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ON INITIAL VALUE PROBLEM FOR AN INTEGRO–DIFFERENTIAL EQUATION

BY

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Abstract. In this paper we shall prove that the initial value problem for the integro–differential equation

$$y'(t) = f\left(t, \int_0^t K(t,s,y(s))ds\right) \text{ for almost all } t \in J = [0,\infty),\ y(0) = 0,$$

has at least one solution, where $K : J\times J\times \mathbb{R}^N \longrightarrow \mathbb{R}^N$, $f : J\times \mathbb{R}^N \longrightarrow \mathbb{R}^N$. The solution is obtained via the Schauder–Tychonoff fixed point theorem.

Introduction. In the past few years, several papers have been devoted to the study of initial value and boundary value problems for integro–differential equations under different conditions on the nonlinerity and the kernel (see for instance [2], [3], [4], [6], [7], [8], [9], [10], [11], [12], [16] and [17]). For an application of integro–differential equations to problems arising from physics see for example [1], [2] and [15].

The fundamental tools used in the existence proofs of all above mentioned works are essentially fixed point theorems: Schauder's theorem in [9]; Schauder–Tychonoff's theorem in [13], [14]; Leray–Schauder nonliner alternative type in [13], [14]; topological transversality theorem of Granas in [2], [12]; Banach fixed point theorem in [16] and monotone iterative combined with upper and lower solutions in [6], [7], [8], [10], [11].

In this paper, we shall be concerned with the existence of solutions of the initial value problem for first order integro–differential equation:

$$(1.1)\quad y'(t) = f\left(t, \int_0^t K(t,s,y(s))ds\right),\ \text{for a.a. } t \in J := [0,\infty),\ y(0) = 0$$

where $K : J\times J\times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and $f : J\times \mathbb{R}^N \longrightarrow \mathbb{R}^N$.

We will provide sufficient conditions on the nonlinearity f and the kernel K in order to insure the existence of solutions of the problem (1.1).

We will deal with the case where f is an L^1–Carathéodory function. The method we are going to use is to reduce the problem (1.1) to the search for fixed points of a suitable operator on the Fréchet space $C(J, \mathbb{R}^N)$ and we use the Schauder–Tychonoff fixed point theorem [5].

2. Preliminaries. In this section, we introduce notations, definitions, and results which are used throughout the paper. In the following we will denote by:

K_m the compact real interval $[0, m]$.

$\mathbb{R}^N$ the space of real N – vectors normed by $|y| := |y_1| + \cdots + |y_N|$ where $y = (y_1, ..., y_N) \in \mathbb{R}^N$.

$L^1(J, \mathbb{R}^N)$ the Lebesgue space of all integrable functions defined on J with values in $\mathbb{R}^N$.

$C(J, \mathbb{R}^N)$ the linear metric Fréchet space of all continuous functions $y : J \longrightarrow \mathbb{R}^N$ with the metric (see [4])

$$d(y, z) = \sum_{m=0}^{\infty} \frac{2^{-m}\rho_m(y - z)}{1 + \rho_m(y - z)} \text{ where } \rho_m(y) := \sup\{|y(t)| : t \in K_m\}.$$

The convergence in $C(J, \mathbb{R}^N)$ is the uniform convergence on compact intervals, i.e. $y_j \longrightarrow y$ in $C(J, \mathbb{R}^N)$ if and only if for each $m \in \mathbb{N}$, $\|y_j - y\|_m \longrightarrow 0$ in $C(K_m, \mathbb{R}^N)$ as $j \longrightarrow \infty$.

$M \subseteq C(J, \mathbb{R}^N)$ is a bounded set iff there exists a positive function $\phi \in C(J, \mathbb{R}^N)$ such that

$$|y(t)| \leq \phi(t) \text{ for all } t \in J \text{ and } y \in M.$$

$M \subseteq C(J, \mathbb{R}^N)$ is a compact set iff for each $m \in \mathbb{N}$, M is a compact set in the Banach space $(C(K_m, \mathbb{R}^N), \|\cdot\|_m)$.

