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A NOTE ON STABLY FIBRE HOMOTOPIC TRIVIAL VECTOR BUNDLES

BY

DĂNUŢ MARCU

Abstract. In this paper, we study relations among Chern classes of a stably fibre homotpic trivial complex vector bundle.

Let τ be a stably fibre homotopic trivial vector bundle. A classical result of Thom states that the Stiefel–Whitney classes of τ vanish, and one way to prove this is as follows. Let u be the Thom class of τ in $mod\ 2$ cohomology. Then, by [2], u is stably spherical and, therefore, all stable cohomology operations vanish on u, showing that $w_i(\tau)u = Sq^i u = 0$.

In this note, we shall apply this same method, using complex cobordism and Landweber-Novikov operations, to study relations among Chern classes of a stably fibre homotopic trivial complex vector bundle. Thus, we will obtain, in a unified way, certain strong $mod\, p$ conditions for every prime p.

Let MU denote the complex cobordism spectrum. Then, for every finite sequence E of nonnegative integers there exists the Landweber–Novikov operation s_E and we may define the total operation s_t to be $\Sigma_E s_E t^E$, where $t_1, t_2, ...$ are variables, t_i has degree $-2i$, and t^E is the monomial corresponding to the sequence E. If $c_1, c_2, ...$ denote the universal Chern classes in MU cohomology and c_E is the polynomial associated with E as in [3, Chapter 16], write $c_t = \Sigma_E c_E t^E$. Then, if u is the canonical MU cohomology Thom class of a complex vector bundle τ, we have the formula $s_t u = c_t(\tau)u$ (see Part I of [1]).

Let τ be a complex vector bundle of complex dimension k over a connected finite CW-complex X and assume that τ is stably fibre homotopic trivial as a real vector bundle. Then, by [2], there exists a stable map $f : M(\tau) \longrightarrow S^{2k}$ of degree one on the bottom cell. Then, if σ is the canonical

generator in $MU^{2k}(S^{2k})$, we have that $f^*(\sigma)$ is a Thom class for τ. Therefore, $f^*(\sigma) = (I+x)u$, where u is the canonical Thom class of τ and x belongs to $\widetilde{MU}^0(X)$. Applying s_t to this equation, we obtain $s_t(I+x)c_t(\tau) = I+x$.

Now, let μ be the canonical map of ring spectra $MU \to H$, where H denotes the Eilenberg-MacLane spectrum for singular integral cohomology. Mapping our last equation into $H^*(X)$ via μ, we obtain $\mu(s_t(I+x))c_t(\tau) = I$, where c_t is the series described above but with coefficients the singular cohomology Chern classes c_E.

If E is a nonzero finite sequence of nonnegative integers and $F \leq E$ is also nonzero, let $\lambda_{E,F}$ be the index of the subgroup $im(s_F)$ of MU^0 (point) and let λ_E be the greatest common divisor of all $\lambda_{E,F}$ for all nonzero $F \leq E$. Now, we state our main result.

Theorem. *If τ is a complex vector bundle as above, then $c_E(\tau)$ is divisible by λ_E.*

Recall, from [4], that $\widetilde{MU}^*(X)$ is generated over MU^* (point) by a finite set $x_1, x_2, ..., x_m$ of positive dimensional elements. Write $x = \Sigma a_i x_i$, where $a_i \in MU^*$ (point). Observe that if $G \neq 0$, then $s_G(I+x) = s_G(x) = \Sigma\Sigma s_I(a_i)s_J(x_i)$, where the first sum is over i and the second over $I+J=G$. Since $\mu(s_I(a_i)) = 0$ unless $s_I(a_i)$ has dimension zero, applying μ to the previous equation, we see that $\mu(s_G(I+x))$ is divisible by λ_E. Now, from the last equation before the statement of the theorem, we obtain $c_E(\tau) = -\Sigma\mu(s_G(I+x))c_H(\tau)$, where the sum is over $G+H=E$ with $G \neq 0$, and this implies that $c_E(\tau)$ is divisible by λ_E, which is our result. ∎

There remains the problem of calculating λ_E. We leave to the interested readers the exercise of showing that $\lambda_E = I$ unless there exists a prime p such that $i+I$ is a power of p for all nonzero coordinates e_i of E, and that $\lambda_E = p$ in this case.

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Str. Pasului 3, Sect.2
70241 Bucureşti
ROMANIA