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Tomul XLIV, s.I.a, Matematică, 1998, f1

CLASSICAL SOLUTIONS TO NONLINEAR HYPERBOLIC FUNCTIONAL PARTIAL DIFFERENTIAL EQUATIONS

BY

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Abstract: Local existence of classical (of class C^1) solutions to the Cauchy problem for nonlinear functional partial differential equations of first order are investigated. The proof is based on the bicharacteristics and successive approximations methods.

1. Introduction. Let $B=[-b_0,0]\times[-b,b]$ where $b=(b_1,...,b_n)\in\mathbb{R}^n_+$, $\mathbb{R}_+=[0,\infty)$, and $b_0\in\mathbb{R}_+$. For any function $z:[-b_0,a]\times\mathbb{R}^n\to\mathbb{R}$ with $a>0$ and a fixed $(x,y)=(x,y_1,...,y_n)\in[0,a]\times\mathbb{R}^n$ we define the function $z_{(x,y)}:B\to\mathbb{R}$ by $z_{(x,y)}(t,s)=z(x+t,y+s)$, $(t,s)\in B$. We denote by $C^0(B,R)$ the set of all real continuous functions on B. We put $E_0=[-b_0,0]\times\mathbb{R}^n$.

Suppose that $f:[0,a]\times\mathbb{R}^n\times C^0(B,R)\times\mathbb{R}^n\to\mathbb{R}$, $\varphi:E_0\to\mathbb{R}$, and let us consider the Cauchy problem

$$D_xz(x,y)=f(x,y,z_{(x,y)},D_yz(x,y)), \tag{1}$$

$$z(x,y)=\varphi(x,y),\quad (x,y)\in E_0, \tag{2}$$

where $D_yz=(D_{y_1},...,D_{y_n})$.

In this paper we extend the method of quasilinerization to problem (1), (2) in which we construct two uniformly convergent sequences: the first converges to a solution of the problem, the second to its derivative. The technique essentially involves the method of bicharacteristics.

Classical solutions to nonlinear partial differential–functional equations of first order have been studied by P.B r a n d i, R.C e p p i t e l– l i [1],[2], D. J a r u s z e w s k a–W a l c z a k [4], Z. K a m o n t [5]–[7], A. S a l v a d or i [8], J. S z ar s k i [9] and K. Z i m a [11]. They used different methods of the proofs. Generalized (or weak) solutions (i.e. solutions which are absolutely continuous and satisfy a differential equation almost everywhere) of problem (1), (2) were investigated by P. B r a n d i, Z. K a m o n t, A. S a l v a d o r i [3].

2. Preliminaries. For $y \in \mathbb{R}^n$ we write $\|y\| = \sum_{i=1}^n |y_i|$. Let $\|\cdot\|_0$ denote the supremum norm in $C^0(B, R)$. We will denote by $C^{0,L}(B, R)$ the class of all functions $w \in C^0(B, R)$ such that

$$\|w\|_L = \sup\{|w(t,s) - w(t,\bar{s})| \cdot \|s - \bar{s}\|^{-1} : \ (t,s), \ (t,\bar{s}) \in B\} < \infty.$$

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For $w \in C^{0,L}(B, R)$ we put $\|w\|_{0,L} = \|w\|_0 + \|w\|_L$.

Let $C^1(B, R)$ be the set of all continuous functions $w : B \to \mathbb{R}$ such that the derivatives $D_s w = (D_{s_1}, ..., D_{s_n})$ exist and $D_s w \in C^0(B, R^n)$. For $w \in C^1(B, R)$ we put $\|w\|_1 = \|w\|_0 + \max\{\|D_s w(t,s)\| : \ (t,s) \in B\}$.

We denote by $C^{1,L}(B, R)$ the set of all functions $w \in C^1(B, R)$ such that $D_s w \in C^{0,L}(B, R)$.

For $w \in C^{1,L}(B, R)$ we write $\|w\|_{1,L} = \|w\|_1 + \|D_s w\|_L$.

Let Θ be the class of all nondecreasing functions $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$.

We put $\Omega^{(i)} = [0,a] \times \mathbb{R}^n \times C^i(B,R) \times \mathbb{R}^n$ for $i = 1,2$, $\Omega^{(1,L)} = [0,a] \times \mathbb{R}^n \times C^{1,L}(B,R) \times \mathbb{R}^n$, $E_t = [-b_0, t] \times \mathbb{R}^n$ and $E_t^* = [0,t] \times \mathbb{R}^n$ where $0 \le t \le a$. We denote by $\|\cdot\|_{C(E_t,R)}$, $\|\cdot\|_{C(E_t,R^n)}$ and $\|\cdot\|_{C(E_t^*,R^n)}$ the supremum norms in the spaces $C(E_t, R)$, $C(E_t, R^n)$ and $C(E_t^*, R^n)$, respectively.

Given $S = (S_0, S_1, S_2) \in \mathbb{R}_+^3$, we denote by $C^{1,L}[S]$ the set of all functions $\varphi : E_0 \to \mathbb{R}$ such that

(i) $\varphi \in C(E_0, R)$ and there exists $D_y\varphi(x,y)$ for $(x,y) \in E_0$;

(ii) $|\varphi(x,y)| \le S_0$, $\|D_y\varphi(x,y)\| \le S_1$ on E_0, and
$\|D_y\varphi(x,y) - D_y\varphi(x,\bar{y})\| \le S_2\|y - \bar{y}\|$, for $(x,y), (x,\bar{y}) \in E_0$.

