For the arbitrary fixed A in $L^s(X,Y;K)$, the multivalued map

$$(A_K)^{-1}(y) = \{x \in K; Ax = y\}, \quad y \in Y$$

is a *closed convex process* (under the convention in Robinson [18]). So, by the main result in that paper, its associated norm

$$\|(A_K)^{-1}\| = \sup\{\inf\{\|x\|; x \in (A_K^{-1})(y)\}; \|y\| = 1\}$$

is well defined, as an element of $]0,\infty[$. Denote

$$\text{(2D5)} \quad \gamma_K(A) = \|(A_K)^{-1}\|, \quad A \in L^s(X,Y;K).$$

So, by definition,

$$\text{(2.1)} \quad \begin{cases} \gamma_K(A) = \inf\{\mu > 0; \text{ for each } y \in Y \text{ there exists} \\ x \in K \text{ with } Ax = y, \|x\| \leq \mu\|y\|\}. \end{cases}$$

Remember that T is called *Gâteaux differentiable* at $x \in X$ when there exists an element $x \vdash T'(x)(h)$ in $L(X,Y)$ (called the *Gâteaux derivative* of T at x) with

$$\text{(2D6)} \quad \begin{cases} \Lambda_T(x;\tau,h) = \|\tau^{-1}(T(x+\tau h) - Tx) - T'(x)(h)\| \to 0 \\ \text{as } \tau \to 0+, \text{ for any (direction) } h. \end{cases}$$

Now, assume in the following

(2H5) T is Gâteaux differentiable over X,

as well as (for some $x_0 \in M, \rho_0 > 0$),

$$\text{(2H6)} \quad \begin{cases} T'(x) \in L^s(X,Y;K), x \in M(x_0;K,\rho_0) = (x_0+K) \cap M(x_0,\rho_0) \\ \text{and } \beta_0 = \sup\{\gamma_K(T'(x)); x \in M(x_0;K,\rho_0)\} < \infty. \end{cases}$$

Denote, for $\rho \in]0,\rho_0[,\ \beta > \beta_0,\ \alpha \in]0,1[$,

$$\text{(2D7)} \quad M_T(x_0,K,\rho;\beta,\alpha) = \{z \in M(x_0;K,\rho); \frac{\beta}{1-\alpha}\|Tz\| + \|z-x_0\| < \rho\}.$$

Assume in the following

(2H7) $M_T(x_0,K,\rho;\beta,\alpha) \neq \emptyset$, for some (ρ,β,α) as above,

and take a certain point z in this subset (of M). Let x be arbitrary fixed in

$$M[z;K,(\beta/(1-\alpha))\|Tz\|\,] \subseteq M(x_0;K,\rho).$$

By (2H6) and the choice of β, there exists $h \in K$ with

$$T'(x)(h) = -Tx, \quad \|h\| \leq \beta\|Tx\|.$$

This gives the second half of (2H4) as well as – by (2H3) and the definition of the Gâteaux derivative – the first half of the same (whenever $\varepsilon \in M(x,h)$ is small enough). We therefore proved

Theorem 2.2. *Suppose the triplet (ρ,β,α) is taken as in* (2H7) *and let z be some point in $M_T(x_0,K,\rho;\beta,\alpha)$. Then, the operator equation* (OE) *has at least one solution in $M[z;K,(\beta/(1-\alpha))\|Tz\|\,] \subseteq M(x_0;K,\rho)$.*

As a rule, the nonemptiness condition (2H7) is difficult to be checked in terms of (ρ,β,α). The only hope is to assure its validity via $\|Tz\|$ being small enough; i.e., z being an *almost* solution to (OE).(For a distinct approach, we refer to Section 3 below).It remains now to say what happens with (2H6). Technically speaking, the possibility of handling it is related to

i) the *algebraic characterization* of those members in $L(X,Y)$ which belong to $L^s(X,Y;K)$

ii) the *continuity properties* of the map $A \vdash \gamma_K(A)$ from $L^s(X,Y;K)$ to $]0,\infty[$.

Concerning the first of these, the following is to be noted. Assume $(Y, \|\cdot\|)$ is to be written as the product structure of two (non–degenerate) normed spaces $(Y_1, \|\cdot\|)$, $(Y_2, \|\cdot\|)$:

(2H8) $Y = Y_1 \times Y_2$, $\|(y_1, y_2)\| = \max(\|y_1\|, \|y_2\|)$, $y_i \in Y_i, i = 1, 2$.

Then, necessarily, $L(X, Y) = L(X, Y_1) \times L(X, Y_2)$; i.e., each $A \in L(X, Y)$ is representable as

$$A = (A_1, A_2), \quad \text{where} \quad A_i \in L(X, Y_i), \ i = 1, 2.$$

A natural question is to characterize the K–surjectivity of this operator in terms of its components. To this end, put

$$X(A) = \{x \in X; Ax = 0\} \text{ (the } \textit{nullspace} \text{ of } A).$$

We then have (cf. T u r i n i c i [20]):

Proposition 2.1. *The bounded linear operator $A = (A_1, A_2)$ is K–surjective, iff*

(2H9) *A_1 is K-surjective, $K \cap X(A_1) \neq \{0\}$, and A_2 is $K \cap X(A_1)$–surjective.*

In particular, if

(2H10) $Y_2 = R^m$, *for some* $m \geq 1$,

we have

$$A_2 = (A_2^1, ..., A_2^m), \text{ where } A_2^i \in X^*, \ 1 \leq i \leq m;$$

and the $K \cap X(A_1)$–surjectivity of A_2 means

$$(2\mathrm{H}9') \quad \begin{cases} \textit{for each } i \in \{1, ..., m\}, \ K \cap X(A_1) \cap (\cap_{r \neq i} X(A_2^r)) \textit{ has a} \\ \textit{nonempty intersection with both } X^{(+)}(A_i) \textit{ and } X^{(-)}(A_i). \end{cases}$$

Here, by convention, for each $x^* \in X^*$,

(2D8) $X^{(+)}(x^*) = \{x \in X; x^*(x) > 0\}$, $X^{(-)}(x^*) = -X^{(+)}(x^*)$.

In particular, if $K = X$, this condition may be simply written as (cf. T u r i n i c i [op. cit.])

(2H9″) $\{A_2^1, ..., A_2^m\}$ are independent over $X(A_1)$.

We are now passing to the second of the questions above. The following intermediate statement is to be noted (again under (2H1)–(2H3)):

Theorem 2.3. *Suppose the point $z \in M$ is such that, a K–surjective bounded linear operator $A : X \to Y$, a $\eta > \gamma_K(A)$ and a $\delta > 0$ with $\delta^{-1} > \eta$ may be found so as: for each x in $M_z = M[z; K, (\eta/(1-\delta\eta))\|Tz\|\,]$ and each $h \in K$, the condition*

$$\text{(2H4')} \quad \|\varepsilon^{-1}(T(x+\varepsilon h) - T(x)) - Ah\| \le \delta\|h\|$$

is fulfilled, for some ε in $M(x_0, h)$. Then, $Tx = 0$ for at least one x in M_z.