Finally $W^{1,1}(J, \mathbb{R}^N)$ the space of continuous functions $y : J \to \mathbb{R}^N$ such that y' (which exists almost everywhere on J) is an element of $L^1(J, \mathbb{R}^N)$ normed by

$$\|y\|_{W^{1,1}} = \|y\|_{L^1} + \|y'\|_{L^1} \text{ for all } y \in W^{1,1}(J, \mathbb{R}^N).$$

We say that $f : [a,b]\times\mathbb{R}^N \longrightarrow \mathbb{R}^N$ is an L^1–Carathéodory function if f satisfies the following properties:

(i) the map $t \longmapsto f(t,y)$ is measurable for all $y \in \mathbb{R}^N$,

(ii) the map $y \longmapsto f(t,y)$ is continuous for almost all $t \in [a,b]$,

(iii) for given $k > 0$ there exists $h_k \in L^1([a,b],\mathbb{R}_+)$ such that $|y| \leq k$ implies $|f(t,y)| \leq h_k(t)$ for almost $t \in [a,b]$.

Recall that a map is said to be completely continuous if it is continuous and the image of each bounded set in the domain is relatively compact.

We shall show that if K and the L^1–Carathéodory function f satisfy the conditions

H1 There exists a continuous function $\alpha : J \longrightarrow \mathbb{R}_+$ such that

$$|f(t,y)| \leq \alpha(t)|y|;$$

H2 There exists a nondecreasing continuous function $\psi : [0,+\infty) \longrightarrow (0,+\infty)$, a continuous function $\beta : J \longrightarrow \mathbb{R}_+$ such that

$$\left|\int_0^t K(t,s,y(s))ds\right| \leq \beta(t)\psi\left(\int_0^t |y(s)|ds\right) \text{ for all } t\in J \text{ and } y\in C(J,\mathbb{R});$$

H3 $\displaystyle\int_0^{+\infty} \frac{du}{\psi(u)} = \infty.$

Then the initial value problem (1.1) has at least one solution.

Remark 2.1. By a solution of (1.1), we shall mean a function y which belongs to

$$W_0^{1,1}(J,\mathbb{R}) := \{y \in W^{1,1}(J,\mathbb{R}^N) : y(0) = 0\}$$

and satisfies the integro–differential equation $y'(t) = f\left(t, \displaystyle\int_0^t K(t,s,y(s))ds\right)$ almost everywhere on J.

The following theorem is crucial in the proof of our main result:

Theorem 2.1. (Schauder–Tychonoff [5]**)** *Let K be a closed convex subset of a locally convex Hausdorff space E. Assume that $N : K \longrightarrow K$ is*

continuous and that $N(K)$ is relatively compact in E. Then N has at least one fixed point in K.

3. Main result. Now, we are in position to formulate our main result.

Theorem 3.1. *Suppose that conditions $(H1)$, $(H2)$ and $(H3)$ are satisfied. Then the initial value problem* (1.1) *has at least one solution.*

Proof. Set

$$C_0(J, \mathbb{R}^N) := \{y \in C(J, \mathbb{R}^N) : y(0) = 0\}.$$

A solution to (1.1) is a fixed point of the operator $N : C_0(J, \mathbb{R}^N) \longrightarrow C_0(J, \mathbb{R}^N)$ defined by

$$(Ny)(t) = \int_0^t f\left(s, \int_0^s K(s, u, y(s))du\right) ds.$$

Let

$$K := \left\{ y \in C_0(J, \mathbb{R}^N) : \int_0^t |y(s)|ds \leq a(t), \text{ and } |y(t)| \leq \int_0^t \alpha(s)\beta(s)\psi(a(s))ds, \ t \in J \right\}$$

where

$$a(t) = I^{-1}\left(\int_0^t \left(\int_0^x \alpha(s)\beta(s)\psi(a(s))ds\right) dx\right)$$

and

$$I(z) = \int_0^z \frac{du}{\psi(u)}.$$

Clearly, K is a convex subset of $C_0(J, \mathbb{R}^N)$.

We shall show tht K is closed and the operator N defined on K has values in K and it is compact. The proof will be given in several steps.

Step 1. K is closed.

Let $y_n \in K$ with $\rho_m(y_n) \longrightarrow \rho_m(y)$ (i.e. y_n converges uniformly to y on $[0, t_m]$) for each $m \in \{1, 2, ...\}$. Then for each fixed $t \in [0, t_m]$ we have

$$\int_0^t |y_n(s)|ds \leq a(t)$$

which implies

$$\int_0^t |y(s)|ds \leq a(t).$$

So, $y \in K$.