Let $\varphi \in C^{1,L}[S]$ be given and let $0 < c \le a$, $P = (P_0, P_1)$, $Q = (Q_0, Q_1, Q_2)$, where $P_j, Q_i \in \mathbb{R}_+$, $Q_i \ge S_i$ for $j = 0,1$, $i = 0,1,2$. We denote by $C^{1,L}[c, \varphi, P, Q]$ the set of all functions $z \in C([-b_0, c] \times R^n, R)$ such that

(i) $z(x,y) = \varphi(x,y)$ for $(x,y) \in E_0$;

(ii) there exists $D_y z(x,y)$ for $(x,y) \in [0,c] \times \mathbb{R}^n$;

(iii) $|z(x,y)| \le Q_0$ and $\|D_y z(x,y)\| \le Q_1$ for $(x,y) \in [-b_0,c] \times \mathbb{R}^n$;

(iv) for $(x,y),(\bar{x},y),(\bar{x},\bar{y}) \in [0,c] \times \mathbb{R}^n$ we have

$$|z(x,y) - z(\bar{x},y)| \le P_0|x-\bar{x}|,$$
$$\|D_y z(x,y) - D_y z(\bar{x},\bar{y})\| \le P_1|x-\bar{x}| + Q_2\|y-\bar{y}\|.$$

Let $\tilde{Q} = (\tilde{Q_1}, \tilde{Q_2}),\ \tilde{P_1}, \tilde{Q_1}, \tilde{Q_2} \in \mathbb{R}_+$. We denote by $C^{0,L}[c,\tilde{P_1},\tilde{Q}]$ the class of all functions $v \in C([0,c] \times R^n, R^n)$ such that

(i) $\|v(x,y)\| \le \tilde{Q_1}$ for $(x,y) \in [0,c] \times \mathbb{R}^n$;

(ii) $\|v(x,y) - v(\bar{x},\bar{y})\| \le \tilde{P_1}|x-\bar{x}| + \tilde{Q_2}\|y-\bar{y}\|$ for $(x,y),(\bar{x},\bar{y}) \in [0,c] \times \mathbb{R}^n$.

Let $CL(B,R)$ denote the set of all linear continuous operators defined on $C(B,R)$ and taking values in $\mathbb{R}$. The norm in $CL(B,R)$ we denote by $\|\cdot\|_{CL(B,R)}$.

3. Assumptions and main results.

Assumption H_1. Suppose that

1^0 $f \in C^0(\Omega^{(0)}, \mathbb{R})$ and there is $\tilde{\gamma} \in \Theta$ such that $|f(x,y,w,q)| \le \tilde{\gamma}(\|w\|_0)$ on $\Omega^{(0)}$;

2^0 for every $(x,y,w,q) \in \Omega^{(1)}$ there exist $D_y f(x,y,w,q) \in \mathbb{R}^n$ and $D_w f(x,y,w,q) \in CL(B,R)$;

3^0 there are $\alpha_0,\ \beta_0 \in \Theta$ such that

$$\|D_y f(x,y,w,q)\| \le \alpha_0(\|w\|_1),\ \|D_w f(x,y,w,q)\|_{CL(B,R)} \le \beta_0(\|w\|_1),$$

on $\Omega^{(1)}$;

4^0 there are $\alpha_1,\ \beta_1 \in \Theta$ such that

$$\|D_y f(x,y,w,q) - D_y f(x,\bar{y},w+h,\bar{q})\| \le$$
$$\le \alpha_1(\|w\|_{1,L})[\|y-\bar{y}\| + \|h\|_1 + \|q-\bar{q}\|],$$

and

$$\|D_w f(x,y,w,q) - D_w f(x,\bar{y},w+h,\bar{q})\|_{CL(B,R)} \le$$
$$\le \beta_1(\|w\|_{1,L})[\|y-\bar{y}\| + \|h\|_1 + \|q-\bar{q}\|],$$

for $(x,y,w,q) \in \Omega^{(1,L)},\ \bar{y},\bar{q} \in \mathbb{R}^n,\ h \in C^1(B,R)$.

Assumption H_2. Suppose that

1^0 $D_q f \in C^0(\Omega^{(1)}, \mathbb{R}^n)$ and there is $\gamma_0 \in \Theta$ such that on $\Omega^{(1)}$ we have

$$\|D_q f(x,y,w,q)\| \le \gamma_0(\|w\|_1),$$

2^0 there is $\gamma_1 \in \Theta$ such that

$$\begin{aligned}&\|D_q f(x,y,w,q) - D_q f(x,\bar{y},w+h,\bar{q})\| \le \\ &\le \gamma_1(\|w\|_{1,L})[\|y-\bar{y}\| + \|h\|_1 + \|q-\bar{q}\|],\end{aligned}$$

for $(x,y,w,q) \in \Omega^{(1,L)},\ \bar{y},\bar{q} \in \mathbb{R}^n$ and $h \in C^1(B,R)$.