Proof. Let $x \in M_z$ be arbitrary fixed. By the definition of $\gamma_K(A)$ and the choice of η

$$Ah = -Tx, \ \|h\| \le \eta\|Tx\|, \text{ for some } h \in K.$$

But then, (2H4) follows at once, with $\alpha = \delta\eta$, $\beta = \eta$. This, along with Theorem 2.1, ends the argument. **q.e.d.**

As a direct consequence, we have the following useful fact.

Theorem 2.4. *Suppose that the point $x_0 \in M$, the number $\rho > 0$, the K–surjective bounded linear operator $A : X \to Y$ and the number $\delta > 0$ with $\delta^{-1} > \gamma_K(A)$ were taken so as: for each x in $M_0 = M(x_0; K, \rho)$ and each $h \in K$, (2H4′) is holding for some ε in $M(x, h)$. Then, for each y with $\|Tx_0 - y\| < \rho(\gamma_K(A)^{-1} - \delta)$, the operator equation $Tx = y$ has a solution in M_0.*

Proof. Let y be as in the statement. For $\eta > \gamma_K(A)$ sufficiently close $\gamma_K(A)$, we have

$$\|Tx_0 - y\| < \rho(\eta^{-1} - \delta); \text{ or, equivalently, } (\eta/(1-\delta\eta))\|T_y(x_0)\| < \rho,$$

where

$$T_y(x) = T(x) - y, \ y \in X \ \ (\text{the } y - -\textit{translate} \text{ of T}).$$

Hence, all conditions of Theorem 2.3 are fulfilled (with T_y in place of T). And, in such a case,

$$\text{(2.3)} \quad \begin{cases} T_y(x) = 0(\text{ i.e., } T(x) = y), \text{ for at least one } x \text{ in} \\ M[x_0; K, (\eta/(1-\delta\eta))\|T_y(x_0)\|\,] \subseteq M(x_0; K, \rho). \end{cases}$$

This ends the argument. **q.e.d.**

Now, suppose that, in addition,

(2H11) T is linear (hence $T \in L(X,Y)$).

Then, condition (2H4′) becomes

(2H4″) $\|Th - Ah\| \le \delta\|h\|, \quad h \in K$.

And, with $M = K$, $x_0 = 0$, conclusion of the statement above may be rephrased as

$$(2.4)\quad \begin{cases} \text{for each } y \text{ with } \|y\| < \gamma_K(A)^{-1} - \delta, \text{ the operator} \\ \text{equation } Tx = y \text{ has a solution } x \in K \text{ with } \|x\| < 1. \end{cases}$$

This firstly means that T is K–surjective too; and, secondly, that (cf. (2.1))

$$\gamma_K(T) \le \gamma_K(A)/(1 - \delta\gamma_K(A)).$$

Moreover, if $\delta > 0$ is taken so small that $\delta^{-1} > 2\gamma_K(A)$ then (by this evaluation)

$$\gamma_K(T) \le 2\gamma_K(A) \quad (\text{hence } \delta^{-1} > \gamma_K(T)).$$

So, the above statement is applicable (with T and A interchanged) to get

$$\gamma_K(A) \le \gamma_K(T)/(1 - \delta\gamma_K(T)).$$

Summing up, we derived

Theorem 2.5. *The subclass $L^s(X,Y;K)$ is open in $L(X,Y)$ and the map $A \vdash \gamma_K(A)$ is cotinuous from $L^s(X,Y;K)$ to $]0,\infty[$. Precisely, given $A \in L^s(X,Y;K)$ then*

$$(2.5)\quad \begin{cases} \textit{each } B \in L(X,Y) \textit{ with } \|B - A\| < 1/\gamma_K(A) \\ \textit{is in } L^s(X,Y;K) \textit{ too; and} \end{cases}$$

(2.6) $\gamma_K(B) \le \gamma_K(A)/(1 - \|B - A\|\gamma_K(A))$.

With these informations at hand, let us now return to the context of Theorem 2.2. Denote, for each $z \in X$, $\rho > 0$,

(2D9) $\Delta_T(z,\rho) = \sup\{\|T'(x) - T'(z)\|; x \in X(z,\rho)\}$.

This will be referred to as the *sup–global oscillation* of $x \vdash T'(x)$ over $X(z,\rho)$. Note that, by Theorem 2.5, the assumption

(2H12) $T'(z) \in L^s(X,Y,;K)$ and $\Delta_T(z,\rho) < 1/\gamma_K(T'(z))$

gives immediately

(2.5′) $T'(w) \in L^s(X, Y, ; K), \quad w \in X(z, \rho)$

(2.6′) $\sup\{\gamma_K(T'(w)); w \in X(z, \rho)\} \le \gamma_K(T'(z))/(1 - \Delta_T(z, \rho)\gamma_K(T'(z)))$.

Denote further, for each $x \in X$,

(2D10) $\Delta_T(z; 0+) = \inf\{\Delta_T(z, \rho); \rho > 0\}$;

it will be termed the *sup–local oscillation* of $x \vdash T'(x)$ at z. Finally, when

(2D11) $T'(z) \in L^s(X, Y; K)$ and $\Delta_T(z; 0+) < 1/\gamma_K(T'(z))$,

the point in question will be referred to as a $K - regular$ one (for T').

With these conventions, the following practical consequence of Theorem 2.2 is to be noted.

Theorem 2.6. *Suppose the K–regular point $x_0 \in M$ is such that, for a sufficiently small $\rho > 0$,*

(2H13) $\|Tx_0\|\gamma_K(T'(x_0))/(1 - \Delta_T(x_0; \rho)\gamma_K(T'(x_0))) < \rho$

Then, $Tx = 0$, for at least one $x \in M(x_0; K, \rho)$.

In particular, the regularity condition imposed upon x_0 is trivially fulfilled under

(2H14) $T'(x_0) \in L^s(X, Y; K)$ and $\Delta_T(x_0; 0+) = 0$.

Note that, the second part of this condition says, in particular, that T is *Fréchet differentiable* at x_0, in the sense

(2D12) $\dfrac{1}{\|h\|}(T(x_0 + h) - T(x_0) - T'(x_0)(h)) \to 0$ as $h \to 0$.

This follows at once from the Denjoy–Bourbaki *mean value theorem* (see, for instance, D i e u d o n n é [8, ch. 8, Section 5]). On the other hand, a basic choice for M so that (2H3) be fulfilled is

(2H3’) $\begin{cases} M \text{ is } \textit{algebraically } K - \textit{open} \text{ (for each x} \in \text{M, h} \in \text{K,} \\ M(x, h) \text{ includes an interval starting from 0).} \end{cases}$

Note that, in this case, we have (cf. I l i o i [10, ch. 1, Sect.1])

(2.7) $M = X, \quad$ whenever $K = X$;

and the results above may be viewed as direct extensions of the ones in T u r i n i c i [20]. For a number of historical aspects see N a s h e d [14] and the references therein.

3. Nonsmooth implicit functions. As already precised, the specific assumption (2H3) imposed upon (M, K) is essential for the above developments. So, it is natural asking of to what extent are the solvability results above retainable in the absence of such a condition. It is our aim in the following to provide an appropriate answer in the trivial quasi–ordering $(K = X)$ setting. The general case will be discussed elsewhere.

