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ON THE THEORY OF FLUID MIXTURES

BY

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Abstract. In this paper we study the linear thermodynamical theory of inviscid fluid mixtures. First we establish a reciprocal theorem. Then we show that the reciprocity relation implies a uniqueness result and variational theorems.

Introduction. The continuum theory of mixtures has been a subject of intensive study in recent years. For an extensive review of the literature on mixtures the reader is referred to the reviews by BOWEN [1], ATKIN and CRAINE [2], BEDFORD and DRUMHELLER [3] and RAJAGOPAL and WINEMAN [4]. In [5], GURTIN and VARGAS have shown that the classical theory of fluid mixtures can be formulated within the framework of modern continuum thermodynamics. The linear theory appropriate to small departures from a strong equilibrium state was studied in various papers. In [6], GURTIN and VILLAGGIO have established, within this framework, that the statical stability criterion of Gibbs is both necessary and sufficient for the dynamical stability of mixture. A uniqueness result is also derived.

In this paper we consider the linear thermodynamic theory of inviscid fluid mixture. First, we establish a reciprocal theorem. Then, we show that the reciprocity relation implies a uniqueness result with no definiteness assumption on the constitutive coefficients and variational principles.

2. Basic equations. We refer to the motion of the continuum to a system of rectangular Cartesian axex Ox_i $(i = 1, 2, 3)$. We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers (1,2,3), summation over repeated subscripts is implied, and subscripts preceded by a comma denote partial

differentiation with respect to the corresponding Cartesian coordinate. Letters in boldface stand for tensors of an order $p \geq 0$, and if $\mathbf{v}$ has the order p, we write $v_{ij...s}$ (p subscripts) for the components of $\mathbf{v}$ in the Cartesian coordinate system.

In this paper we consider the classical theory of fluid mixture. The mixture is assumed to contain $N+1$ constituents. We use the following notations: ρ' is the density, $v' = 1/\rho'$ is the specific volume, $\mathbf{v}'$ is the velocity, p' is the pressure, $\mathbf{q}$ is the heat flux, c'_α is the concentration of constituent $\alpha, \mathbf{h}_\alpha$ is the relative mass flux of constituent α, m_α is the mass supplied to constituent α, μ'_α is the relative chemical potential of constituent α, θ' is the absolute temperature ($\theta' > 0$) and η' is the entropy. These fields are defined for all $\mathbf{x}$ in the region of space B occupied by the mixture and for all time t. In what follows the Greek subscripts have the range 1 to N. We write grad and div for the gradient and divergence with respect to $\mathbf{x}$ holding t fixed. Let

$$S = (p', \theta', \mu'_1, \mu'_2, ..., \mu'_N).$$

The linear theory of inviscid fluid mixtures appropriate to small departures from a strong equilibrium state $S_0 = (p_0, \mu^0_1, ..., \mu^0_N)$ has been established by GURTIN and VARGAS [5]. The linearized constitutive equations are

$$\begin{aligned} &v = Ap + D\theta + A_\alpha \mu_\alpha, && q_i = -k\theta_{,i} - \ell_\alpha \mu_{\alpha,i}, \\ &\eta = -Dp + C\theta + B_\alpha \mu_\alpha, && h_{\alpha i} = -\lambda_\alpha \theta_{,i} - \lambda_{\alpha\beta}\mu_{\beta,i}, \\ &c_\alpha = -A_\alpha p + B_\alpha \theta + C_{\alpha\beta}\mu_\beta, && m_\alpha = -P_{\alpha\beta}\mu_\beta, \end{aligned} \tag{2.1}$$

where we have used the notations

$$\begin{aligned} &p = p' - p_0, && \theta = \theta' - \theta_0, && \mu_\alpha = \mu'_\alpha - \mu^0_\alpha, && c_\alpha = c'_\alpha - c^0_\alpha, \\ &v = v' - v_0, && \eta = \eta' - \eta_0, && v_0 = v(S_0), && \eta_0 = \eta(S_0), \\ &c^0_\alpha = c_\alpha(S_0), && \rho_0 = 1/v_0. \end{aligned} \tag{2.2}$$

We assume that A, D, C, A_α, B_α, $C_{\alpha\beta}$, k, ℓ_α, λ_α, $\lambda_{\alpha\beta}$, $P_{\alpha\beta}$, p_0, θ_0, μ^0_α, c^0_α, η_0 and ρ_0 are given constants. We introduce the matrices

$$\mathcal{A} = \begin{pmatrix} -A & -D & -A_1 & \cdots & -A_N \\ -D & C & B_1 & \cdots & B_N \\ -A_1 & B_1 & C_{11} & \cdots & C_{1N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -A_N & B_N & C_{N1} & \cdots & C_{NN} \end{pmatrix},$$

$$\mathcal{K} = \begin{pmatrix} k/\theta_0 & \ell_1/\theta_0 & \cdots & \ell_N/\theta_0 \\ \lambda_1 & \lambda_{11} & \cdots & \lambda_{1N} \\ \vdots & & & \\ \lambda_N & \lambda_{N1} & \cdots & \lambda_{NN} \end{pmatrix}, \quad \mathcal{P} = (P_{\alpha\beta}). \tag{2.3}$$

We note (cf. GURTIN and VARGAS [5] and GURTIN and VILLAGGIO [6] that the matrix $\mathcal{A}$ is symmetric, and the matrices $\mathcal{K}$ and $\mathcal{P}$ are positive semi–definite, i.e.

$$\mathcal{A} = \mathcal{A}^{\top},\ \mathcal{K} \geq 0,\ \mathcal{P} \geq 0. \tag{2.4}$$

The relations (2.4) are consequences of the second law of thermodynamics.