Let us choose constants P_0 and P_1 such that

$$\begin{aligned}&P_0 > S_1\gamma_0(Q')G + \tilde{\gamma}(Q_0) + \tilde{Q}_1\gamma_0(Q'),\\ &P_1 > S_2\gamma_0(Q')G + \alpha_0(Q') + \tilde{Q}_1\beta_0(Q')\end{aligned}$$

where

$$\begin{aligned}&G = \exp\{c\bar{Q}\gamma_1(Q^*)\}, &&Q' = Q_0 + Q_1,\\ &\overline{Q} = 1 + Q_1 + Q_2 + \tilde{Q}_2, &&Q^* = Q_0 + Q_1 + Q_2.\end{aligned}$$

Put

$$\begin{aligned}&H = \bar{Q}\alpha(Q^*) + \tilde{Q}_1\bar{Q}\beta_1(Q^*) + \tilde{Q}_2\beta_0(Q'),\\ &K = \alpha_0(Q') + Q_1\beta_0(Q') + 2\tilde{Q}_2\gamma_0(Q') + \tilde{Q}_1\bar{Q}\gamma_1(Q^*),\\ &M = (S_2 + cH)\gamma_1(Q^*)G + \alpha_1(Q^*) + \tilde{Q}_1\beta_1(Q^*),\end{aligned}$$

Assumption H_3. Suppose that

1^0 $\varphi \in C^{1,L}[S]$ and $Q_i > S_i,\ i = 0,1,2,$ and $P_1 = \tilde{P}_1,\ Q_k = \tilde{Q}_k,\ k = 1,2;$

2^0 the constant $c \in (0,a]$ is small enough to satisfy

$$\begin{aligned}&c\beta_0(Q') < 1,\ Mc < 1,\ [1 - c\beta_0(Q')]^{-1}[S_1 + c\alpha_0(Q')] \le \tilde{Q}_1,\\ &S_0 + c[\tilde{\gamma}(Q_0) + \tilde{Q}_1\gamma_0(Q')] \le Q_0,\\ &cK\gamma_0(Q')G \le P_0 - [S_1\gamma_0(Q')G + \tilde{\gamma}(Q_0) + \tilde{Q}_1\gamma_0(Q')],\\ &cH\gamma_0(Q')G \le P_1 - [S_2\gamma_0(Q')G + \alpha_0(Q') + \tilde{Q}_1\beta_0(Q')],\\ &(S_1 + cH)G \le Q_2.\end{aligned}$$

Theorem. *Let Assumptions H_1-H_3 hold. Then for every $\varphi \in C^{1,L}[S]$ there exists a solution $\bar{z} : [-b_0, c] \times \mathbb{R}^n \to \mathbb{R}$ of problem* (1), (2).

In order to prove the main result we need to prepare a number of lemmas.

4. Bicharacteristics. Suppose that $\varphi \in C^{1,L}[S]$, $z \in C^{1,L}[c,\varphi,P,Q]$, $u \in C^{0,L}[c,\tilde{P}_1,\tilde{Q}]$. We consider the Cauchy problem

$$\eta'(t) = -D_q f(t,\eta(t), z_{(t,\eta(t))}, u(t,\eta(t))), \quad \eta(x) = y, \tag{3}$$

and denote by $g[z,u](\cdot,x,y)$ its solution.

Lemma 1. *Let* $\varphi,\bar\varphi\in C^{1,L}[S]$, $z\in C^{1,L}[c,\varphi,P,Q]$, $\bar z\in C^{1,L}[c,\bar\varphi,P,Q]$, $u,\bar u\in C^{0,L}[c,\tilde P_1,\tilde Q]$, *be given, and let Assumption* H_2 *be satisfied. Then the solutions* $g[z,u](\cdot,x,y)$ *and* $g[\bar z,\bar u](\cdot,x,y)$ *of* (3) *exist on* $[0,c]$, *they are unique and we have estimates*

$$\|g[z,u](t;x,y)-g[z,u](t;\bar x,\bar y)\|\le G[\gamma_0(Q')|x-\bar x|+\|y-\bar y\|], \tag{4}$$

and

$$\begin{aligned}\|g[z,u](t;x,y)-g[\bar z,\bar u](t;x,y)\|\le \gamma_1(Q^*)G[\|z-\bar z\|_{C(E_t,R)}+\\ +\|D_yz-D_y\bar z\|_{C(E_t,R^n)}+\|u-\bar u\|_{C(E_t^*,R^n)}].\end{aligned} \tag{5}$$

Proof. The existence and uniqueness of the solution of (3) follows from classical theorems.

Now, we put

$$P[z,u](\tau;x,y)=(\tau,g[z,u](\tau;x,y),z_{(\tau,g[z,u](\tau;x,y))},u(\tau,g[z,u](\tau;x,y))).$$

It follows from Assumption H_2 and (3) that for $(t;x,y)\in[0,c]\times[0,c]\times\mathbb{R}^n$, $\bar y\in\mathbb{R}^n$ we get

$$\begin{aligned}&\|g[z,u](t;x,y)-g[z,u](t;x,\bar y)\|\le\\ &\le\|y-\bar y\|+|\textstyle\int_x^t\|D_qf(P[z,u](\tau;x,y))-D_qf(P[z,u](\tau;x,\bar y))\|d\tau|\le\\ &\le\|y-\bar y\|+\bar Q\gamma_1(Q^*)|\textstyle\int_x^t\|g[z,u](\tau;x,y)-g[z,u](\tau;x,\bar y)\|d\tau|.\end{aligned}$$

An application of the well–known Gronwall inequality implies

$$\|g[z,u](t;x,y)-g[z,u](t;x,\bar y)\|\le G\|y-\bar y\|.$$

For $t,x,\bar x\in[0,c],\ y\in\mathbb{R}^n$, we have

$$\begin{aligned}&\|g[z,u](t;x,y)-g[z,u](t;\bar x,y)\|\le|\int_x^{\bar x}\|D_qf(P[z,u](\tau;\bar x,y))\|d\tau+\\ &+|\int_x^t\|D_qf(P[z,u](\tau;x,y))-D_qf(P[z,u](\tau;\bar x,y))\|d\tau|\le\\ &\le\gamma_0(Q')|x-\bar x|+\bar Q\gamma_1(Q^*)|\int_x^t\|g[z,u](\tau;x,y)-g(\tau;\bar x,y)\|d\tau|.\end{aligned}$$

