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MONOTONE FLOWS AND MIXED MONOTONE REACTION DIFFUSION SYSTEMS

BY

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Dedicated to Professor Constantin Corduneanu on the occasion of his 70th birthday

Abstract. We consider the reaction–diffusion systems which possesses the mixed quasimonotone property. Employing the coupled upper and lower solutions of the corresponding elliptic problem and the theory of monotone flows, we investigate the convergence of the solutions of the mixed monotone reaction–diffusion systems to the solutions of the corresponding elliptic systems.

1. Introduction. Consider the following weakly coupled parabolic system

$$\begin{array}{ll} u_t - Lu = g(x,u), & (t>0,\ x\in\Omega), \\ Bu = h(x), & (t>0,\ x\in\partial\Omega), \\ u(0,x) = u_0(x), & (x\in\Omega), \end{array} \tag{1.1}$$

and its corresponding elliptic system

$$\begin{array}{ll} -Lu = g(x,u), & (x\in\Omega), \\ Bu = h(x), & (x\in\partial\Omega), \end{array} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$. Here and throughout this paper, we assume that Lu is elliptic and boundary operator Bu is given by

$$L_i u_i = \sum_{j,k=1}^{N} a_{jk,i}(x)\ \frac{\partial^2 u_i}{\partial x_j \partial x_k} + \sum_{j=1}^{N} b_{j,i}(x)\ \frac{\partial u_i}{\partial x_j}$$

$$B_i u_i = \alpha_i(x)\frac{\partial u_i}{\partial \gamma} + \beta_i(x) u_i$$

for each $i = 1, 2, ..., n$, where $\dfrac{\partial}{\partial\gamma}$ denotes the outward normal derivative on $\partial\Omega$.

We also assume that Ω is the class $C^{2+\alpha}$ and for each i, the coefficients in L_i are Hölder continuous in $\overline{\Omega}$, $\beta_i \in C^{1+\alpha}(\partial\Omega)$, and either $\alpha_i(x) = 0$ and $\beta_i(x) = 1$ (Dirichlet condition), or $\alpha_i(x) = 1$ and $\beta_i(x) \geq 0$ (Neumann or Robin condition). The functions $g(x, u), h(x)$ and $u_0(x)$ are Hölder continuous in x in these respective domains and $u(x)$ satisfies the compatibility condition. The above assumptions guarantee the existence of a classical solution to (1.1). See [1], [4], [5] for details. In Section 2, we consider the reaction–diffusion system when $g(x, u)$ in (1.1) possesses quasimonotone nondecreasing property. Employing the upper and lower solutions of system (1.2) and the theory of monotone flows [2], [3] we prove the convergence of solutions of (1.1) to the solutions of system (1.2). In Section 3, we discuss the mixed monotone reaction–diffusion systems. We prove that the solutions of the mixed monotone reaction–diffusion systems converge to the solutions of the corresponding mixed monotone elliptic systems by using the idea of projection operator and expanding the given problem suitably in order to transform the problem to that of quasimonotone systems.

2. Monotone flows and quasimonotone systems. We shall begin with a result that offers convergence results for quasimonotone reaction–diffusion systems.

Theorem 2.1. *Assume that*

(A_0) $g \in C[\overline{\Omega}\times\mathbb{R}^n, \mathbb{R}^n]$, $g(x, u)$ *is Hölder continuous in* x, *is quasimonotone nondecreasing in* u *and*

$$|g(x, u_1) - g(x, u_2)| \leq L|u_1 - u_2|, \ L > 0;$$

(A_1) $\alpha_0, \beta_0 \in C^{2+\alpha}[\overline{\Omega}, \mathbb{R}^n]$, $\alpha_0(x) \leq \beta_0(x)$,

$$\begin{aligned} -L\alpha_0 &\leq g(x, \alpha_0), \quad (x \in \overline{\Omega}), \\ B\alpha_0 &\leq h(x), \qquad (x \in \partial\Omega), \end{aligned}$$

and

$$\begin{aligned} -L\beta_0 &\geq g(x, \beta_0), \quad (x \in \overline{\Omega}), \\ B\beta_0 &\geq h(x), \qquad (x \in \partial\Omega). \end{aligned}$$