The linear theory of fluid mixture is based on the following balance equations: balance of mass

$$\rho_o \dot{v} = \operatorname{div} \mathbf{v}, \tag{2.5}$$

balance of momentum

$$\rho_0 \dot{\mathbf{v}} = -\operatorname{grad} p + \rho_0 \mathbf{f}, \tag{2.6}$$

balance of mass for each constituent

$$\rho_0 \dot{c}_\alpha = -\operatorname{div} \mathbf{h}_\alpha + m_\alpha,\ (\alpha = 1, 2, ..., N), \tag{2.7}$$

balance of energy

$$r_0 \theta_0 \dot{\eta} = -\operatorname{div} \mathbf{q} + \rho_0 r. \tag{2.8}$$

Here $\mathbf{f}$ is the body force, r is the heat supply and a superposed dot denotes the material derivative with respect to the time.

To the system of field equations (2.1), (2.5)–(2.8) we must adjoin boundary conditions and initial conditions. We consider the following initial conditions

$$\begin{array}{ll} v(\mathbf{x}, 0) = a(\mathbf{x}), & \mathbf{v}(\mathbf{x}, 0) = \mathbf{b}(\mathbf{x}), \\ c_\alpha(\mathbf{x}, 0) = \xi_\alpha(\mathbf{x}), & \eta(\mathbf{x}, 0) = \zeta(\mathbf{x}), \quad \mathbf{x} \in \overline{B}, \end{array} \tag{2.9}$$

where $a, \mathbf{b}, \xi_\alpha$ and ζ are given.

We assume that

(i) $\mathbf{f}$ and r are continuous on $\overline{B} \times I$, where $I = [0, \infty)$;
(ii) $a, \mathbf{b}, \xi_\alpha$ and ζ are continuous on $\overline{B}$.

3. Reciprocity and uniqueness. Let f and g be scalar fields on $B \times I$ that are continuous in time. We denote by $f * g$ the convolution of f and g

$$[f * g](\mathbf{x}, t) = \int_0^t f(\mathbf{x}, t - s) g(\mathbf{x}, s) ds.$$

We introduce the function

$$\gamma(t) = 1, \ t \in I. \tag{3.1}$$

If g is a continuous function on $\overline{B}\times I$, then we write $\bar{g}$ for $\gamma * g$, i.e.

$$\bar{g}(\mathbf{x},t) = \int_0^t g(\mathbf{x},s)ds = [\gamma * g](\mathbf{x},t). \tag{3.2}$$

We define the functions M, M_α and W on $\overline{B}\times I$ by

$$M = a, \ M_\alpha = -\xi_\alpha, \ W = -\zeta - \bar{r}/\theta_0, \ \mathbf{G} = \gamma * \mathbf{f} + \mathbf{b}. \tag{3.3}$$

An immediate consequence of the above definitions is the following proposition

Lemma 3.1. *The functions $v, \mathbf{v}, c_\alpha, \mathbf{h}_\alpha, m_\alpha, \eta$ and $\mathbf{q}$ of class C^1 on $B\times I$ satisfy the equations* (2.5)–(2.8) *and the initial conditions* (2.9) *if and only if*

$$\rho_0 v = \operatorname{div} \bar{\mathbf{v}} + \rho_0 M, \tag{3.4}$$

$$\rho_0 \mathbf{v} = -\operatorname{grad} \bar{p} + \rho_0 \mathbf{G}, \tag{3.5}$$

$$\rho_0 c_\alpha = -\operatorname{div} \bar{\mathbf{h}}_\alpha + \bar{m}_\alpha - \rho_0 M_\alpha, \tag{3.6}$$

$$\rho_0 \theta_0 \eta = -\operatorname{div} \bar{\mathbf{q}} - \rho_0 \theta_0 W, \tag{3.7}$$

on $B\times I$.

We consider two systems of loading $L^{(\nu)}$ $(\nu = 1, 2)$. Let $s^{(\nu)} = \{p^{(\nu)}, \theta^{(\nu)}, \mu_\alpha^{(\nu)}, v^{(\nu)}, \mathbf{v}^{(\nu)}, c_\alpha^{(\nu)}, \eta^{(\nu)}, \mathbf{h}_\alpha^{(\nu)}, \mathbf{q}^{(\nu)}, m_\alpha^{(\nu)}\}$ be a process corresponding to $L^{(\nu)}$. We define

$$\begin{aligned} &M^{(\nu)} = a^{(\nu)}, \ M_\alpha^{(\nu)} = -\xi_\alpha^{(\nu)}, \ W^{(\nu)} = -\zeta^{(\nu)} - \bar{r}^{(\nu)}/\theta_0, \\ &\mathbf{G}^{(\nu)} = \gamma * \mathbf{f}^{(\nu)} + \mathbf{b}^{(\nu)}, \qquad\qquad (\nu = 1, 2). \end{aligned} \tag{3.8}$$

Let

$$I^{(\nu\kappa)}(r,s) =$$

$$\begin{aligned} &= \int_{\partial B} \left[p^{(\nu)}(r)\cdot\bar{\mathbf{v}}^{(\kappa)}(s) + \frac{1}{\theta_0}\theta^{(\nu)}(r)\bar{\mathbf{q}}^{(\kappa)}(s) + \mu_\alpha^{(\nu)}(r)\bar{\mathbf{h}}_\alpha^{(\kappa)}(s) \right] \cdot \mathbf{n}\, d\mathbf{x} - \\ &- \int_B \rho_0 [\mathbf{f}^{(\nu)}(r)\cdot\bar{\mathbf{v}}^{(\kappa)}(s) + M^{(\nu)}(r)p^{(\kappa)}(s) + W^{(\nu)}(r)\theta^{(\kappa)}(s) + \\ &\quad + M_\alpha^{(\nu)}(r)\mu_\alpha^{(\kappa)}(s)]d\mathbf{x} + \int_B \rho_0 \dot{\mathbf{v}}^{(\nu)}(r)\cdot\bar{\mathbf{v}}^{(\kappa)}(s)d\mathbf{x} + \\ &+ \int_B \left[\frac{1}{\theta_0}\bar{\mathbf{q}}^{(\nu)}(r)\cdot\operatorname{grad}\theta^{(\kappa)}(s) + \bar{\mathbf{h}}_\alpha^{(\nu)}(r)\cdot\operatorname{grad}\mu_\alpha^{(\kappa)}(s) \right] d\mathbf{x} + \\ &\quad + \int_B P_{\alpha\beta}\mu_\alpha^{(\nu)}(r)\bar{\mu}_\beta^{(\kappa)}(s)d\mathbf{x}, \end{aligned} \tag{3.9}$$

for all $r, s \in I$. Here $\mathbf{n}$ is the outward unit normal of ∂B and, for convenience, we have suppressed the argument $\mathbf{x}$.