In view of the Gronwall inequality we see that

$$\|g[z,u](t;x,y)-g[z,u](t;\bar x,y)\|\le\gamma_0(Q')G|x-\bar x|.$$

So the required (4) follows by the above estimates. Moreover, for $z \in C^{1,L}[c,\varphi,P,Q]$, $\bar{z} \in C^{1,L}[c,\bar{\varphi},P,Q]$ and $u,\bar{u} \in C^{0,L}[c,\tilde{P}_1,\tilde{Q}]$ we have

$$\begin{aligned}&\|g[z,u](t;x,y)-g[\bar{z},\bar{u}](t;x,y)\| \le c\gamma_1(Q^*)[\|z-\bar{z}\|_{C(E_t,R)}+\\&+\|D_y z - D_y\bar{z}\|_{C(E_t,R^n)}+\|u-\bar{u}\|_{C(E_t^*,R^n)}]\\&+\gamma_1(Q^*)\bar{Q}|\int_x^t \|g[z,u](\tau;,x,y)-g[\bar{z},\bar{u}](\tau;x,y)\|d\tau|.\end{aligned}$$

From the Gronwall inequality we deduce (5). The proof of Lemma 1 is complete.

5. Successive approximations. For given

$$\varphi \in C^{1,L}[S],\ z \in C^{1,L}[c,\varphi,P,Q] \text{ and } u \in C^{0,L}[c,\tilde{P}_1,\tilde{Q}],$$

we define

$$\begin{aligned}F[z,u](x,y)=\varphi(0,g(0;x,y))+\int_0^x [f(P[z,u](\tau;x,y))-\\-\sum_{k=1}^n D_{q_k} f(P[z,u](\tau;x,y))u_k(\tau,g(\tau;x,y))]d\tau\end{aligned}$$

and

$$\begin{aligned}T_i[z,u](x,y)=D_{y_i}\varphi(0,g(0;x,y))+\int_0^x [D_{y_i} f(P[z,u](\tau;x,y))\\+D_w f(P[z,u](\tau;x,y))(u_i)_{(\tau,g(\tau;x,y))}]d\tau,\ i=1,...,n.\end{aligned}$$

Now we consider the system of integral–functional equations which are generated by (1) and (2):

$$\begin{gathered}z=F[z,u],\quad u=T[z,u],\quad z=\varphi \text{ on } E_0\\ g(t;x,y)=y+\int_t^x D_q f(P[z,u](\tau;x,y))d\tau.\end{gathered}$$

We construct successive approximations. Suppose that $\varphi \in C^{1,L}[S]$ and that Assumptions $H_1 - H_3$ are satisfied. We define the sequence $\{z^{(m)},u^{(m)}\}$ in the following way. We put first

$$\begin{aligned}&z^{(0)}(x,y)=\varphi(x,y) && \text{for } (x,y)\in E_0,\\&z^{(0)}(x,y)=\varphi(0,y) && \text{for } (x,y)\in[0,c]\times \mathbb{R}^n,\\&u^{(0)}(x,y)=D_y\varphi(0,y) && \text{for } (x,y)\in[0,c]\times \mathbb{R}^n.\end{aligned}$$

It is easy to see that $u^{(0)} \in C^{0,L}[c, \tilde{P}_1, \tilde{Q}]$.

Now, if $(z^{(m)}, u^{(m)}) \in C^{1,L}[c, \varphi, P, Q] \times C^{0,L}[c, \tilde{P}_1, \tilde{Q}]$ are known functions, then $u^{(m+1)}$ is a solution of the equations

$$u = T^{(m)}[u], \tag{6}$$

$$z^{(m+1)} = F^{(m)}[z^{(m)}, u^{(m+1)}], \tag{7}$$

where $T^{(m)} = (T_1^{(m)}, ..., T_n^{(m)})$ is defined by

$$T_i^{(m)}[u](x,y) = D_{y_i}\varphi(0, g(0; x, y)) + \int_0^x [D_{y_i} f(P[z^{(m)}, u](\tau; x, y)) \\ + D_w f(P[z^{(m)}, u](\tau; x, y))(u_i^{(m)})_{(\tau, g(\tau; x, y))}] d\tau,$$

$i = 1, ..., n$. We denote by g the solution of

$$g(t; x, y) = y + \int_t^x D_q f(P[z^{(m)}, u](\tau; x, y)) d\tau$$

and for simplicity we ignore the dependence of g on $z^{(m)}$ and u.

Lemma 2. *If Assumptions H_1–H_3 are satisfied, then for any $m \geq 0$ we have*

a_m) $\quad z^{(m)} \in C^{1,L}[c, \varphi, P, Q],\ u^{(m)} \in C^{0,L}[c, \tilde{P}_1, \tilde{Q}]$;

b_m) $\quad D_y z^{(m)}(x,y) = u^{(m)}(x,y)$ *in* $[0, c] \times \mathbb{R}^n$.