Step 2. $N(K) \subseteq K$.
Let $y \in K$ and fix $t \in J$. We must show $Ny \in K$.
Let $x < t$, then

$$\begin{aligned} |(Ny)(x)| &\leq \int_0^x \left| f\left(s, \int_0^s K(s,u,y(u))du\right)\right| ds \\ &\leq \int_0^x \alpha(s)\beta(s)\psi\left(\int_0^s |y(u)|du\right) ds \\ &\leq \int_0^x \alpha(s)\beta(s)\psi(a(s))ds \end{aligned}$$

and

$$\begin{aligned} \int_0^t |(Ny)(x)|dx &\leq \int_0^t \left(\int_0^x \alpha(s)\beta(s)\psi(a(s))ds\right) dx \\ &= \int_0^t a'(x)dx \\ &= a(t) \end{aligned}$$

since

$$\int_0^{a(x)} \frac{du}{\psi(u)} = \int_0^x \left(\int_0^s \alpha(u)\beta(u)du\right) ds.$$

Thus, $Ny \in K$, so $N : K \longrightarrow K$.

Step 3. N is continuous.
Let $y_n \longrightarrow y$ in $C_0(J, \mathbb{R}^N)$. We will show that

$$Ny_n \longrightarrow Ny \text{ in } C_0(J, \mathbb{R}^N).$$

Now $\rho_m(y_n) \longrightarrow \rho_m(y)$ implies that there exists $r > 0$ such that

$$\rho_m(y_n) \leq r \text{ and } \rho_m(y) \leq r.$$

Since $K : J \times J \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is continuous, then there exists a constant $M > 0$ such that

$$\sup\left\{\left|\int_0^s K(s,u,y(u))du\right| : 0 \leq u \leq s \leq t_m\right\} \leq M.$$

Also there exists $\mu_M \in L^1_{\rm loc}(J, \mathbb{R}_+)$ such that

$$\left| f\left(s, \int_0^s K(s,u,y(u))du\right)\right| \leq \mu_M(s) \text{ for almost all } s \in [0,t_m].$$

We then have for $t \in [0,t_m]$

$$f\left(s, \int_0^s K(s,u,y_n(u))du\right) \longrightarrow f\left(s, \int_0^s K(s,u,y(u))du\right) \text{ for a.a. } s \in [0,t_m].$$

The Lebesgue dominated convergence theorem implies that

$$Ny_n(s) \longrightarrow Ny(s) \text{ pointwise on } [0,t_m].$$

If we show that the convergence is uniform then N is continuous in $C_0(J, \mathbb{R}^N)$. Let $t_1, t_2 \in [0,t_m]$ with $t_1 < t_2$. Then

$$\begin{aligned} |Ny_n(t_2) - Ny_n(t_1)| &\leq \int_{t_1}^{t_2} \left| f\left(s, \int_0^s K(s,u,y_n(u))du\right)\right| ds \\ &\leq \int_{t_1}^{t_2} \mu_M(s)ds. \end{aligned}$$

Also we obtain

$$|Ny(t_2) - Ny(t_1)| \leq \int_{t_1}^{t_2} \mu_M(s)ds.$$

Thus for any $\varepsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in [0,t_m]$ and $|t_2 - t_1| \leq \delta$ implies

$$|Ny_n(t_2) - Ny_n(t_1)| \leq \varepsilon \text{ for all } n \in \mathbb{N} \tag{3.1}$$

and

$$|Ny(t_2) - Ny(t_1)| \leq \varepsilon. \tag{3.2}$$

Now (3.1) and (3.2) together with the fact that $Ny_n(s) \longrightarrow Ny(s)$ pointwise on $[0,t_m]$ implies that the convergence is uniform.

Consequently, $N : C_0(J, \mathbb{R}^N) \longrightarrow C_0(J, \mathbb{R}^N)$ is continuous.

Step 4. N send bounded sets into bounded sets.

Let $B \subseteq C_0(J, \mathbb{R}^N)$ be a bounded set. We will show that NB is uniformly bounded. There exists $r > 0$ with $\rho_m(v) \leq r$ for all $v \in B$.

Let $t \in [0, t_m]$ we have

$$\begin{aligned} |N(v)(t)| &\leq \int_0^t \left| f\left(s, \int_0^s K(s,u,v(u))du\right)\right| ds \\ &\leq \int_0^{t_m} \mu_M(s)ds < \infty \text{ for all } v \in B \end{aligned}$$

Step 5. N send bounded sets into equicontinuous sets on $[0, t_m]$.

This follows immediately from the same reasoning as that used to prove (3.1).

In view of Step 3, Step 4 and Step 5 together with the Arzelà–Ascoli theorem, the operator $N : K \to K$ is completely continuous.

As a consequence of Schauder–Tychonoff theorem we can conclude that N has a fixed point y in K which is a solution of (1.1). This completes the proof of the theorem.

Remark 3.1. The divergence of the integral in (H3) is assumed to obtain an existence result for the problem (1.1) on all J.

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