Theorem 3.1. *Assume that*

$$\mathcal{A} = \mathcal{A}^{\top}. \tag{3.10}$$

Then

$$I^{(\nu\kappa)}(r,s) = I^{(\kappa\nu)}(s,r), \quad (\nu,\kappa = 1,2), \ \textit{for all } r,s \in I. \tag{3.11}$$

Proof. Let

$$J^{(\nu\kappa}(r,s) = p^{(\nu)}(r)v^{(\kappa)}(s) - \theta^{(\nu)}(r)\eta^{(\kappa)}(s) - \mu_{\alpha}^{(\nu)}(r)c_{\alpha}^{(\kappa)}(s).$$

It follows from(2.1) and (3.11) that

$$\begin{aligned} J^{(\nu\kappa)}(r,s) &= Ap^{(\nu)}(r)p^{(\kappa)}(s) + D[p^{(\nu)}(r)\theta^{(\kappa)}(s) + p^{(\kappa)}(s)\theta^{(\nu)}(r)] - \\ &- C\theta^{(\nu)}(r)\theta^{(\kappa)}(s) + A_{\alpha}[p^{(\nu)}(r)\mu_{\alpha}^{(\kappa)}(s) + p^{(\kappa)}(s)\mu_{\alpha}^{(\nu)}(r)] - \\ &- B_{\alpha}[\mu_{\alpha}^{(\nu)}(r)\theta^{(\kappa)}(s) + \mu_{\alpha}^{(\kappa)}(r)] - C_{\alpha\beta}\mu_{\alpha}^{(\nu)}(r)\mu_{\beta}^{(\kappa)}(s). \end{aligned} \tag{3.12}$$

By (3.10) and (3.12)

$$J^{(\nu\kappa)}(r,s) = J^{(\kappa\nu)}(s,r). \tag{3.13}$$

On the other hand, in view of (2.1), (2.6), (3.4)–(3.7), we obtain

$$\begin{aligned} J^{(\nu\kappa)}(r,s) &= p^{(\nu)}(r)M^{(\kappa)}(s) + \mu_{\alpha}^{(\nu)}(r)M^{(\kappa)}(s) + \theta^{(\nu)}(r)W^{(\kappa)}(s) + \\ &+ \frac{1}{\rho_0}\operatorname{div}\left[p^{(\nu)}(r)\bar{\mathbf{v}}^{(\kappa)}(s) + \frac{1}{\theta_0}\theta^{(\nu)}(r)\bar{\mathbf{q}}^{(\kappa)}(s) + \mu_{\alpha}^{(\nu)}(r)\bar{\mathbf{h}}_{\alpha}^{(\kappa)}(s)\right] + \\ &+ \dot{\mathbf{v}}^{(\nu)}(r)\cdot\bar{\mathbf{v}}^{(\kappa)}(s) - \mathbf{f}^{(\nu)}(r)\cdot\bar{\mathbf{v}}^{(\kappa)}(s) - \frac{1}{\rho_0\theta_0}\bar{\mathbf{q}}^{(\kappa)}(s)\cdot\operatorname{grad}\theta^{(\nu)}(r) - \\ &- \frac{1}{\rho_0}\bar{\mathbf{h}}_{\alpha}^{(\kappa)}(r)\cdot\operatorname{grad}\mu_{\alpha}^{(\nu)}(s) + \frac{1}{\rho_0}P_{\alpha\beta}\mu_{\alpha}^{(\nu)}(r)\bar{\mu}_{\beta}^{(\kappa)}(s). \end{aligned} \tag{3.14}$$

If we integrate the relation (3.14) over B, we obtain, with the aid of the divergence theorem and (3.13), the desired result. ∎

We introduce the notations

$$U^{(\nu)} = \mathbf{v}^{(\nu)}\cdot\mathbf{n}, \ Q^{(\nu)} = \mathbf{q}^{(\nu)}\cdot\mathbf{n}, \ H_{\alpha}^{(\nu)} = \mathbf{h}_{\alpha}^{(\nu)}\cdot\mathbf{n}, \ (\nu = 1,2). \tag{3.15}$$

As in [7], Theorem 3.1 forms the basis of the following reciprocal theorem.

Theorem 3.2. *Assume that*

$$\mathcal{A} = \mathcal{A}^\top,\ \mathcal{K} = \mathcal{K}^\top,\ \mathcal{P} = \mathcal{P}^\top. \tag{3.16}$$

Then

$$\begin{aligned}
&\int_{\partial B} \gamma * \left[p^{(1)} * U^{(2)} + \frac{1}{\theta_0}\theta^{(1)} * Q^{(2)} + \mu_\alpha^{(1)} * H_\alpha^{(2)} \right] d\mathbf{x} - \\
&- \int_B \rho_0 \left[G_i^{(1)} * v_i^{(2)} + M^{(1)} * p^{(2)} + W^{(1)} * \theta^{(2)} + M_\alpha^{(1)} * \mu_\alpha^{(2)} \right] d\mathbf{x} = \\
&= \int_{\partial B} \gamma * \left[p^{(2)} * U^{(1)} + \frac{1}{\theta_0}\theta^{(2)} * Q^{(1)} + \mu_\alpha^{(2)} * H_\alpha^{(1)} \right] d\mathbf{x} - \\
&- \int_B \rho_0 \left[G_i^{(2)} * v_i^{(1)} + M^{(2)} * p^{(1)} + W^{(2)} * \theta^{(1)} + M_\alpha^{(2)} * \mu_\alpha^{(1)} \right] d\mathbf{x}.
\end{aligned} \tag{3.17}$$