Proof. We will prove a_m) and b_m) by induction. It follows from the definition of $z^{(0)}$ and $u^{(0)}$ that a_0) and b_0) are satisfied. Suppose now that conditions a_m) and b_m) hold for given $m \geq 0$. First we will prove that

$$T^{(m)} : C^{0,L}[c, \tilde{P}_1, \tilde{Q}] \to C^{0,L}[c, \tilde{P}_1, \tilde{Q}]. \tag{8}$$

Indeed, it follows from Assumptions H_1–H_3 and Lemma 1 that for $u \in C^{0,L}[c, \tilde{P}_1, \tilde{Q}]$ and $(x,y), (\bar{x}, \bar{y}) \in [0, c] \times \mathbb{R}^n$, we have

$$\|T^{(m)}[u](x,y)\| \leq S_1 + c[\alpha_0(Q') + \tilde{Q}_1 \beta_0(Q')] \leq \tilde{Q}_1,$$

and

$$
\begin{aligned}
&\|T^{(m)}[u](x,y) - T^{(m)}[u](\bar{x},\bar{y})\| \leq \\
&\leq \|D_y\varphi(0,g(0;x,y)) - D_y\varphi(0,g(0;\bar{x},\bar{y}))\| + \\
&+ \int_0^x \|D_y f(P[z^{(m)},u](\tau;x,y)) - D_y f(P[z^{(m)},u](\tau;\bar{x},\bar{y}))\| d\tau + \\
&+ \int_x^{\bar{x}} \|D_y f(P[z^{(m)},u](\tau;\bar{x},\bar{y}))\| d\tau| + \\
&+| \int_x^{\bar{x}} \sum_{k=1}^n D_w f(P[z^{(m)},u](\tau;\bar{x},\bar{y}))(u_k^{(m)})_{(\tau,g(\tau;\bar{x},\bar{y}))} d\tau| + \\
&+ \int_0^x \sum_{k=1}^n |D_w f(P[z^{(m)},u](\tau;x,y))(u_k^{(m)})_{(\tau,g(\tau;x,y))} - \\
&-D_w f(P[z^{(m)},u](\tau;\bar{x},\bar{y}))(u_k^{(m)})_{(\tau,g(\tau;\bar{x},\bar{y}))}| d\tau \leq \\
&\leq (S_2 + cH)G[\gamma_0(Q')|x-\bar{x}| + \|y-\bar{y}\|][\alpha_0(Q') + \tilde{Q}_1\beta_0(Q')]|x-\bar{x}| \leq \\
&\leq \tilde{P}_1|x-\bar{x}| + \tilde{Q}_2\|y-\bar{y}\|.
\end{aligned}
$$

By the above inequality we obtain (8).

From Assumptions H_1–H_3, applying Lemma 1, we get

$$
\begin{aligned}
&\|T^{(m)}[u](x,y) - T^{(m)}[\bar{u}](x,y)\| \leq \\
&\leq \|D_y\varphi(0,g(0;x,y)) - D_y\varphi(0,\bar{g}(0;x,y))\| + \\
&+ \int_0^x \|D_y f(P[z^{(m)},u](\tau;x,y)) - D_y f(P[z^{(m)},\bar{u}](\tau;x,y))\| d\tau + \\
&+ \sum_{k=1}^n \int_0^x |D_w f(P[z^{(m)},u](\tau;x,y))(u_k^{(m)})_{(\tau,g(\tau;x,y))} - \\
&-D_w f(P[z^{(m)},\bar{u}](\tau;x,y))(\bar{u}_k^{(m)})_{(\tau,\bar{g}(\tau;x,y))}| d\tau \leq Mc\|u-\bar{u}\|_{C(E_c^*,R^n)}.
\end{aligned}
$$

Hence, by assumption 2^0 of H_3, using the Banach fixed point theorem we conclude that there is exactly one $u^{(m+1)} \in C^{0,L}[c,\tilde{P}_1,\tilde{Q}]$ satisfying (6).

Now we prove that the function $z^{(m+1)}$ given by (7) satisfies b_{m+1}.

Put

$$
\begin{aligned}
\Delta(x,y,\bar{y}) &= z^{(m+1)}(x,\bar{y}) - z^{(m+1)}(x,y) - \\
&-\sum_{k=1}^n u_k^{(m+1)}(x,y)(\bar{y}_k - y_k).
\end{aligned}
$$

Utilizing Hadamard mean value theorem we obtain

$$\Delta(x,y,\bar{y}) = \varphi(0,g(0;x,\bar{y})) - \varphi(0,g(0;x,y)) -$$
$$-\sum_{k=1}^{n} D_{y_k}\varphi(0,g(0;x,y))(\bar{y}_k - y_k)+$$
$$+\int_0^x \sum_{k=1}^{n} \int_0^1 D_{y_k} f(Q^{(m)}(s,\tau))ds[g_k(\tau;x,\bar{y}) - g_k(\tau;x,y)]d\tau+$$
$$+\int_0^x \int_0^1 D_w f(Q_{(m)}(s,\tau))[z^{(m)}_{\tau,g(\tau;x,\bar{y}))} - z^{(m)}_{\tau,g(\tau;x,y))}]d\tau+$$
$$+\int_0^x \sum_{k=1}^{n} \int_0^1 D_{q_k} f(Q^{(m)}(s,\tau)ds[u_k^{(m+1)}(\tau,g(\tau;x,\bar{y}))-$$
$$-u_k^{(m+1)}(\tau,g(\tau;x,y))]d\tau-$$
$$-\int_0^x \sum_{k=1}^{n}\{D_{q_k} f(P[z^{(m)},u^{(m+1)}](\tau;x,\bar{y}))u_k^{(m+1)}(\tau,g(\tau;x,\bar{y}))-$$
$$-D_{q_k} f(P[z^{(m)},u^{(m+1)}](\tau;x,y))u_k^{(m+1)}(\tau,g(\tau;x,y))\}d\tau-$$
$$-\int_0^x \sum_{k=1}^{n}\{D_{y_k} f(P[z^{(m)},u^{(m+1)}](\tau;x,y))+$$
$$+D_w f(P[z^{(m)},u^{(m+1)}](\tau;x,y))(u_k^{(m)})_{(\tau,g(\tau;x,y))}\}d\tau(\bar{y}_k - y_k),$$