Then

(i) $\alpha_0 \leq u(t, x, \alpha_0) \leq u(t, x, \beta_0) \leq \beta_0$, *where* $u(t, x, \alpha_0)$ *and* $u(t, x, \beta_0)$ *are solutions of* (1.1) *with* $u_0 = \alpha_0$ *and* $u_0 = \beta_0$ *respectively;*

(ii) *$u(t,x,\alpha_0)$ is nondecreasing in t and $u(t,x,\beta_0)$ is nondecreasing in t,*

$$\lim_{t\to\infty} u(t,x,\alpha_0) = \underline{u}(x), \quad \lim_{t\to\infty} u(t,x,\beta_0) = \bar{u}(x)$$

exist and $\underline{u}(x)$, $\bar{u}(x)$ are solutions of (1.2)*;*

(iii) *for any u_0 with $\alpha_0 \le u_0 \le \beta_0$, we have $\underline{u}(x) \le u^*(x) \le \bar{u}(x)$ on $\overline{\Omega}$, if*

$$\lim_{t\to\infty} u(t,x,u_0) = u^*(x) \quad \text{on} \quad \overline{\Omega}.$$

Proof. It is easy to see that $\alpha_0(x), \beta_0(x)$ are also lower and upper solutions of (1.1) provided $\alpha_0(x) \le u_0(x) \le \beta_0(x)$. Since $g(x,u)$ is Lipschitzian in u, it follows from [1] that there exists a unique solution $u(t,x)$ of (1.1) such that $\alpha_0(x) \le u(t,x,u_0) \le \beta_0(x)$ on $R\times\overline{\Omega}$. Using the comparison theorem of [1], we see that $\alpha_0 \le u(t,x,\alpha_0)$, where $u(t,x,\alpha_0)$ is the unique solution of (1.1) with $u(0,x) = \alpha_0(x)$. Similarly we get $\beta_0 \ge u(t,x,\beta_0)$, where $u(t,x,\beta_0)$ is the unique solution of (1.1) with $u(0,x) = \beta_0(x)$. This proves (i).

To prove (ii), we let, for fixed $s > 0$, $v(t,x) = u(t+s,x,\beta_0)$ and $u(t,x) = u(t,x,\beta_0)$. Then using (i), we have

$$\begin{aligned} &v_t - Lv = g(x,v), && (t>0,\ x\in\Omega),\\ &Bv = h, && (t>0,\ x\in\partial\Omega),\\ &v(0,x) = u(s,x,\beta_0) \le \beta_0(x), && (x\in\Omega) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &u_t - Lu = g(x,u), && (t>0,\ x\in\Omega),\\ &Bu = h, && (t>0,\ x\in\Omega),\\ &u(0,x) = \beta_0(x), && (x\in\Omega). \end{aligned} \tag{2.4}$$

Hence by applying the comparison theorem in [1], we obtain that $v(t,x) \le u(t,x)$, which implies $u(t+s,x,\beta_0) \le u(t,x,\beta_0)$ on $R_+\times\overline{\Omega}$. This proves $u(t,x,\beta_0)$ is nondecreasing in t. Similarly, one can prove $u(t,x,\alpha_0)$ is nondecreasing in t. We therefore have the estimate

$$\alpha_0(x) \le u(t,x,\alpha_0) \le u(t,x,\beta_0) \le \beta_0(x) \quad \text{on} \quad R_+\times\overline{\Omega}.$$