Proof. If we take in (3.9) $r = \tau$ and $s = t - \tau$ and integrate from 0 to t, we arrive at

$$\begin{aligned}
&\int_0^t I^{(\nu\kappa)}(\tau, t-\tau) d\tau = \\
&= \int_{\partial B} \gamma * \left[p^{(\nu)} * U^{(\kappa)} + \frac{1}{\theta_0}\theta^{(\nu)} * Q^{(\kappa)} + \mu_\alpha^{(\nu)} * H_\alpha^{(\kappa)} \right] d\mathbf{x} - \\
&- \int_B \rho_0 \left[G_i^{(\nu)} * v_i^{(\kappa)} + M^{(\nu)} * p^{(\kappa)} + W^{(\nu)} * \theta^{(\kappa)} + M_\alpha^{(\nu)} * \mu_\alpha^{(\kappa)} \right] d\mathbf{x} + \\
&+ \int_B \rho_0 v_i^{(\nu)} * v_i^{(\kappa)} d\mathbf{x} - \int_B \gamma * \big[\frac{1}{\theta_0} k \theta_{,i}^{(\nu)} * \theta_{,i}^{(\kappa)} + \frac{1}{\theta_0} \ell_\alpha \mu_{\alpha,i}^{(\nu)} * \theta_{,i}^{(\kappa)} + \\
&+ \lambda_\alpha \theta_{,i}^{(\nu)} * \mu_{\alpha,i}^{(\kappa)} + \lambda_{\alpha\beta} \mu_{\beta,i}^{(\nu)} * \mu_{\alpha,i}^{(\kappa)} \big] d\mathbf{x} + \int_B \gamma * P_{\alpha\beta} \mu_\alpha^{(\nu)} * \mu_\beta^{(\kappa)} d\mathbf{x}.
\end{aligned} \tag{3.18}$$

From (2.3), (3.11), (3.16) and (3.18) we obtain the desired result. ∎

The reciprocity relation (3.17) can be used to obtain representations of Somigliana type. In the next section we use Theorem 3.1 to obtain variational theorems.

A consequence of Theorem 3.1 is the following result.

Theorem 3.3. *Assume that the matrices $\mathcal{A}$, $\mathcal{K}$ and $\mathcal{P}$ satisfy the*

relations (3.16). *Let*

$$K = \frac{1}{\theta_0} k\bar{\theta}_{,i}\bar{\theta}_{,i} + \left(\frac{1}{\theta_0}\ell_\alpha + \lambda_\alpha\right)\bar{\mu}_{\alpha,i}\bar{\theta}_{,i} + \lambda_{\alpha\beta}\bar{\mu}_{\alpha,i}\bar{\mu}_{\beta,i} + P_{\alpha\beta}\bar{\mu}_\alpha\bar{\mu}_\beta,$$

$$\Lambda(r,s) = \int_{\partial B}\left[p(r)\overline{U}(s) + \frac{1}{\theta_0}\theta(r)\overline{Q}(s) + \mu_\alpha(r)\overline{H}_\alpha(s)\right]d\mathbf{x} - \tag{3.19}$$

$$-\int_B \rho_0[\mathbf{f}(r)\cdot\bar{\mathbf{v}}(s) + M(t)p(s) + W(r)\theta(s) + M_\alpha(r)\mu_\alpha(s)]d\mathbf{x},$$

for all $r, s \in I$. *Then*

$$\frac{d}{dt}\left[\int_B \rho_0\bar{\mathbf{v}}^2 d\mathbf{x} + \int_0^t\int_B K\, d\mathbf{x}\, ds\right] = \tag{3.20}$$

$$= \int_0^t [\Lambda(t+s, t-s) - \Lambda(t-s, t+s)]ds + \int_B \rho_0 v_i(0)\bar{\mathbf{v}}_i(2t)d\mathbf{x}.$$

Proof. By theorem 3.1,

$$\int_0^t I^{(11)}(t+s, t-s)ds = \int_0^t I^{(11)}(t-s, t+s)ds. \tag{3.21}$$

Now we apply the relation (3.21) to the process

$$\{p, \theta, \mu_\alpha, v, \mathbf{v}, c_\alpha, \eta, \mathbf{h}_\alpha, \mathbf{q}, m_\alpha\}.$$

It follows from (3.9), (3.15) and (3.19) that

$$\int_0^t I^{(11)}(t+s, t-s)ds = \int_0^t \Lambda(t+s, t-s)ds +$$

$$+\int_0^t\int_B \rho_0\dot{v}_i(t+s)\bar{v}_i(t-s)d\mathbf{x}\, ds + \int_0^t\int_B [P_{\alpha\beta}\mu_\alpha(t+s)\bar{\mu}_\beta(t-s) - \tag{3.22}$$

$$-\frac{1}{\theta_0}k\bar{\theta}_{,i}(t+s)\theta_{,i}(t-s) - \frac{1}{\theta_0}\ell_\alpha\bar{\mu}_{\alpha,i}(t+s)\theta_{,i}(t-s) -$$

$$-\lambda_\alpha\bar{\theta}_{,i}(t+s)\mu_{\alpha,i}(t-s) - \lambda_{\alpha\beta}\bar{\mu}_{\beta,i}(t+s)\mu_{\alpha,i}(t-s)]d\mathbf{x}\, ds.$$