where

$$\begin{aligned} Q^{(m)}(s,\tau) \quad &= \quad (\tau, g(\tau;x,y) + s[g(\tau;x,\bar{y}) - g(\tau;x,y)], z^{(m)}_{(\tau,g(\tau;x,y))}+ \\ &\quad +s[z^{(m)}_{(\tau,g(\tau;x,\bar{y}))} - z^{(m)}_{(\tau,g(\tau;x,y))}], u^{(m+1)}(\tau,g(\tau;x,y))+ \\ &\quad +s[u^{(m+1)}(\tau,g(\tau;x,\bar{y})) - u^{(m+1)}(\tau,g(\tau;x,y))]). \end{aligned}$$

It is easily to show that Δ can be written in the form

$$\Delta(x,y,\bar{y}) = \sum_{i=0}^{4} \Delta_i(x,y,\bar{y}) + \sum_{i=0}^{3} \bar{\Delta}_i(x,y,\bar{y}),$$

where

$$\Delta_0(x,y,\bar{y}) = \varphi(0,g(0;x,\bar{y})) - \varphi(0,g(0;x,y))-$$
$$-\sum_{k=1}^{n} D_{y_k}\varphi(0,g(0;x,y))[g_k(0;x,\bar{y}) - g_k(0;x,y)],$$
$$\Delta_1(x,y,\bar{y}) = \int_0^x \sum_{k=1}^{n} \int_0^1 D_{y_k} f(Q^{(m)}(s,\tau))-$$
$$-D_{y_k} f(P[z^{(m)},u^{(m+1)}](\tau;x,y))]ds[g_k(\tau;x,\bar{y}) - g_k(\tau;x,y)]d\tau,$$

$$\Delta_2(x,y,\bar{y}) = \int_0^x \int_0^1 [D_w f(Q_{(m)}(s,\tau))-$$
$$-D_w f(P[z^{(m)},u^{(m+1)}](\tau;x,y))]ds[z^{(m)}_{(\tau,g(\tau;x,\bar{y}))} - z^{(m)}_{(\tau,g(\tau;x,y))}]d\tau,$$

$$\Delta_3(x,y,\bar{y}) = \int_0^x \sum_{k=1}^n \int_0^1 [D_{q_k} f(Q^{(m)}(s,\tau)) - D_{q_k} f(P[z^{(m)},u^{(m+1)}]$$
$$\cdot(\tau;x,\bar{y}))]ds[u_k^{(m+1)}(\tau,g(\tau;x,\bar{y})) - u_k^{(m+1)}(\tau,g(\tau;x,y))]d\tau,$$

$$\Delta_4(x,y,\bar{y}) = \int_0^x D_w f(P[z^{(m)},u^{(m+1)}](\tau;x,y))[z^{(m)}_{(\tau,g(\tau;x,\bar{y}))} - z^{(m)}_{(\tau,g(\tau;x,y))}$$
$$-\sum_{k=1}^n (u_k^{(m)})_{(\tau,g(\tau;x,y))}(g_k(\tau;x,\bar{y}) - g_k(\tau;x,y))]d\tau,$$

and

$$\bar{\Delta}_0(x,y,\bar{y}) = \sum_{k=1}^n D_{y_k}\varphi(0,g(0;x,y))[g_k(0;x,\bar{y}) - g_k(0;x,y) - (\bar{y}_k - y_k)],$$

$$\bar{\Delta}_1(x,y,\bar{y}) = \int_0^x \sum_{k=1}^n \{D_{y_k} f(P[z^{(m)},u^{(m+1)}](\tau;x,y))[g_k(\tau;x,\bar{y})$$
$$-g_k(\tau;x,y) - (\bar{y}_k - y_k)]\}d\tau,$$

$$\bar{\Delta}_2(x,y,\bar{y}) = \int_0^x D_w f(P[z^{(m)},u^{(m+1)}](\tau;x,y))\{\sum_{k=1}^n (u_k^{(m)})_{(\tau,g(\tau;x,y))}\cdot$$
$$\cdot[g_k(\tau;x,\bar{y}) - g_k(\tau;x,y) - (\bar{y}_k - y_k)]\}d\tau,$$

$$\bar{\Delta}_3(x,y,\bar{y}) = \int_0^x \sum_{k=1}^n [D_{q_k} f(P[z^{(m)},u^{(m+1)}](\tau;x,\bar{y}))-$$
$$-D_{q_k} f(P[z^{(m)},u^{(m+1)}](\tau;x,y))]u_k^{(m+1)}(\tau,g(\tau;x,y))d\tau.$$

Similarly as in [3], we can check that

$$\sum_{i=0}^3 \bar{\Delta}_i(x,y,\bar{y}) = 0,\ \ (x,y),(x,\bar{y}) \in [0,c]\times\mathbb{R}^n.$$

Hence, we have

$$\Delta(x,y,\bar{y}) = \sum_{i=0}^4 \Delta_i(x,y,\bar{y}).$$

Assumptions H_1 and H_2 imply the estimate

$$|\Delta(x,y,\bar{y})|\le C^*\|y-\bar{y}\|^2,$$

where

$$C^* = S_2 + cG^2\{\beta_0(Q')Q_2 + [\alpha_1(Q^*) + \beta_1(Q^*)Q_1 + \gamma_1(Q^*)\tilde{Q}_2]\bar{Q}\}.$$

Consequently

$$D_y z^{(m+1)}(x,y)=u^{(m+1)}(x,y) \text{ for } (x,y)\in[0,c]\times\mathbb{R}^n.$$

It is easily to show that $z^{(m+1)} \in C^{1,L}[c,\varphi,P,Q]$.
The proof of Lemma 2 is complete.