We start with the following local mapping statement derived from Theorem 1.1. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be a couple of Banach spaces. Take a continuous mapping $x \vdash Tx$ from X to Y and a closed part M of X.

Theorem 3.1. *Suppose the point $z \in M$ is such that, a couple of constants $\alpha \in]0, 1[$ and $\beta > 0$ may be found so as: for each x in $M_z = M[z, (\beta/(1-\alpha))\|Tz\|\,]$, the conditions*

$$\text{(3H1)}\quad \|\varepsilon^{-1}(Tx' - Tx) + Tx\| \le \alpha\|Tx\|,\ \|\varepsilon^{-1}(x' - x)\| \le \beta\|Tx\|.$$

are satisfied for some ε in $]0, 1]$ and some x' in M. Then, (OE) *has at least one solution in M_z.*

Now, it would be necessary determining the appropriate setting in which (3H1) be fulfilled. This will necessitate a new convention. Denote, for each $z \in M$,

$$\text{(3D1)}\quad C(z; M) = \{w \in X; \lim_{\substack{M \ni x \to z \\ \tau \to 0+}} \tau^{-1} d(x + \tau w, M) = 0\}.$$

This will be referred to as the *Clarke tangent cone* [5] of M at z. Note that, by a result in P e n o t [16]

$$\text{(3.1)}\quad C(z; M) = \{w \in X; \limsup_{\substack{x \to z \\ \tau \to 0+}} \tau^{-1}(d(x + \tau w, M) - d(x, M)) \le 0\}.$$

The used terminology is, technically, appropriate, as results from

Proposition 3.1. *Under the precised setting,*

(3.2) $C(z; M)$ *is a closed cone* (*in* X).

We refer to the quoted paper for details; see also W a r d [22].

Now, let $(X, \|\cdot\|)$ be a Banach space and R^p (for some $p > 1$) the *standard* p–dimensional space endowed with e.g., the usual maximum norm. Take any continuous and Gâteaux differentiable map T from X to R^p and let M be a closed part of X. Fix $x_0 \in M$ with

(3H3) $\begin{cases} Tx_0 = 0 \text{ and } x_0 \text{is strongly } K-\text{regular} \\ (T'(x_0) \in L^s(X, R^p; K) \text{ and } \Delta_T(x_0; 0+) < 1/2\gamma_K(T'(x_0))), \\ \text{where } K = C(x_0, M). \end{cases}$

Hence, for some $\rho_0 > 0$ (fixed in the sequel)

(3.5) $\Delta_T(x_0, \rho_0) < 1/2\gamma_K(T'(x_0)) < 1/\gamma_K(T'(x_0))$.

Denote, for simplicity

(3D2) $\beta_0 = \gamma_K(T'(x_0))/(1 - \Delta_T(x_0, \rho_0)\gamma_K(T'(x_0)))$,
$\alpha_0 = (1 + \beta_0\Delta_T(x_0, \rho_0))/2$.

The first inequality in (3.5) gives

$$\beta_0\Delta_T(x_0, \rho_0) < 1 \quad (\text{hence } \beta_0\Delta_T(x_0, \rho_0) < \alpha_0 < 1).$$

And, the relation between the extreme terms of the same yields (by the developments in the preceding section)

(3.6) $T'(x) \in L^s(X, Y; K),\ x \in X(x_0, \rho_0)$,

(3.7) $\sup\{\gamma_K(T'(x)); x \in X(x_0, \rho_0)\} \leq \beta_0$.

Fix a couple (β, α) with

(3H4) $\beta > \beta_0,\ \alpha_0 < \alpha < 1,\ \beta\Delta_T(x_0, \rho_0) < \alpha_0$.

For each ρ in $]0, \rho_0[$ define

(3D3) $M_T(x_0, \rho; \beta, \alpha) = \{z \in M(x_0, \rho); \dfrac{\beta}{1-\alpha}\|Tz\| + \|z - x_0\| < \rho\}$.

All these sets are nonempty, by (3H3) (the first half); precisely

(3.8) $x_0 \in M_T(x_0, \rho; \beta, \alpha)$, for each ρ as before.

We are now in position to give an appropriate answer to the posed question.

Theorem 3.2. *Let the hypotheses above be admitted. Then, a* σ_0 *in* $]0, \rho_0[$ *may be determined so as: for each* σ *in* $]0, \sigma_0[$ *and each* z *in* $M_T(x_0, \sigma; \beta, \alpha)$*, the operator equation* (OE) *has at least a solution in*

$$M[z, (\beta/(1-\alpha))\|Tz\|\,] \subseteq M(x_0, \sigma).$$

Proof. Let λ be arbitrary fixed in $]0, 1[$ and put

$$\eta = \min\{\beta - \beta_0, \lambda(\alpha - \alpha_0)/\|T'(x_0)\|, (1-\lambda)(\alpha - \alpha_0)\}.$$

We start from the evident fact

(3.9) $R^p_{(1)} = \{y \in R^p; \|y\| = 1\}$ is compact (in R^p).

So, for this η, there must be a finite part $\{v_i; i \in I\}$ in $R^p_{(1)}$ with

(3.10) for each $y \in R^p_{(1)}$ there exists $i \in I$ with $\|v_i + y\| \leq \eta$.

Further, by (3.6)+(3.7), there may be determined a finite part $\{u_i; i \in I\}$ of $K = C(x_0, M)$ with

(3.11) $T'(x_0)(u_i) = v_i$ and $\|u_i\| \leq \beta_0\|v_i\| = \beta_0, \quad i \in I$.

As $\{u_i; i \in I\}$ are elements of $C(x_0, M)$, it results that for η above there must be some σ_0 in $]0, \rho_0[$ and $\varepsilon_0 > 0$ with

(3.12) $i \in I, x \in M(x_0, \sigma_0),\ 0 < \tau < \varepsilon_0 \Rightarrow d(x + \tau u_i, M) < \tau\eta$.

In addition, ε_0 may be taken so small that

(3H5) $\varepsilon_0 < (\rho_0 - \sigma_0)/\beta$.

It may now be proved, in a standard way, that σ_0 is our desired element. Hence the conclusion. **q.e.d.**

The following aspect is to be noted. Precisely, whenever

(3H6) $x_0 \in M[z, \dfrac{\beta}{1-\alpha}\|Tz\|\,]$ (i.e., $\|z - x_0\| \leq \dfrac{\beta}{1-\alpha}\|Tz\|$)

the solution of (OE) claimed in Theorem 3.2 is just x_0. So, it is legitimate to ask of to what extent are these sets identical with $\{x_0\}$. For an appropriate answer, we need a convention. Let the closed part M of X be given. Denote, for each $z \in M$,

(3D4) $B(z; M) = \{w \in X; \lim\limits_{\tau \to 0+} d(z + \tau w, M) = 0\}$.

This (cf. U r s e s c u [21]) will be referred to as the *Bouligand tangent cone* of M at z. Its relationships with the Clarke tangent cone are characterized as

(3.13) $\liminf\limits_{M \ni x \to z} B(x; M) \subseteq C(z; M) \subseteq B(z; M), \quad z \in M$;

see, for instance, P e n o t [op.cit.]. As a matter of fact, the used convention is not entirely appropriate; because

(3.14) $B(z; M)$ is closed and positively homogeneous.