Similarly,

$$\int_0^t I^{(11)}(t+s, t-s)ds = \int_0^t \Lambda(t-s, t+s)ds +$$

$$+\int_0^t\int_B \rho_0\dot{v}_i(t-s)\bar{v}_i(t+s)d\mathbf{x}\, ds + \int_0^t\int_B [P_{\alpha\beta}\mu_\alpha(t-s)\bar{\mu}_\beta(t+s) - \tag{3.23}$$

$$-\frac{1}{\theta_0}k\bar{\theta}_{,i}(t-s)\theta_{,i}(t+s) - \frac{1}{\theta_0}\ell_\alpha\bar{\mu}_{\alpha,i}(t-s)\theta_{,i}(t+s) -$$

$$-\lambda_\alpha\bar{\theta}_{,i}(t-s)\mu_{\alpha,i}(t+s) - \lambda_{\alpha\beta}\bar{\mu}_{\beta,i}(t-s)\mu_{\alpha,i}(t+s)]d\mathbf{x}\, ds.$$

If we use the relations

$$
\begin{aligned}
&\int_0^t \ddot{f}(t+s)h(t-s)ds = \dot{f}(2t)h(0) - \dot{f}(t)h(t) + \int_0^t \dot{h}(t-s)\dot{f}(t+s)ds,\\
&\int_0^t \ddot{h}(t-s)f(t+s)ds = \dot{h}(t)f(t) - \dot{h}(0)f(2t) + \int_0^t \dot{h}(t-s)\dot{f}(t+s)ds,\\
&\int_0^t f(t+s)\dot{h}(t-s)ds = -h(0)f(2t) + f(t)h(t) + \int_0^t \dot{f}(t+s)h(t-s)ds,\\
&\int_0^t f(t-s)\dot{h}(t+s)ds = h(2t)f(0) - h(t)f(t) + \int_0^t h(t+s)\dot{f}(t-s)ds,
\end{aligned}
\tag{3.24}
$$

then we obtain

$$
\begin{aligned}
&\int_0^t \dot{v}_i(t+s)\bar{v}_i(t-s)ds = -v_i\bar{v}_i + \int_0^t v_i(t-s)v_i(t+s)ds,\\
&\int_0^t \dot{v}_i(t-s)\bar{v}_i(t+s)ds = v_i\bar{v}_i - v_i(0)\bar{v}_i(2t) +\\
&\qquad + \int_0^t v_i(t-s)v_i(t+s)ds,\\
&\int_0^t \big[P_{\alpha\beta}\mu_\alpha(t+s)\bar{\mu}_\beta(t-s) - \frac{1}{\theta_0}k\bar{\theta}_{,i}(t+s)\theta_{,i}(t-s) -\\
&\quad -\frac{1}{\theta_0}\ell_\alpha\bar{\mu}_{\alpha,i}(t+s)\theta_{,i}(t-s) - \lambda_\alpha\bar{\theta}_{,i}(t+s)\mu_{\alpha,i}(t-s) -\\
&\quad -\lambda_{\alpha\beta}\bar{\mu}_{\beta,i}(t+s)\mu_{\alpha,i}(t-s)\big]ds =\\
&\quad = -K + \int_0^t \big[P_{\alpha\beta}\bar{\mu}_\alpha(t+s)\mu_\beta(t-s) - \frac{1}{\theta_0}k\theta_{,i}(t+s)\bar{\theta}_{,i}(t-s) -\\
&\quad -\frac{1}{\theta_0}\ell_\alpha\mu_{\alpha,i}(t+s)\bar{\theta}_{,i}(t-s) - \lambda_\alpha\theta_{,i}(t+s)\bar{\mu}_{\alpha,i}(t-s) -\\
&\quad -\lambda_{\alpha\beta}\mu_{\beta,i}(t+s)\bar{\mu}_{\alpha,i}(t-s)\big]ds.
\end{aligned}
\tag{3.25}
$$

From (3.16), (3.21), (3.22), (3.23) and (3.25) we obtain the desired result. ∎

Theorem 3.3 can be used to obtain continuous dependence results. Here we use Theorem 3.3 to obtain a uniqueness result.

We consider for the remainder of the paper the following boundary conditions (cf. [2])

$$
\begin{array}{llll}
p = 0 & \text{on } \overline{S}_1\times I, & \mathbf{v}\cdot\mathbf{n} = 0 & \text{on } S_2\times I,\\
\theta = 0 & \text{on } \overline{S}_3\times I, & \mathbf{q}\cdot\mathbf{n} = 0 & \text{on } S_4\times I,\\
\mu_\alpha = 0 & \text{on } \overline{S}_5\times I, & \mathbf{h}_\alpha\cdot\mathbf{n} = 0 & \text{on } S_6\times I,
\end{array}
\tag{3.26}
$$

where S_r $(r = 1, 2, ..., 6)$ are non empty subsets of ∂B so that $\overline{S}_1 \cup S_2 = \overline{S}_3 \cup S_4 = \overline{S}_5 \cup S_6 = \partial B$, $S_2 \cap S_2 = S_3 \cap S_4 = S_5 \cap S_6 = \emptyset$.

By a solution of the mixed boundary value problem we mean a class C^2 process $\{p, \theta, \mu_\alpha, v, \mathbf{v}, c_\alpha, \eta, \mathbf{h}_\alpha, \mathbf{q}, m_\alpha\}$ on $B \times I$ that satisfies the equations (2.1), (2.5)–(2.8), the initial conditions (2.9), and the boundary conditions (3.26).

Theorem 3.4. *Assume that*

(i) *the relations* (3.16) *hold;*
(ii) ρ_0 *is strictly positive;*
(iii) *the matrix* $\mathcal{P}$ *is positive definite, and the matrix* $\mathcal{K}$ *is positive semidefinite;*
(iv) k *is strictly positive.*

Then the mixed boundary value problem has at most one solution.