Hence the limits $\lim_{t\to\infty} u(t,x,\alpha_0) = \underline{u}(x)$ and $\lim_{t\to\infty} u(t,x,\beta_0) = \bar{u}(x)$ exists on $\overline{\Omega}$. In order to prove that the limits $\bar{u}(x)$ and $\underline{u}(x)$ are solutions of (1.2),

we follow similar arguments as given in [4]. Let $w(x)$ be the generalized solutions of

$$\begin{aligned} -Lw + cw &= G(x, \bar{u}) \text{ in } \Omega, \\ Bw &= h(x), \text{ on } \partial\Omega, \end{aligned} \tag{2.5}$$

where c is a positive constant and

$$G(x, u) = cu + g(x, u).$$

Now setting $W(t, x) = u(t, x, \beta_0) - w(x)$, it is easy to see that $W(t, x)$ satisfies

$$\begin{aligned} (W)_t - LW + cW &= q(t, x) \\ BW &= h(t, x) \equiv 0, \\ W(0, x) &= W_0(x), \end{aligned}$$

where $q(t, x) = cw(t, x) + g(x, u(t, x, \beta_0)) - g(x, \bar{u}(x))$,

$$W(0, x) = \bar{u}(x) - w(x).$$

Since $\lim_{t\to\infty} u(t, x, \beta_0) = \bar{u}(x)$, we have $q(t, x) \to 0$ as $t \to \infty$. By Theorem 2.1.3 in [4], we get $W(t, x) \to 0$ in $L_2(\Omega)$ as $t \to \infty$. This proves $\bar{u}(x)$ is the generalized solution of (2.5) which immediately shows that $\bar{u}(x)$ is the generalized solution of (1.2). Now, using Lemma 3.1.2 in [4], $\bar{u} \in C^{1+\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. This implies $g(x, \bar{u}(x))$ is the $C^{\alpha}(\overline{\Omega})$. Moreover, using Theorem 3.1.3 in [4], we see that $\bar{u} \in C^{2+\alpha}(\overline{\Omega})$ and therefore $\bar{u}(x)$ is the solution of (1.2). Similarly one can show that $\underline{u}(x) = \lim_{t\to\infty} u(t, x, \alpha_0)$ is the solution of (1.2). Thus we have proved (ii).

To prove (iii), let $\alpha_0(x) \le u_0(x) \le \beta_0(x)$ and consider the solution $u(t, x, u_0)$ of (1.1). By repeated application of comparison theorem in [1], it follows that

$$u(t, x, \alpha_0) \le u(t, x, u_0) \le u(t, x, \beta_0) \text{ on } R_+ \times \overline{\Omega}.$$

Hence if $\lim_{t\to\infty} u(t, x, u_0) = u^*(x)$ exists, then because of (ii), we obtain

$$\underline{u}(x) \le u^*(x) \le \bar{u}(x) \text{ on } \overline{\Omega},$$

proving (iii). The proof is complete.

3. Monotone flows and mixed monotone systems. In this section, we consider the mixed monotone reaction–diffusion system given by

$$\begin{aligned} (u)_t - Lu &= H(x, u), && (t > 0,\ x \in \Omega), \\ Bu &= h(x), && (t > 0,\ x \in \partial\Omega), \\ u(0, x) &= u_0(x), && (x \in \Omega), \end{aligned} \tag{3.1}$$

where H satisfies the same assumptions as g in (1.1) except that $H(x,u)$ is mixed monotone in u. The corresponding steady state systems is

$$\begin{aligned} &-Lu = H(x,u), \text{ on } \overline{\Omega}, \\ &Bu = h(x) \text{ on } \partial\Omega. \end{aligned} \tag{3.2}$$

We can now prove the following results relative to (3.1) and (3.2).

Theorem 3.1. *Assume that*

(B_0) *there exists a function* $F \in C[\Omega\times R^n\times R^n, R^n]$ *such that* $F(x,u,u) = H(x,u)$, $F(x,u,v)$ *is Hölder continuous in* x, $F(x,u,v)$ *is quasi-monotone nondecreasing in* u *and monotone nonincreasing in* v, *and* $F(x,u,v)$ *is Lipschitzian in* u *and* v;