In other words, $B(z;M)$ is not in general additive (hence a cone). It, however, has such a property whenever M itself is convex. Precisely,

(3.15) M=convex $\Longrightarrow B(z;M)=C(z;M), \quad z\in M;$

we refer to C l a r k e [op. cit.] for details.

Now, let X,Y be a couple of Banach spaces and $T:X\to Y$, a continuous Gâteaux differentiable map. Take a closed part M of X and some point $x_0\in M$ with the property

$$\text{(3H3}'\text{)}\quad \begin{cases} Tx_0=0 \text{ and } \Delta_T(x_0;0+)<\infty \text{ (hence } \Delta_T(x_0,\rho_0)<\infty \\ \text{for some } \rho_0>0 \text{ (fixed in the sequel)).} \end{cases}$$

Theorem 3.3. *Let the nonzero direction $h\in X$ be such that*

(3H7) $h\in B(x_0;M), \quad T'(x_0)(h)=0.$

Then, for each σ_0 in $]0,\rho_0[$ and each η in $]0,1[$ there exists $\theta=\theta(\sigma_0,\eta)$ in $]0,\sigma_0/3\|h\|\,[$ and a mapping $\tau\vdash z(\tau)$ from $[0,\theta]$ to $M(x_0,\sigma_0)$ with

$$(3.16)\quad \begin{cases} z(0)=x_0, \|z(\tau)-x_0-\tau h\|<\tau\eta\|h\| \text{ (hence} \\ 0<\|z(\tau)-x_0\|<2\tau\|h\|) \text{ and } \|Tz(\tau)\|<\tau\eta\|h\|,\ 0\le\tau\le\theta. \end{cases}$$

Consequently, for each $\beta>0$ and each α in $]0,1[$

(3.17) $z(\tau)\in M(x_0,3\tau\|h\|;\beta,\alpha),\ 0<\tau\le\theta,$ *whenever* $\eta<(1-\alpha)/\beta.$

Proof. By (3H7) (the first half) there may be determined some $\theta=\theta(\sigma_0,\eta)$ in $]0,\sigma_0/3\|h\|\,[$ and a mapping $\tau\vdash z(\tau)$ from $[0,\theta]$ to M with the properties (3.16) (the first part) and

(3.16′) $\|z(\tau)-x_0-\tau h\|<\tau\eta\|h\|/2(1+\Delta_T(x_0,\rho_0)+\|T'(x_0)\|),\ 0<\tau\le\theta.$

As a consequence, the second part of (3.16) follows at once. This gives

$$z(\tau)\ne x_0,\ 0<\tau\le\theta \text{ (because } 0<\eta<1)$$

as well as (cf. the choice of θ)

$$\|z(\tau)-x_0\|\le\|z(\tau)-x_0-\theta h\|+\theta\|h\|\le 2\tau\|h\|<\sigma_0,\ 0<\tau\le\theta.$$

Now (by diminishing eventually θ) one may assume that (cf. (3H7) above)

$$\text{(3H8)}\quad \begin{cases} \Lambda_T(x_0,\tau,h)<(\eta/2)\|h\| \text{ (hence} \\ \|T(x_0+\tau h)\|=\tau\Lambda_T(x_0,\tau,h)<(\tau\eta/2)\|h\|), \text{whenever } 0<\tau\le\theta. \end{cases}$$

On the other hand, a mean value argument gives (via (3.16′))

$$\begin{cases} \|T(z(\tau)) - T(x_0 + \tau h)\| \leq (\Delta_T(x_0, \sigma_0) + \|T'(x_0)\|)\|z(\tau) - x_0 - \tau h\| \\ < (\tau\eta/2)\|h\|, \quad 0 < \tau \leq \theta. \end{cases}$$

Combining these, gives

$$\begin{cases} \|T(z(\tau)\| \leq \|T(z(\tau)) - T(x_0 + \tau h)\| + \|T(x_0 + \tau h)\| \\ < \tau\eta\|h\|, \quad 0 < \tau \leq \theta. \end{cases}$$

and this proves the last part of (3.16). Finally, (3.17) is but a reformulation of this relation. The proof is complete. **q.e.d.**

We are now in position to state the basic result of this section. Let X, R^p, T, M be introduced as before. Fix some point $x_0 \in M$ according to (3H3); hence, for some $\rho_0 > 0$ as in (3.5) and (β_0, α_0) as in (3D2), the relations (3.6)+(3.7) are fulfilled. Let $m \geq 1$ be a positive integer and R^m stand for the usual m–dimensional space endowed, e.g., with the maximum norm. The subset

(3D5) $R^m_+ = \{z = (\zeta_1, ..., \zeta_m); \zeta_i \geq 0, 1 \leq i \leq m\}$

is a *pointed cone* of R^m; let $(\leq)$ stand for its associated ordering in R^m. On the other hand, its interior

(3D5′) $(R^m_+)^0 = \{z = (\zeta_1, ..., \zeta_m); \zeta_i > 0,\ 1 \leq i \leq m\}$

is not a cone. So, its associated relation $(<)$ is not an order; we, however, call it the *usual non–reflexive order* over R^m. Let $S = (S_1, ..., S_m)$ be a function from X to R^m with

(3H9) $S(x_0) = 0$ and S is Fréchet differentiable (on X).

Note that, by the definition of the involved notion, one has, for $1 \leq i \leq m$,

(3.18) $S_i(x) - S_i(x_0) = S'_i(x_0)(x - x_0) + E_i(x - x_0)\|x - x_0\|,\ x \in X$

where $S'(x_0) = (S'_1(x_0), ..., S'_m(x_0))$ stands for the Fréchet derivative and $E = (E_1, ..., E_m)$ fulfils

(3.19) $E_i(u) \to 0$ as $u \to 0$, for $1 \leq i \leq m$.

Theorem 3.4. *Let the nonzero direction h and the number ε be such that*

(3H7') $h \in B(x_0; M), T'(x_0)(h) = 0, S'(x_0)(h) > 0$

(3H10) $0 < \varepsilon < \min\{S_i'(x_0)(h)/\|S_i'(x_0)\| \cdot \|h\|; 1 \leq i \leq m\}$

Then, there exists $\theta = \theta(h, \varepsilon) > 0$ and a mapping $\tau \vdash w(\tau)$ from $[0, \theta]$ to $M(x_0, \rho_0)$ with the properties

(3.20) $T(w(\tau)) = 0, \quad S(w(\tau)) \geq 0, \quad 0 \leq \tau \leq \theta$

(3.21) $w(0) = 0$ *and* $\|\tau^{-1}(w(\tau) - x_0) - h\| \leq \varepsilon \|h\|, \quad 0 < \tau \leq \theta.$

Proof. By the same way as in T u r i n i c i [20], it may be shown that the second part of (3.20) follows from the first part of the same, plus (3.21). Let the couple (β, α) be taken as in (3H4), and σ_0 in $]0, \rho_0[$ be that indicated by Theorem 3.2. Further, let η be a number fulfiling