Proof. Suppose that there are two solutions. Then their difference $\{p, \theta, \mu_\alpha, v, \mathbf{v}, c_\alpha, \eta, \mathbf{h}_\alpha, \mathbf{q}, m_\alpha\}$ corresponds to null data. The relation (3.20) leads to

$$\int_B \rho_0 \bar{\mathbf{v}}^2 d\mathbf{x} + \int_0^t \int_B K \, d\mathbf{x} \, ds = 0, \ (0 \leq t < \infty). \tag{3.27}$$

In view of (ii) and (iii), (3.27) implies that

$$\mathbf{v} = 0, \ \mu_\alpha = 0 \text{ on } \overline{B} \times I. \tag{3.28}$$

Since v vanishes initially, from (2.5) and (3.28) we obtain $v = 0$ on $B \times I$. It follows from (2.6), (3.26) and (3.28) that $p = 0$ on $B \times I$. In view of (3.28), the relation (3.27) reduces to

$$\int_0^t \int_B k \overline{\theta}_{,i} \overline{\theta}_{,i} d\mathbf{x} \, ds = 0.$$

This relation and (iv) yield $\theta_{,i} = 0$ on $B \times I$. Then, from (3.26) and (2.1) we conclude that $\theta = 0$, $c_\alpha = 0$, $\mathbf{q} = 0$, $\mathbf{h}_\alpha = 0$, $m_\alpha = 0$ on $B \times I$ and the proof is complete. ∎

We note that we made no definiteness assumption concerning the matrix $\mathcal{A}$. The uniqueness result established in [6] is based on the assumption that $\mathcal{A}$ is positive definite.

4. Variational principles. In the first part of this section we establish a variational theorem of Gurtin type [8]. Then, we use the results from [9], [10] to derive a minimum principle.

The equations (2.1), (3.4)–(3.7) combine to yield the basic equations expressed in terms of the functions p, v_i, θ and μ_α

$$
\begin{aligned}
&\rho_0(Ap + D\theta + A_\alpha\mu_\alpha) - \bar{v}_{i,i} = \rho_0 M,\\
&\rho_0 v_i + \bar{p}_{,i} = \rho_0 G_i,\\
&\frac{1}{\theta_0}(k\bar{\theta}_{,ii} + \ell_\alpha\bar{\mu}_{\alpha,ii}) - \rho_0(C\theta - Dp + B_\alpha\mu_\alpha)) = \rho_0 W,\\
&\lambda_\alpha\bar{\theta}_{,ii} + \lambda_{\alpha\beta}\bar{\mu}_{\beta,ii} - P_{\alpha\beta}\bar{\mu}_\beta - \rho_0(C_{\alpha\beta}\mu_\alpha + B_\alpha\theta - A_\alpha p) = \rho_0 M_\alpha,
\end{aligned}
\tag{4.1}
$$

on $B\times I$. The boundary conditions (3.26) can be written in the form

$$
\begin{array}{llll}
p = 0 & \text{on } \overline{S}_1\times I, & v_i n_i = 0 & \text{on } S_2\times I,\\
\theta = 0 & \text{on } \overline{S}_3\times I, & (k\theta_{,i} + \ell_\alpha\mu_{\alpha,i})n_i = 0 & \text{on } S_4\times I,\\
\mu_\alpha = 0 & \text{on } \overline{S}_5\times I, & (\lambda_\alpha\theta_{,i} + \lambda_{\alpha\beta}\mu_{\beta,i})n_i = 0 & \text{on } S_6\times I.
\end{array}
\tag{4.2}
$$

In view of Lemma 3.1, the initial problem has been reduced to the study of the equations (4.1) with the boundary conditions (4.2).

We introduce the notion of an admissible state $U = (p, v_i, \theta, \mu_\alpha)$ by which we mean an ordered array of functions $p, v_i, \theta, \mu_\alpha$ of class C^2 on $B\times(0,\infty)$ and of class $C^{1,0}$ on $\overline{B}\times I$ (for the definition of the class $C^{M,N}$ see [6], p. 24).

Let $\mathcal{D}$ be the set of all admissible states U that satisfy the conditions (4.2). We introduce the operators Φ_s $(s = 1, 2, ..., 5)$ and Ψ_α on $\mathcal{D}$, defined by

$$
\begin{aligned}
&\Phi_1 U = \rho_0(Ap + D\theta + A_\alpha\mu_\alpha) - \bar{v}_{i,i},\\
&\Phi_{i+1} U = \rho_0 v_i + \bar{p}_{,i},\\
&\Phi_5 U = \frac{1}{\theta_0}(k\bar{\theta}_{,ii} + \ell_\alpha\bar{\mu}_{\alpha,ii}) - \rho_0(C\theta - Dp + B_\alpha\mu_\alpha),\\
&\Psi_\alpha U = \lambda_\alpha\bar{\theta}_{,ii} + \lambda_{\alpha\beta}\bar{\mu}_{\beta,ii} - P_{\alpha\beta}\bar{\mu}_\beta - \rho_0(C_{\alpha\beta}\mu_\beta + B_\alpha\theta - A_\alpha p),
\end{aligned}
\tag{4.3}
$$

for every $U = (p, v_i, \theta, \mu_\alpha) \in \mathcal{D}$. If we introduce the notations

$$
\begin{aligned}
&\mathcal{L}U = (\Phi_1 U, \Phi_2 U, ..., \Phi_5 U, \Psi_1 U, ..., \Psi_N U),\\
&\mathcal{F} = (\rho_0 M, \rho_0 G_1, \rho_0 G_2, \rho_0 W, \rho_0 M_1, ..., \rho_0 M_N),
\end{aligned}
\tag{4.4}
$$

then the equations (4.1) can be written in the form

$$
\mathcal{L}U = \mathcal{F}. \tag{4.5}
$$

Clearly, the problem consists in the determination of an admissible state U belonging to $\mathcal{D}$ that satisfies (4.5).