7. Uniformly convergence. Put

$$\begin{array}{rcl} L_0 &=& [S_2 + cH + c\beta_0(Q')\tilde{Q}_2]\gamma_1(Q^*)G + c[\alpha_1(Q^*) + \tilde{Q}_1\beta_1(Q^*)], \\ L_1 &=& L_0 + \beta_0(Q'), \\ N_0 &=& (S_1 + cK)\gamma_1(Q^*) + \tilde{Q}_1(Q^*), \\ N_1 &=& [2\gamma_0(Q') + \tilde{Q}_1\gamma_1(Q^*)]\exp\{cL_0\}. \end{array}$$

Assumption H_4. Suppose that $c \in [0,a]$ is such small constant that

$$cL_1 exp\{cL_0\} < 1,\ c\{N_0 + cL_1[N_0 + 2\gamma(Q')]exp\{cL_0\} + \beta_0(Q')\} < 1.$$

Lemma 3. *If Assumptions H_1–H_4 are satisfied then the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ are uniformly convergent on $[0,c]\times\mathbb{R}^n$.*

Proof. For $t \in [0,c]$ and $m \ge 1$ we put

$$\begin{array}{l} Z^{(m)}(t) = sup\{|z^{(m)}(x,y) - z^{(m-1)}(x,y)| \ :\ (x,y) \in [0,t]\times\mathbb{R}^n\}, \\ U^{(m)}(t) = sup\{\|u^{(m)}(x,y) - u^{(m-1)}(x,y)\| \ :\ (x,y) \in [0,t]\times\mathbb{R}^n\}. \end{array}$$

According to (6) and Assumptions H_1, H_2, we have on $[0,c]\times\mathbb{R}^n$

$$\begin{array}{l} |u_i^{(m+1)}(x,y) - u_i^{(m)}(x,y)| \le \\ \le |D_{y_i}\varphi(0,g^{(m+1)}(0;x,y)) - D_{y_i}\varphi(0,g^{(m)}(0;x,y))| + \\ + \displaystyle\int_0^x |D_{y_i}f(P[z^{(m)},u^{(m+1)}](\tau;x,y)) - D_y f(P[z^{(m-1)},u^{(m)}](\tau;x,y))|d\tau + \\ + \displaystyle\int_0^x |D_w f(P[z^{(m)},u^{(m+1)}](\tau;x,y))(u_i^{(m)})_{(\tau,g^{(m+1)}(\tau;x,y))} - \\ - D_w f(P[z^{(m-1)},u^{(m)}](\tau;x,y))(u_i^{(m-1)})_{(\tau,g^{(m)}(\tau;x,y))}|d\tau \le \\ \le L_0 \int_0^x U^{(m+1)}(\tau)d\tau + c[L_0 Z^{(m)}(x) + L_1 U^{(m)}(x)]. \end{array}$$

An application of the Gronwall inequality implies

$$U^{(m+1)}(x) \le c[L_0 Z^{(m)}(x) + L_1 U^{(m)}]exp\{cL_0\}. \tag{9}$$

It follows from (7), (9) and Assumptions H_1, H_2, that we have for $(x,y) \in [0,c] \times \mathbb{R}^n$

$$\begin{aligned}
&|z^{(m+1)}(x,y) - z^{(m)}(x,y)| \le S_1 \|g^{(m+1)}(0;x,y) - g^{(m)}(0;x,y) + \\
&+ \int_0^x \{\alpha_0(Q')\|g^{(m+1)}(\tau;x,y) - g^{(m)}(\tau;x,y)\| + \\
&+\beta_0(Q')[Z^{(m)}(\tau) + Q_1\|g^{(m+1)}(\tau;x,y) - g^{(m)}(\tau;x,y)\|]+ \\
&+\gamma_0(Q')[U^{(m+1)}(\tau) + \tilde{Q}_1\|g^{(m+1)}(\tau;x,y) - g^{(m)}(\tau;x,y)\|\}d\tau + \\
&+ \int_0^x \{\tilde{Q}_1\gamma_1(Q^*)[Z^{(m)}(\tau) + U^{(m)}(\tau) + U^{(m+1)}(\tau) + \\
&+\bar{Q}\|g^{(m+1)}(\tau;x,y) - g^{(m)}(\tau;x,y)\|]+ \\
&+\gamma_0(Q')[U^{(m+1)}(\tau) + \tilde{Q}_2\|g^{(m+1)}(\tau;x,y) - g^{(m)}(\tau;x,y)\|\}d\tau \le \\
&\le c[N_0 + \beta_0(Q') + cL_0N_1]Z^{(m)}(x) + c[N_0 + cL_1N_1]U^{(m)}(x).
\end{aligned}$$

Hence, we get

$$Z^{(m+1)}(x) \le c[N_0+\beta_0(Q')+cL_0N_1]Z^{(m)}(x)+c[N_0+cL_1N_1]U^{(m)}(x). \tag{10}$$

From definitions of $z^{(0)}$ and $u^{(0)}$ and from Assumptions H_1, H_2 we have $Z^{(1)}(x) \le c[\tilde{\gamma}(Q_0) + \gamma_0(Q')\tilde{Q}_1]$, $U^{(1)}(x) \le c[\alpha_0(Q') + \beta_0(Q')\tilde{Q}_1]$.