(3H11) $0 < \eta < (\varepsilon/2) \min\{(1 - \alpha)/\beta, 1\}.$

By Theorem 3.3, there exists for these data some $\theta = \theta(\sigma_0, \eta)$ in $]0, \sigma_0/3\|h\|[$ and a mapping $\tau \vdash z(\tau)$ from $[0, \theta]$ to $M(x_0, \sigma_0)$ fulfiling (3.16)+(3.17) above. This, along with Theorem 3.2, shows that one may determine a map $\tau \vdash w(\tau)$ from $[0, \theta]$ to $M(x_0, \sigma_0)$ so as (3.20) (the first part) be fulfilled, as well as

(3.22) $w(0) = x_0, \ \|w(\tau) - z(\tau)\| \leq \dfrac{\beta}{1 - \alpha} \|T(z(\tau)\|, \quad 0 < \tau \leq \theta.$

Combining with the choice of η yields

$$(3.23) \quad \begin{cases} \|w(\tau) - x_0 - \tau h\| \leq \|w(\tau) - z(\tau)\| + \|z(\tau) - x_0 - \tau h\| \\ \leq \dfrac{\beta}{1 - \alpha} \|T(z(\tau))\| + \tau\eta\|h\| \leq 2\tau\eta\|h\| \leq \tau\varepsilon\|h\|, \quad 0 < \tau \leq \theta; \end{cases}$$

and so, (3.21) follows at once. This completes the argument. **q.e.d.**

We therefore deduced that – under the notations of the introductory part–for each initial point x_0 in M_{TS}, each nonzero direction $h \in B(x_0; M)$ and each prescribed accuracy ε taken as before, there may be constructed a *path* $\tau \vdash w(\tau)$ from $[0, \theta]$ to M_{TS} with the properties (3.21). This will be referred to as an *implicit function* over the restriction set M_{TS}. In particular, when

(3H12) $M = X$ (hence $C(x_0; M) = B(x_0; M) = X$)

the result above reduces to the one in T u r i n i c i [op. cit.] proved via similar methods; see also A l t m a n [1].

4. A variant of the Farkas-Minkowski lemma. We are now passing to the algebraic part of our developments. Let X be a Banach space and X^*, its topological dual. The couple (X, X^*) is to be viewed as a *dual system* with respect to the *bilinear functional*

(4D1) $< x, x^* >= x^*(x), \quad x \in X,\ x^* \in X^*.$

Denote, as usually,

(4D2) $\begin{cases} \sigma(X, X^*) \text{ (resp., } \sigma(X^*, X)) = \text{ the weak (resp., weak}^*) \\ \text{topology over } X (\text{resp., } X^*) \end{cases}$

These are coarser than the norm topology on X (resp., X^*). Moreover, an identity between these is to be reached iff X is finite dimensional; see P r e c u p a n u [17, ch. 7, Sect. 1] for details.

Now, let us define the *polar* of any (nonempty) set M of X as

(4D3) $M^0 = \{x^* \in X^*; x^*(x) \geq -1,\ x \in M\}.$

Clearly, M^0 is convex, weak*–closed and contains the origin. By convention, any subset of X^* with these properties will be referred to as *admissible*. When M is supposed to satisfy additional conditions, the information about its polar may be completed as follows.

i) Let $X_{(1)}$ and $X^*_{(1)}$ stand for the unit sphere in X and X^*, respectively.

It is not hard to see that

(4.1) $(X_{(1)})^0 = X^*_{(1)};$

and so, $X^*_{(1)}$ is an admissible set in X^*. In fact, we have the stronger property

$$X^*_{(1)} \text{ is weak}^* - \text{compact in } X^*;$$

see, for instance, B a r b u and P r e c u p a n u [3, ch. 1, Sect. 2].

ii) Let K be a cone in X. We have

(4.2) $K^0 = \{x^* \in X^*; x^*(x) \geq 0, \text{for all } x \in K\}.$

Hence, K^0 is a weak*–closed cone in X^* (called the *dual cone* of K). An interesting particular case is (cf. the notations in Section 1)

$$K = X^{[+]}(x^*)(= X^{(+)}(x^*) \cup X(x^*)),\ \text{when } x^* \in X^*.$$

Precisely, one has (cf. C r i s t e s c u [7, ch. 3, Sect. 3])

(4.3) $(X^{[+]}(x^*))^0 = \mathrm{cone}\,(x^*)$.

(Here, "cone" stands for the *conical covering*).

iii) Let P be a subspace of X. Then

(4.4) $P^0 = \{x^* \in X^*; x^*(x) = 0, \text{for all } x \in P\}$.

Hence, P^0 is a weak*–closed subspace of X^* (referred to as the *annihilator* of P). The following particular case is to be noted. Let Y be another Banach space, and A be an element of $L(X, Y)$ which is, in addition, surjective. Then (cf. the notations in Section 2)

(4.5) $(X(A))^0 = A^*(Y^*)(= Y^* \circ A)$,

where A^* is the *adjoint* of A. To verify this, see, e.g., I o f f e and T i k h o m i r o v [11, ch. 0, Sect. 1]; we do not give details.

Returning to the general framework, the following properties of the polar operator must be recorded (cf. P r e c u p a n u [op. cit., ch. 6, Sect. 2]):

iv) (The Bipolar Theorem). We have

(4.6) $M^{00} = \mathrm{cl}\ \mathrm{cone}\,(M \cup \{0\}),\ M \subseteq X$.

Note that, as a consequence,

(4.6′) $K^{00} = K$, whenever K is a closed cone of X.

And this gives (in the precised context)

(4.6″) $h \in X, x^*(h) \geq 0$ for all $x^* \in K^0 \Rightarrow h \in K$.

v) (The Polar Intersection Property). For each finite system $\{P_1,..., P_n\}$ of subsets in X,

(4.7) $(P_1 \cap ... \cap P_n)^0 = w^*\mathrm{cl}\,(\mathrm{conv}\,(P_1^0 \cup ... \cup P_n^0))$.

Hence, in particular, for each finite system $\{K_1, ..., K_n\}$ of cones in X

(4.7′) $(K_1 \cap ... \cap K_n)^0 = w^*\mathrm{cl}\,(K_1^0 + ... + K_n^0)$.

An interesting problem to be discussed is to what extent can we delete the weak*–closure operator in this relation. For example, we have, for each system $\{x_1^*, ..., x_n^*\}$ in X^*,

(4.8) $(X^{[+]}(x_1^*) \cap ... \cap X^{[+]}(x_n^*))^0 = \mathrm{cone}\,\{x_1^*, ..., x_n^*\}$.

Some other aspects are to be derived from the result below. Given a cone M^* of X^*, any part M_1^* of M^* with

(4D4) $0 \notin M_1^*$ and $M^* = \cup\{\lambda M_1^*; \lambda \geq 0\}$

is called a *sole* of M. Further, given a linear subspace N^* of X^*, denote by $(N^*)_C$ its *quotient map*

(4D5) $(N^*)_C(x^*) = x^* + N^*, \quad x^* \in X^*;$

and, by $X^*|N^*$, the associated quotient space.