Let F be a functional defined on $\mathcal{D}$, and $U, V \in \mathcal{D}$. Clearly, $U+\lambda V \in \mathcal{D}$ for every scalar λ. We introduce the notation

$$\delta_V F\{U\} = \frac{d}{d\lambda} F\{U + \lambda V\}\big|_{\lambda-0}. \tag{4.6}$$

We write $\delta F\{U\} = 0$ if $\delta_V F\{U\}$ exists and equals zero for every choice of $V \in \mathcal{D}$.

Let $U = (p, v_i, \theta, \mu_\alpha)$ and $V = (p', v_i', \theta', \mu_\alpha')$ be two $(N+5)$–dimensional vector fields, continuous on $\overline{B}\times I$. We introduce the notation

$$(U \circledast V) = \int_B [p * p' + v_i * v_i' + \theta' + \mu_\alpha * \mu_\alpha'](\mathbf{x}, t) d\mathbf{x}. \tag{4.7}$$

We assume for the remainder of the paper that (3.16) holds.

If $U, V \in \mathcal{D}$, then Theorem 3.2 implies that

$$(V \circledast \mathcal{L}U) = (U \circledast \mathcal{L}V). \tag{4.8}$$

We consider the functional $\mathcal{H}$ on $\mathcal{D}$ defined by

$$\mathcal{H}\{U\} = (U \circledast \mathcal{L}U) - 2(U \circledast \mathcal{F}). \tag{4.9}$$

By (4.6) and (4.8) we obtain

$$\delta_V \mathcal{H}\{U\} = 2((\mathcal{L}U - \mathcal{F}) * V). \tag{4.10}$$

In view of a fundamental lemma established by GURTIN ([11], Sect.64, Lemma 1) it follows that $\delta\mathcal{H}\{U\} = 0$, if and only if $U \in \mathcal{D}$ satisfies the equation (4.5).

From (4.2) and (4.3), with the aid of the divergence theorem, we obtain

$$\begin{aligned}
\mathcal{H}(U) = \int_B \big\{ & \rho_0 [Ap * p + 2Dp * \theta + 2A_\alpha p * \mu_\alpha - 2B_\alpha \mu_a * \theta - \\
& - C\theta * \theta - C_{\alpha\beta}\mu_\alpha * \mu_\beta] + 2\gamma * p_{,i} * v_i + \rho_0 v_i * v_i - \\
& - \gamma * \Big[\frac{1}{\theta_0} (k\theta_{,i} + \ell_\alpha \mu_{\alpha,i}) * \theta_{,i} + (\lambda_\alpha \theta_{,i} + \lambda_{\alpha\beta}\mu_{\beta,i}) * \mu_{\alpha,i} - \\
& - P_{\alpha\beta}\mu_\beta * \mu_\alpha \Big] \big\} d\mathbf{x} - 2 \int_B \rho_0 (M * p + G_i * v_i + W * \theta + M_\alpha * \mu_\alpha) d\mathbf{x}.
\end{aligned} \tag{4.11}$$

Thus we have the following result

Theorem 4.1. *Assume that the relations* (3.16) *hold. Let* $\mathcal{H}$ *be the functional on* $\mathcal{D}$ *defined by* (4.1). *Then*

$$\delta\mathcal{H}\{U\} = 0,$$

at $U \in \mathcal{D}$ *if and only if* U *is a solution of the problem.*

The case of non–zero boundary data can be treated as in [12].

Now we show that Theorem 3.1 implies a minimum principle. We say that φ is bounded at infinity if $\lim_{t\to\infty} \varphi(\mathbf{x}, t)$ exists for each $\mathbf{x}$ in the domain of definition of φ. Let X be the set of all functions φ on $\overline{B}\times I$ with the following properties:

(i) φ is of class C^2 on $B\times(0,\infty)$ and of class $C^{1,0}$ on $\overline{B}\times I$;
(ii) $\varphi, \varphi_{,i}, \dot{\varphi}$ and $\varphi_{,ij}$ are bounded at infinity.

Following [9], [10], we introduce the set Γ of admissible weight functions. We say that $g \in \Gamma$ if g is a function on I with the following properties:

$$(\alpha) \qquad \int_0^\infty \int_0^\infty g^{[k]}(t+s)dt\,ds \text{ exists for } k \geq 0;$$

$$(\beta) \qquad g(t) = \int_0^\infty G(p)e^{-pt}dp,\ t \in I,$$

where $g^{[k]}(t) = \partial^k g/\partial t^k$, and G is continuous and positive on I and has a finite limit at infinity. An example of weight function is $g(t) = (t+a)^{-n}$, $n > 2$, $a > 0$, with $G(t) = [t^{n-1}\exp(-at)]/(n-1)!$.

Let Y be the set of all continuous functions on $\overline{B}\times I$ that are bounded at infinity. Let $\varphi, \psi \in Y$. We introduce the notation

$$[\varphi, \psi]_g = \int_0^\infty \int_0^\infty \int_B g(t+s)\varphi(\mathbf{x}, t)\psi(\mathbf{x}, s)d\mathbf{x}\,dt\,ds, \tag{4.12}$$

where g is any fixed function of Γ. We note that

$$[\varphi, \psi]_g = [\psi, \varphi]_g.$$

Let A be a linear operator from X into Y. We denote by D_A the domain of definition of the operator A. We say that the operator A is g-symmetric if

$$[A\varphi, \psi]_g = [\varphi, A\psi]_g, \tag{4.13}$$

for any $\varphi, \psi \in \Gamma$. A g-symmetric operator A is called positive if

$$[A\varphi, \varphi]_g \geq 0, \tag{4.14}$$

for any $\varphi \in D_A$. We consider the equation

$$Au = f, \tag{4.15}$$

where $f \in Y$. Let F_g be the functional on D_A defined by

$$F_g\{u\} = [Au, u]_g - 2[u, f]_g, \tag{4.16}$$

for every $u \in D_A$. Let $u, u^0 \in D_A$ and define $w = u^0 - u$. Clearly, if A is g–symmetric, then

$$F_g\{u^0\} = F_g\{u\} + 2[Au - f, w]_g + [Aw, w]_g.$$

An immediate consequence of this relation is the following well–known result

Lemma 4.1. *Let A be a g–positive operator, and let F_g be the functional on D_A defined by* (4.16). *Further, let u be a solution of the equation* (4.15). *Then*

$$F_g\{u\} \leq F_g\{u^0\},$$

for every $u^0 \in D_A$.