(B_1) $\alpha_0, \beta_0 \in C^{2+\alpha}[\overline{\Omega}, R^n]$, $\alpha_0(x) \le \beta_0(x)$ *on* $\overline{\Omega}$,

$$\begin{aligned} &L\alpha_0 \le F(x,\alpha_0,\beta_0), && x\in\overline{\Omega}, \\ &B\alpha_0 \le h(x), && x\in\partial\Omega, \end{aligned}$$

and

$$\begin{aligned} &-L\beta_0 \ge F(x,\beta_0,\alpha_0), && x\in\overline{\Omega}, \\ &B\beta_0 \ge h(x), && x\in\partial\Omega, \end{aligned}$$

Then

(a) $\alpha_0(x) \le u(t,x,\alpha_0) \le u(t,x,\beta_0) \le \beta_0(x)$, $(t,x) \in R_+\times\overline{\Omega}$, *where* $u(t,x,\alpha_0)$, $u(t,x,\beta_0)$ *are the solutions of* (3.1);

(b) $u(t,x,\alpha_0)$ *is nondecreasing in* t, $u(t,x,\beta_0)$ *is nonincreasing in* t, $\lim\limits_{t\to\infty} u(t,x,\alpha_0) = \underline{u}$ *and* $\lim\limits_{t\to\infty} u(t,x,\beta_0) = \bar{u}$ *exist and* $\underline{u}, \bar{u}$ *are solutions of* (3.2);

(c) *if* $u(t,x,u_0)$ *is the solution of* (3.1) *with* $\alpha_0 \le u_0 \le \beta_0$ *then*

$$u(t,x,\alpha_0) \le u(t,x,u_0) \le u(t,x,\beta_0)$$

and if $\lim\limits_{t\to\infty} u(t,x,u_0) = u^*(x)$ *exists, then* $\underline{u}(x) \le u^*(x) \le \bar{u}(x)$ *on* $\overline{\Omega}$.

Proof. We define $g \in C[\overline{\Omega}\times R^{2n}, R^{2n}]$ by

$$g(x,z) = g(x,u,v) = (F(x,u,v), F(x,v,u)).$$

Introduce the ordering in $K = R^n\times\{-R^n\}$ by $Z_1 \le Z_2$, where $Z_1 = (u_1, v_1)$, $Z_2 = (u_2, v_2)$ iff $u_1 \le u_2$, $v_1 \le v_2$.

Define the reflection operator $P : R^{2n} \to R^{2n}$ by

$$P(u, v) = (v, u),$$

and let $\xi_0 = (\alpha_0, \beta_0)$, $\eta_0 = P\xi_0$, $Z = (u, v)$ and $Z(t, x, \xi_0) = (u(t, x, \alpha_0), u(t, x, \beta_0))$. Then $g(x, z)$ satisfies the assumptions of Theorem 2.1. As a result, conclusions (a), (b) follow from Theorem 3.1. Since $\eta_0 = P\xi_0$, $Z(t, x, \eta_0) = PZ(t, x, \xi_0)$, we get $P\underline{Z} = \overline{Z}$, where $\underline{Z} = (\bar{u}, \underline{u})$. To prove (c), let $Z_0 = (u_0, u_0)$ so that $\xi_0 \leq Z_0 \leq \eta_0$, which implies from Theorem 2.1, that $u(t, x, \alpha_0) \leq u(t, x, u_0) \leq u(t, x, \beta_0)$, which proves (c). Hence the proof of the theorem is complete.

Remark 3.1. Suppose that $H(x, u)$ is mixed monotone in u, $H_i(x, u_i, [u]_{p_i}, [u]_{q_i})$ is nondecreasing in $[u]_{p_i}$ and nonincreasing in $[u]_{q_i}$, where $u = (u_i, [u]_{p_i}, [u]_{q_i})$ for each i with $p_i + q_i = n - 1$. In this case, we can choose F of Theorem 3.1 as follows:

$$F(x, u, v) = (H(x, u_i, [u]_{p_i}, [M - v]_{q_i}), -H(x, M - u_i, [M - v]_{p_i}, [u]_{q_i})$$

where $M \geq \beta_0$. See [5], where similar results are discussed using monotone iterative method for mixed monotone systems.

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