Proposition 4.1. *Let the couple P^*, Q^* of (nonempty) parts of X^* be such that, one of the conditions below is holding*

(4H1) *P^*, Q^* are admissible and Q^* is weak*–compact*

(4H2) $\begin{cases} P^*, Q^* \text{ are weak}^* - \text{closed cones, } Q^* \text{ has a} \\ \text{weak}^* - \text{compact sole and } P^* \cap (-Q^*) = \{0\} \end{cases}$

(4H3) $\begin{cases} P^* \text{ is a weak}^* - \text{closed subspace of } X^* \text{ and} \\ (P^*)_C(Q^*) \text{ is closed in } X^*|P^*. \end{cases}$

Then, $P^* + Q^*$ is weak*–closed in X^*.

The proof is based on the Dieudonné closedness criterion; see, for instance, P r e c u p a n u [op. cit., ch. 5, Sect. 1].

We are now in position to formulate the basic statement of this section. Let (X, X^*) be a dual system and P, some closed cone in X.

Proposition 4.2. *Suppose that the bounded linear functional z^* and the positive number ε are such that*

(4H4) $z^*(h) \geq -\varepsilon\|h\|, \quad h \in P.$

Then, necessarily,

(4.9) $z^* \in P^0 + \varepsilon X^*_{(1)};$ *hence, in particular,*

(4.10) $\|z^* - w^*\| \leq \varepsilon,$ *for some w^* in P^0.*

The proof uses a well known representation formula in B a r b u and P r e c u p a n u [op. cit., ch. 1, Sect. 1]; we do not give details. An interesting particular case of this statement is the following. Let Y be a Banach space and $A : X \to Y$, a bounded linear operator which is, in addition, surjective. Fix a closed cone K in X and take a finite system $\{x_1^*, ..., x_m^*\}$ of bounded linear functionals over X. The set

$$P = K \cap X(A) \cap X^{[+]}(x_1^*) \cap ... \cap X^{[+]}(x_m^*)$$

is a closed cone in X. Hence, conclusion of Proposition 4.2 is applicable to these data. It tells us that, if the bounded linear functional z^* and the positive number ε are taken so as

$$\text{(4H4}'\text{)}\quad \begin{cases} \text{whenever } h \in K \text{ fulfils } A(h) = 0 \text{ and} \\ x_i^*(h) \geq 0,\ 1 \leq i \leq m,\ \text{then } z^*(h) \geq -\varepsilon\|h\|, \end{cases}$$

we necessarily have a relation like

$$\text{(4.10)}\quad z^* \in w^*\mathrm{cl}\,(K^0 + Y^* \circ A + \mathrm{cone}\,\{x_1^*, ..., x_m^*\}) + \varepsilon X^*_{(1)}.$$

The weak*–closure operator may be deleted here whenever

(4H5) $K^0 + Y^* \circ A + \mathrm{cone}\,\{x_1^*, ..., x_m^*\}$ is weak*–closed.

For example, this always happens (cf. Proposition 4.1) when $K = X$. Note that, in this case, the statement above reduces to the one in T u r i n i c i [20]. In particular, when

(4H6) $Y = R^p$ (hence $A = (A_1, ..., A_p)$, where $A_i \in X^*,\ 1 \leq i \leq p$)

one clearly has

$$\text{(4.11)}\quad Y^* \circ A = \mathrm{lin}\,\{A_1, ..., A_p\}$$

(where "lin" means the *linear (space) covering*); and the underlying statement reduces to the one in E k e l a n d [9]. For other aspects we refer to Z a l i n e s c u [23].

5. Main result (almost minimum case). Let us now return to the (nonlinear programming) problem $(P_{\min})$ of the introductory part. It is our objective in the following to get, by the developments in the previous sections, a lot of sufficient conditions for the existence of "almost" solutions to the underlying problem, with "good" variational properties. To this end, we have to specify the working conditions.

Concerning the triplet (X, Y, Z) of normed spaces encountered there, assume

(5H1) X is a Banach space and $Y = R^p, Z = R^m$, for some $p, m \geq 1$.

Further, the conditions for (M, Q) are

(5H2) M is a closed part of X and $Q = R^m_+$.

Finally, assume that the mappings $T = (T_1, ..., T_p) : X \to R^p$ and $S = (S_1, ..., S_m) : X \to R^m$ appearing in that context are taken in such a way that

(5H3) T is continuous and Gâteaux differentiable over X.

(5H4) S is Fréchet differentiable (hence continuous) on X.

To describe the next assumption,we need some conventions. Denote, for each $x \in M$

(5D1) $I(x) = \{i \in \{1, ..., m\}; S_i(x) = 0\} \quad , I^*(x) = \{1, ..., m\} \setminus I(x)$.

(These are nothing but the set of all *active* and, respectively, *inactive* ranks at x. Clearly, the case of $I(x) = \emptyset$ or $I^*(x) = \emptyset$ cannot be excluded). For each nonempty part J of $I(x)$, let T_J be the mapping from X to $Y_J = Y \times R^J$ defined as

(5D2) $T_J(x) = (Tx; (S_i(x); i \in J)), \quad x \in X$.

(Here, R^J stands for R^q where $q = \text{card}\,(J)$; and, the norm of Y_J is the maximum one). When $J = \emptyset$, we complete the above definition as

(5D2$'$) $T_\emptyset = T$ (hence $R^\emptyset = \{0\},\ Y_\emptyset = Y$).

The announced condition may now be written as

(5H5) $\begin{cases} \text{for each } x \text{ in } M_{TS} \text{ and each part } J \text{ of } I(x),\ T'_J(x) \in L^s(X, Y_J; K) \\ \text{and } \Delta_{T_J}(x; 0+) < 1/2\gamma_K(T'_J(x)),\ \text{where } K = C(x; M). \end{cases}$

(Note that the sufficient conditions for the surjectivity portion of this hypothesis were precised in Section 2 above). On the other hand, it is clear that, in the second part of (5H5), γ_K refers to the couple of spaces (X, Y_J), which depends on the current point x). An immediate consequence of these facts is the following. Let $v \in M_{TS}$ be arbitrary fixed and suppose that the nonzero element $h \in X$ fulfils

(5H6) $h \in C(v; M),\ T'(v)(h) = 0,\ S'_i(v)(h) \geq 0,\ i \in I(v)$.

Denote, for simplicity,

(5D3) $I_h(v) = \{i \in I(v); S'_i(v)(h) = 0\}, I^*_h(v) = I(v) \setminus I_h(v)$.

The conditions above may now be written as

(5H6') $T'_{(h,v)}(v)(h) = 0, \quad S'_j(v)(h) > 0,\ j \in I^*_h(v)$;

where, by convention, $T_{(h,v)}$ stands for T_J with $J = I_h(v)$. Moreover, by (5H5), v appears as a strongly K–regular point of the mapping $T_{(h,v)}$, where $K = C(v; M)$. So, by Theorem 3.4., we have promised, for the sufficiently small $\eta > 0$, a $\theta = \theta(h, \eta) > 0$ and a mapping $\tau \vdash x(\tau)$ from $[0, \theta]$ to M with

(5.1) $T_{(h,v)}(x(\tau)) = 0,\ S_j(x(\tau)) \geq 0,\ j \in I^*_h(v), 0 \leq \tau \leq \theta$

(5.2) $x(0) = v$ and $\|\tau^{-1}(x(\tau) - v) - h\| \leq \eta\|h\|,\ 0 \leq \tau \leq \theta.$

The first part of (5.1) gives

$$T(x(\tau)) = 0,\ S_i(x(\tau)) = 0,\ i \in I_h(v),\ 0 \leq \tau \leq \theta.$$

Combining with

$$S_k(v) > 0,\ \text{for all } k \in I^*(v),$$

it follows, via (5H4) that – by eventually diminishing θ – we necessarily have

(5.3) $S_k(x(\tau)) > 0,\ k \in I^*(v),\quad 0 \leq \tau \leq \theta.$

In other words, the above precised path $\tau \vdash x(\tau)$ remains in M_{TS} and fulfils (5.2).