In what follows we denote by φ^* the Laplace transform with respect to time of the function φ. By (β) and (4.12),

$$[\varphi, \psi]_g = \int_0^\infty \int_B G(s)\varphi^*(\mathbf{x}, s)\psi^*(\mathbf{x}, s)d\mathbf{x}\, ds, \tag{4.17}$$

for any $\varphi, \psi \in X$.

Let $U = (p, v_i, \theta, \mu_\alpha)$ and $V = (p', v_i', \theta', \mu_\alpha')$ be two $(N+5)$–dimensional vector fields. We introduce the notation

$$[U, V]_g = [p, p']_g + \sum_{i=1}^{3}[v_i, v_i']_g + [\theta, \theta']_g + \sum_{\alpha=1}^{N}[\mu_\alpha, \mu_\alpha']_g. \tag{4.18}$$

Let D_1 be the set of all admissible states $U = (p, v_i, \theta, a\mu_\alpha)$ from $\mathcal{D}$ with the property that p, v_i, θ, and μ_α belong to X. We define the functional Z on D_1 by

$$\begin{aligned} Z(U) = \int_0^\infty \int_0^\infty \int_B g(t+s)\{\rho_0[Ap(t)p(s) + 2Dp(t)\theta(s) + \\ +2A_\alpha p(t)\mu_\alpha(s) - 2B_\alpha\mu_\alpha(s)\theta(t) - C\theta(t)\theta(s) - C_{\alpha\beta}\mu_\alpha(t)\mu_\beta(s)] + \\ +p_{,i}(t)\bar{v}_i(s) + v_i(t)\bar{p}_{,i}(s) + \rho_0 v_i(t)v_i(s) - \\ -\frac{1}{\theta_0}\theta_{,i}(t)[k\bar{\theta}_{,i}(s) + \ell_\alpha\bar{\mu}_{\alpha,i}(s)] - \mu_{\alpha,i}(t)[\lambda_\alpha\bar{\theta}_{,i}(s) + \lambda_{\alpha\beta}\bar{\mu}_{\beta,i}(s)] - \\ -P_{\alpha\beta}\bar{\mu}_\beta(s)\mu_\alpha(t)\}d\mathbf{x}\, dt\, ds. \end{aligned} \tag{4.19}$$

Clearly, we have

$$Z(U) = \int_0^\infty \int_B G(t)\{\rho_0[Ap^*(t))^2 + 2Dp^*(r)\theta^*(s) + \\ +2A_\alpha p^*(r)\mu^*_\alpha(r) - 2B_\alpha\theta^*(r)\mu^*_\alpha(r) - C(\theta^*(r))^2 - C_{\alpha\beta}\mu^*_\alpha(r)\mu^*_\beta(r)] + \\ +2rp^*_{,i}(r)v^*_i(r) + \rho_0 v^*_i(r)v^*_i(r) - \\ -r\big[\frac{1}{\theta_0}k\theta^*_{,i}(r) + 2\lambda_\alpha\mu^*_{\alpha,i}(r)\theta^*_{,i}(r) + \lambda_{\alpha\beta}\mu^*_\alpha(r)\mu^*_{\beta,i}(r) + \\ +P_{\alpha\beta}\mu^*_\alpha(r)\mu^*_\beta(r)\big]d\mathbf{x}\,dr.$$

Let D_0 be the set of all admissible states $U \in D_1$ that satisfy $Z(U) \geq 0$. We contract the operator $\mathcal{L}$ by replacing its domain of definition $\mathcal{D}$ with the more restricted domain D_0. We denote this contracted operator by L.

Lemma 4.2. *Assume that* (3.16) *holds. Then, the operator L is g–positive.*

Proof. Consider two systems of loading $\Sigma^{(\nu)}$ $(r = 1, 2)$ with null boundary data. Let U and V be solutions corresponding to $\Sigma^{(1)}$ and $\Sigma^{(2)}$, respectively. Then

$$LU = \mathcal{F}^{(1)},\ LV = \mathcal{F}^{(2)},$$

where $\mathcal{F}^{(\nu)} = (\rho_0 M^{(\nu)}, \rho_0 G_1^{(\nu)}, ..., \rho_0 M_N^{(\nu)})$, $(\nu = 1, 2)$. From (3.17) and (4.17) we obtain

$$[LU, V]_g = [U, LV]_g. \tag{4.20}$$

By (4.2) and (4.3) we get

$$[U, LU]_g = Z(U). \tag{4.21}$$

The relations (4.20) and (4.21) imply the desired result. ∎

In view of Leemmas 4.1, 4.2 and (4.21), we obtain

Theorem 4.2. *Assume that the relations* (3.16) *hold. Let $\mathcal{M}$ be the functional on D_0 defined by*

$$\mathcal{M}\{U\} = Z(U) - 2[U, \mathcal{F}]_g,$$

and let $U \in D_0$ be a solution of the equations (4.1). *Then*

$$\mathcal{M}\{U\} \leq \mathcal{M}\{U^0\},$$

for every $U^0 \in D_0$.

Theorem 4.3. *Assume that the relations* (3.16) *hold. Let* T *be the set of all admissible states* $U \in D_1$ *that satisfy* $Z(U) \leq 0$*, and let* $\mathcal{N}$ *be the functional on* T *defined by*

$$\mathcal{N}\{U\} = -Z(U) - 2[U, \mathcal{F}]_g.$$

Further, let $U \in T$ *be a solution of the equations* (4.1)*. Then*

$$\mathcal{N}\{U\} \leq \mathcal{N}\{U'\},$$

for every $U' \in T$.

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