By Assumption H_4 and by (9), (10), we conclude that there exists $\delta \in (0,1)$ such that

$$U^{(m+1)}(x) + Z^{(m+1)}(x) \le \delta[U^{(m)}(x) + Z^{(m)}(x)],\ x \in [0,c],\ m \ge 0.$$

From this we conclude the assertion of Lemma 3.

8. Proof of Theorem. From Lemma 3 we deduce that two functions $\bar{z} \in C^{1,L}[c,\varphi,P,Q]$ and $\bar{u} \in C^{0,L}[c,\tilde{P}_1,\tilde{Q}]$ exist such that $\{z^{(m)}\}$ converges to $\bar{z}$ and $\{u^{(m)}\}$ converges to $\bar{u}$ uniformly on $[0,c] \times \mathbb{R}^n$. Furthermore, we have that $D_y\bar{z}$ exists on $[0,c] \times \mathbb{R}^n$ and $D_y\bar{z} = \bar{u}$. Thus we obtain

$$\begin{aligned}
&\bar{z}(x,y) = \varphi(0,\bar{g}(0;x,y)) + \int_0^x [f(P[\bar{z},D_y\bar{z}](\tau;x,y)) - \\
&-\sum_{k=1}^n D_{q_k} f(P[\bar{z},D_y\bar{z}](\tau;x,y)) D_{y_k}\bar{z}(\tau;x,y))]d\tau,
\end{aligned} \tag{11}$$

where $\bar{g}(t;x,y) = y + \int_t^x D_q f(P[\bar{z}, D_y\bar{z}](\tau;x,y))d\tau$. Now we prove that $\bar{z}$ is a solution of (1). By taking $y=\bar{g}(x;0,\eta)$ in (11), we get

$$\begin{aligned}
&\bar{z}(x,\bar{g}(x;0,\eta)) = \varphi(0,\eta)+\\
&+\int_0^x [f(\tau,\bar{g}(x;0,\eta),\bar{z}_{(\tau,\bar{g}(x;0,\eta))},D_y\bar{z}(\tau,\bar{g}(x;0,\eta)))-\\
&-\sum_{k=1}^n D_{q_k} f(\tau,\bar{g}(x;0,\eta),\bar{z}_{(\tau,\bar{g}(x;0,\eta))},D_y\bar{z}(\tau,\bar{g}(x;0,\eta)))\cdot\\
&\cdot D_{y_k}\bar{z}(\tau,\bar{g}(x;0,\eta))]d\tau,
\end{aligned} \tag{12}$$

since the function $\bar{g}$ satisfies the group property $\bar{g}(t';t,\bar{g}(t;x,y)) = \bar{g}(t';x,y)$ (see Lemma 1). By differentiation of (12) with respect to x, and by putting again $y = \bar{g}(x;0,\eta)$, we obtain that $\bar{z}$ satisfies (1). It follows immediately that $\bar{z}$ satisfies (2). The proof of Theorem is complete.

9. Special cases of equation (1). We list below examples of equations which can be derived from equation (1).

1) Suppose that $\bar{f}: [0,a]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, $\alpha: [0,a]\times\mathbb{R}^n\times C^0(B,R)\to\mathbb{R}$, $\beta: [0,a]\times\mathbb{R}^n\times C^0(B,R)\to\mathbb{R}^n$ are given. Assume that

$$\begin{aligned}
&-b_0 \le \alpha(x,y,w) - x \le 0,\\
&-b \le \beta(x,y,w) - y \le b,\ \ (x,y,w)\in[0,a]\times\mathbb{R}^n\times C^0(B,R).
\end{aligned}$$

We consider the operator f defined by

$$f(x,y,w,q)=\bar{f}(x,y,w[\alpha(x,y,w)-x,\beta(x,y,w)-y],q),\ (x,y,w,q)\in\Omega^{(0)}.$$

In this case equation (1) reduces to the following differential equation with a retarded argument

$$\begin{aligned}
D_x z(x,y) &= \bar{f}(x,y,z(\alpha(x,y,z_{(x,y)}),\beta(x,y,z_{(x,y)})),D_y z(x,y)),\\
z(x,y) &= \varphi(x,y),\ \ (x,y)\in E_0.
\end{aligned} \tag{13}$$

Note that equation (13) cannot be obtained as a particular case from the equation considered in [3], because in that paper α can be depended only on the argument x.

2) Suppose that $\bar{f}: [0,a]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ is given function.
Let $f(x,y,w,q) = \bar{f}(x,y,\int_B w(t,s)dtds, q)$. Then equation (1) reduces to the differential–integral equation

$$\begin{aligned}
D_x z(x,y) &= \bar{f}(x,y,\int_{-b_0}^0\int_{-b}^b z(x+t,y+s)dtds, D_y z(x,y)),\\
z(x,y) &= \varphi(x,y),\quad (x,y)\in E_0.
\end{aligned}$$

3) If we take

$$f(x,y,w,q) = \bar{f}(x,y,(V(I_{(x,y)}w))(x,y)),$$

where $(I_{(x,y)}w)(t,s) = w(t-x,s-y)$, then equation (1) reduces to the differential–functional equation [10]

$$D_x z(x,y) = \bar{f}(x,y,(Vz)(x,y),D_y z(x,y)).$$

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Received: 1.X.1997

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