Finally, concerning the objective function $F : X \to R$, the basic non–differential assumption is

(5H7) $\inf F(M_{TS}) > -\infty$ and F is lower semicontinuous on X.

For the differential one, we need a convention. Let u be arbitrary fixed in X. Any bounded linear functional $F'(u)$ over X for which one has a relation like

(5D4) $F(u + h) - F(u) \leq F'(u)(h) + E(h)\|h\|,\quad h \in X,$

where the map $E = E(u)$ from X to R_+ fulfils

(5D5) $E(h) \to 0,\quad$ as $h \to 0,$

will be referred to as an *upper Fréchet differential* for F at u. Denote

(5D6) $\mathcal{F}'(u)$=the family of all upper Fréchet differentials for F at u.

We say that F is *upper Fréchet differentiable* at u when $\mathcal{F}'(u) \neq \emptyset$. The announced condition may now be written as

(5H8) F is upper Fréchet differentiable on X.

It is to be noted that the second part of (5H7) is not implied by (5H8). But, it is, evidently, a consequence of its stronger counterpart

(5H8') F is Fréchet differentiable on X.

For, in such a situation, one has (from definition)

(5.4) F is continuous on X.

We are now in position to state our main result. Let the above conditions be accepted. Denote, for each $\varepsilon > 0$, (5D7) $M_{TS}(\varepsilon) = \{x \in M_{TS}; F(x) \le \inf F(M_{TS}) + \varepsilon\}$.

Each point of this (nonempty) set will be referred to as an *ε–approximate solution* to the minimization problem ($P_{\min}$). Clearly, the process of determining $M_{TS}(\varepsilon)$ is not differential in nature. So, it is legitimate asking of which points of this set are variationally relevant. In this direction, we have

Theorem 5.1. *Suppose u is an arbitrary fixed point in $M_{TS}(\varepsilon)$. Then, there may be determined some point $v = v(\varepsilon, u)$ in $M_{TS}(\varepsilon)$ with the properties*

(5.4) $\|u - v\| \le \varepsilon^{1/2}, F(u) \ge F(v)$

(5.5) $\begin{cases} \mathcal{F}'(v) \subseteq w^*\text{cl } \left(C(v;M)^0 + \text{lin}\{T_j'(v); 1 \le j \le p\} + \right. \\ \left. \text{cone}\{S_i'(v); i \in I(v)\}\right) + \varepsilon^{1/2} X^*_{(1)}. \end{cases}$

In particular, if

(5H9) $\begin{cases} C(v;M)^0 + \text{lin}\{T_j'(v); 1 \le j \le p\} + \\ \text{cone}\{S_i'(v); i \in I(v)\} \text{ is weak}^* - -\text{closed}, \end{cases}$

we have that, for each $F'(v)$ in $\mathcal{F}'(v)$ there may be determined a bounded linear functional $v^* \in C(v;M)^0$, a vector $(\lambda_1, ..., \lambda_p)$ in R^p and a vector $(\xi_1, ..., \xi_m)$ in R^m with

(5.6) $\xi_i \ge 0, \quad \xi_i S_i(v) = 0, \quad 1 \le i \le m$

(5.7) $\|F'(v) - (v^* + \lambda_1 T_1'(v) + ... + \lambda_p T_p'(v) + \xi_1 S_1'(v) + ... + \xi_m S_m'(v))\| \le \varepsilon^{1/2}$.

Proof. Let us introduce an order ($\le$) on X by the convention

(5D8) $x_1 \le x_2$ iff $\varepsilon^{1/2}\|x_1 - x_2\| \le F(x_1) - F(x_2)$.

It is simply to verify, via (5H7), that

(5.8) $\begin{cases} \text{each ascending sequence in } M_{TS}(\varepsilon) \text{ is a Cauchy one,} \\ \text{bounded from above.} \end{cases}$

So, by the maximality principle in T u r i n i c i [19], there exists, for the starting point $u \in M_{TS}(\varepsilon)$, some point $v = v(\varepsilon, u)$ in $M_{TS}(\varepsilon)$ with the properties (5.4) and

(5.8) $F(w) - F(v) \ge -\varepsilon^{1/2}\|w - v\|, \quad w \in M_{TS}$.

Let $F'(v)$ be arbitrary fixed in $\mathcal{F}'(v)$ and suppose that the nonzero element $h \in X$ is such that (5H6) holds. By the developments above, for the arbitrarily small $\eta > 0$ there exists $\theta = \theta(h, \eta) > 0$ and a path $\tau \vdash x(\tau)$ in M_{TS} fulfiling (5.2). This gives (by the definition of $F'(v)$)

(5.9) $F'(v)(x(\tau)-v)+E(x(\tau)-v)\|x(\tau)-v\| \geq -\varepsilon^{1/2}\|x(\tau)-v\|,\ 0 \leq \tau \leq \theta,$

(where $x \vdash E(x)$ is introduced as in (5D1)). Combining with the relations (valid for $0 < \tau < \theta$)

$$\begin{cases} \|x(\tau)-v\| \leq \|x(\tau)-v-\tau h\| + \tau\|h\| \leq \tau(\eta+1)\|h\| \\ F'(v)(x(\tau)-v) \leq \tau(F'(v)(h) + \|F'(v)\|\eta\|h\|) \end{cases}$$

one gets (dividing by $\tau > 0$)

$$\text{(5.9')}\quad \begin{cases} F'(v)(h) + \varepsilon^{1/2}\|h\| + \eta(\|F'(v) + \varepsilon^{1/2}\|h\|) \geq \\ -(\eta+1)E(x(\tau)-v)\|h\|, \quad 0 < \tau \leq \theta. \end{cases}$$

So, taking the limit as $\tau \to 0$,

$$F'(v)(h) + \varepsilon^{1/2}\|h\| + \eta(\|F'(v)\| + \varepsilon^{1/2})\|h\| \geq 0.$$

But $\eta > 0$ was arbitrarily small; so, passing to limit as $\eta \to 0$, one gets

(5.10) $F'(v)(h) \geq -\varepsilon^{1/2}\|h\|$, for all such h.

Summing up, Proposition 4.2 is applicable to our data. And then, by its conclusion, (5.5) follows at once. The last part is clear. Hence the result. **q.e.d.**

Some remarks are in order. The specific assumptions (5H3)+(5H4) imposed upon T and S may be supposed to hold over a neighborhood of M_{TS} without modifying the argument. On the other hand, the (rather technical) condition (5H5) is necessarily holding under an (easy to verify) regularity condition like

$$\text{(5H5')}\quad \begin{cases} \text{for each } x \in M_{TS} \text{ and each part } J \text{ of } I(x), \text{ it is the case that} \\ T'_J(x) \in L^s(X,Y;K) \text{ and } \Delta_{T_J}(x;0+) = 0, \text{ where } K = C(x;M). \end{cases}$$

Finally, note that, when

(5H10) $M = X$ (hence $C(x;M) = X,\ x \in M$),

this result partially includes the one in T u r i n i c i [20]; see also A l t m a n [2].

6. Exact minimum points. From the developments above, it results that, for each $\varepsilon > 0$ there exists an $\varepsilon-$ *almost* solution to $(P_{\min})$ which satisfies the first order *almost necessary* conditions (5.5).It is natural to see what happens when $\varepsilon \to 0$; i.e., to what extent one may arrive to an *exact* solution of $(P_{\min})$. The reasonable way of answering this question is the Palais–Smale technique [15]. Fix $u^* \in X^*$ and put

$$\varphi(x^*) = \|u^* - x^*\|, \quad x^* \in X^*.$$

This function has convex weak*–compact sections. And then, for each convex weak*–closed part D^* of X^*, the subset

(6D1) $\operatorname{apr}(u^*, D^*) = \{x^* \in D^*; \varphi(x^*) = d(u^*, D^*)\}$

of all *best approximations* for u^* by elements of D^* is nonempty. Now, let us introduce the notation

(6D2) $\begin{cases} \mathcal{F}'_{TS}(x) = \operatorname{apr}(\mathcal{F}'(x); w^*\operatorname{cl}(C(x;M)^0 + \\ \operatorname{lin}\{T'_j(x); 1 \le j \le p\} + \operatorname{cone}\{S'_i(x); i \in I(x)\})), \quad x \in X. \end{cases}$

Here, $x \vdash \mathcal{F}'(x)$ is the upper Fréchet differential operator and (with D^* as before)

(6D3) $\operatorname{apr}(U^*, D^*) = \cup\{\operatorname{apr}(u^*, D^*); u^* \in U^*\},\ U^* \subseteq X^*.$

Let $x \vdash F'(x)$ and $x \vdash F'_{TS}(x)$ stand for a pair of selections from $x \vdash \mathcal{F}'(x)$ and $x \vdash \mathcal{F}'_{TS}(x)$, respectively; call these *compatible*, when

(6D4) $\begin{cases} F'_{TS}(x) \in \operatorname{apr}(F'(x); w^*\operatorname{cl}(C(x;M)^0 + \operatorname{lin}\{T'_j(x); 1 \le j \le p\} + \\ \operatorname{cone}\{S'_i(x); i \in I(x)\})), \quad x \in M_{TS}. \end{cases}$

Suppose we fixed such a couple. Call the map $x \vdash F'(x) - F'_{TS}(x)$, F–*admissible at the origin* if

(6D5) $\begin{cases} (v_n) \in M_{TS},\ F'(v_n) - F'_{TS}(v_n) \to 0 \text{ and } F(v_v) \text{ bounded} \\ \text{imply } (v_n) \text{ is (norm) relatively compact.} \end{cases}$

Let the general conditions of the preceding section be in force, as well as

(6H1) $\begin{cases} \text{a compatible pair of selections } (F', F'_{TS}) \text{ from } (\mathcal{F}', \mathcal{F}'_{TS}) \text{ exists} \\ \text{with } F' - F'_{TS} \text{ being } F - \text{admissible at the origin.} \end{cases}$

Theorem 6.1. *Under the precised hypotheses,* $(P_{\min})$ *has exact solutions; i.e.,* $\inf F(M_{TS})$ *is effectivelly attained* (*in* M_{TS}).

Proof. Let (ε_n) be a (strictly) descending sequence in $]0, \infty[$, converging to zero. By Theorem 5.1, there may be determined a sequence (v_n) in M_{TS} with

(6.1) $F(v_n) \le \inf F(M_{TS}) + \varepsilon_n, \quad n = 1, 2, ...,$

(6.2) $F'(v_n) - F'_{TS}(v_n) \leq \varepsilon_n^{1/2}, \quad n = 1, 2, ...,$

for each compatible pair of selections (F', F'_{TS}) from $(\mathcal{F}', \mathcal{F}'_{TS})$. In particular, this must be valid for that pair appearing in (6H1). But then, from the very definition of the introduced concept, (v_n) has a subsequence (still denoted in this way) converging to some $v \in M_{TS}$. It suffices now taking into account the lower semicontinuity of the mapping F to deduce, via (6.1), that v is the exact solution to $(P_{\min})$. The proof is complete. **q.e.d.**

In particular, when M fulfils (5H10), the result above reduces to the one in Turinici [20]. For a number of related facts we refer to Motreanu [13] or Barbu and Seidman [4].

The following completion of this fact is to be noted. Put

(6D6) $M_{TS}(0) = \{x \in M_{TS};\ F(x) = \inf F(M_{TS})\};$

each point of this is just an *exact solution* to $(P_{\min})$. Assume now

(6H2) $M_{TS}(0)$ is nonempty.

It is natural to ask of which points in $M_{TS}(0)$ are variationally relevant. An appropriate abswer to this is given by

Theorem 6.2. *Suppose v is arbitrary fixed in $M_{TS}(0)$. Then, necessarily,*

(6.3) $\mathcal{F}'(v) \subseteq w^*\mathrm{cl}\,(C(v;M)^0 + \mathrm{lin}\,\{T'_j(v); 1 \leq j \leq p\} +$
$+\mathrm{cone}\,\{S'_i(v); i \in I(v)\}).$

In particular, if (5H9) *is accepted then, for each $F'(v)$ in $\mathcal{F}'(v)$ there may be determined a bounded linear functional $v^* \in C(v;M)^0$, a vector $(\lambda_1, ..., \lambda_p)$ in R^p and a vector $(\xi_1, ..., \xi_m)$ in R^m with the properties* (5.6) *and*

(6.4) $F'(v) = v^* + \lambda_1 T'_1(v) + ... + \lambda_p T'_p(v) + \xi_1 S'_1(v) + ... + \xi_m S'_m(v).$

Proof. Let v be as in the statement. We have, by definition,

(6.5) $F(w) - F(v) \geq 0, \quad w \in M_{TS}.$

Let $F'(v)$ be arbitrary fixed in $\mathcal{F}'(v)$ and suppose the nonzero element $h \in K$ fulfils (5H6). By the same argument as in Theorem 5.1 one gets

(6.6) $F'(v)(h) \geq 0,$ for all such h.

This, combined with Proposition 4.2 (which also applies to the case $\varepsilon = 0$) gives at once (6.3). Hence the conclusion. **q.e.d.**

As above said, the supplementary condition (5H9) holds under (5H10). Note that, in such a case, this result is comparable to the one in T u r i n i c i [op. cit.]. It also improves a similar statement in this area due to A l t m a n [2]; see also B a r b u and P r e c u p a n u [3, ch.3, Sect.1]. For other aspects, we refer to Z e i d l e r [24, ch.38, Sect.10].

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