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FLUX LIMITERS FOURTH, SIXTH AND EIGHTH ORDER HIGH RESOLUTION METHODS FOR HYPERBOLIC CONSERVATION LAWS

BY

HASSAN N.A. ISMAIL* **and FATMA A.A. ZAID****

Abstract. This paper presents a generalization for Harten's theorem for total variation non–increasing implicit method in the case of five, seven up to $2m+1$ points for non zero integer m. Also it obtained the high resolution, total variation non increasing oscillation free of fourth, six and eighth order accurate explicit method by adding suitable number of limiters or antidiffusive flux to a first order scheme. The CFL condition is still satisfied. Also it presents the modification scheme for these methods to give high accuracy. Later, we present a numerical example and a comparison between the methods.

Introduction. In this paper we consider numerical solutions of the scalar conservation law of the form

$$u_t + f(u)_x = 0, \tag{1}$$

where $f(u)$ is a nonlinear function of u, with the initial condition of Riemann problem:

$$u(x,0) = \begin{cases} u_L & x < 0, \\ u_R & x > 0, \end{cases} \tag{2}$$

[MSM1] where u_L and u_R are the states on the left right sides respectively. Eq (1) may take the form:

$$u_t + a(u)u_x = 0, \tag{2}$$

where $a(u)$ is called a *signal speed.* The discontinuous solutions known as weak solutions which satisfy equation (1) in the sense that:

$$\int_0^\infty \int_{-\infty}^\infty [\varphi_t u + \varphi_x f(u)] dx\, dt + \int_{-\infty}^\infty \varphi(x,0)u(x,0)dx = 0$$

holds, where the test function $\varphi(x,t)$ is C^∞. It is known that the piecewise smooth weak solution of equation (1) satisfies it, and it is pointwise in each smooth region. Accross each point of discontinuity the Rankine–Hugoniot relation

$$f(u_R) - f(u_L) = s(u_R - u_L)$$

holds, where s is the speed of propagation of the discontinuity.

A physical relevant weak solution can be obtained from the equation

$$u_t + f(u)_x = \varepsilon u_{xx}$$

with ε the viscosity, the physically relevant weak solution is defined as the stable limit of a vanishing viscosity mechanism [12].

On the other hand, there is the entropy condition to the solution such that a physically relevant solution is obtained. Therefore assume that equation (1) possesses an entropy function $\eta(u)$, which is defined as follows:

(i) $\eta(u)$ satisfies $\eta_u f_u = \psi_u$, where $\psi(u)$ is some other function called the *entropy flux.*

(ii) $\eta(u)$ is a convex function of u.

Admissible weak solutions $u(x,t)$ of equation (1) satisfy, in the weak sense, the following inequality

$$\eta(u)_t + \psi(u)_x \leq 0, \tag{3}$$

which is called an *entropy condition.*

In the following we shall discuss numerical grid by the partition parameters (h,k) and let $v_j^n = v(jh, nk)$. Then the numerical approximation to weak solutions of (1) which are obtained by $2K+1$)–points explicit schemes in conservation form

$$v_j^{n+1} = v_j^n - \lambda \left(F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}} \right), \tag{4}$$

where $F^n_{j+\frac{1}{2}} = F\left(v^n_{j-K+1}, v^n_{j-K}, ..., v^n_{j+K}\right)$.

Here F is the numerical flux function. It can be shown [9] that for consistency of a conservative method, it is sufficient to require the flux function F of the corresponding scheme (4) to be Lipschitz continuous and to satisfy

$$F(u,u,...,u) = f(u).$$

We say that the numerical scheme (4) is consistent with the entropy condition (3) if an inequality of the following kind is satisfied

$$\eta_j^{n+1} \leq \eta_j^n - \lambda\left(\psi_{j+\frac{1}{2}}^n - \psi_{j-\frac{1}{2}}^n\right)$$

where $\eta_j^n = \eta(v_j^n)$, $\psi_{j+\frac{1}{2}}^n = \psi\left(v_{j-K+1}^n, v_{j-K+2}^n, ..., v_{j+K}^n\right)$; here $\psi_{j+\frac{1}{2}}^n$ is a numerical entropy flux, consistent with the entropy flux $\psi(u)$, i.e. $\psi(u,u,...,u) = \psi(u)$.

2. Generalization of Harten's convergence theorem for higher order method. In this section we review the theory that has been developed by HARTEN [6] for explicit TVNI (total variation non increasing) schemes and we will generalize Harten's theorem for TVNI in the case of five, seven and $2m+1$ points finite difference shcemes.

We first consider the linear scalar equation

$$u_t + au_x = 0,\ a > 0.$$

It is well known that the *total variation* of a mesh function v is defined by

$$TV(v) = \sum_{j=-\infty}^{\infty} \left|\Delta v_{j+\frac{1}{2}}\right| \tag{6}$$

where

$$\Delta v_{j+\frac{1}{2}} = v_{j+1} - v_j \tag{7}$$

and we say that the numerical scheme (4) is *TVNI* if

$$TV(v^{n+1}) \leq TV(v^n). \tag{8}$$

If the general scheme (4) is rewritten in the form

$$v_j^{n+1} = v_j^n + C_1\Delta v_{j+\frac{1}{2}}^n - C_2\Delta v_{j-\frac{1}{2}}^n \tag{9}$$

where C_1 and C_2 are date–dependent coefficients, then it is easily shown [6], [7] that sufficient conditions for the scheme to be TVNI are the inequalities:

$$C_1 \leq 0,\ C_2 \geq 0,\ C + 1 + C_2 \leq 1 \tag{10}$$

holds. Now we formulate the sufficient conditions for five, seven and $2m+1$ points schemes in the two following lemmas:

Lemma 1. *Let z be a finite difference operator with 5–point in the form:*

$$(z \cdot v)_j = v_j + C_1 \Delta v_{j+\frac{3}{2}} + C_2 \Delta v_{j+\frac{1}{2}} - C_3 \Delta v^n_{j-\frac{1}{2}} - C_4 \Delta v_{j-\frac{3}{2}}. \tag{11}$$

(i) *If for all j*

$$0 \leq C_1 \leq C_2,\ 0 \leq C_4 \leq C_3,\ C_2 + C_3 \leq 1, \tag{12}$$

then z is TVNI. *Note that the number of conditions is equal to five.*

(ii) *If for all j*

$$-\infty < c \leq C_2 \leq C_1 \leq 0,\ -\infty < c \leq C_3 \leq C_4 \leq 0, \tag{13}$$

then z is TVI. *Note that the number of conditions is equal to four.*

Proof. Subtracting equation (11) at $j = i$ from (11) at $j = i+1$ to obtain

$$\begin{aligned} \Delta w_{i+\frac{1}{2}} = {} & C_1 \Delta v_{i+\frac{5}{2}} + (C_2 - C_1)\Delta v_{i+\frac{3}{2}} + (1 - C_2 - C_3)\Delta v_{i+\frac{1}{2}} + \\ & + (C_3 - C_4)\Delta v_{i-\frac{1}{2}} + C_4 \Delta v_{i-\frac{3}{2}}, \end{aligned} \tag{14}$$

where $w = z \cdot v$, by (12) all the coefficients in equation (14) are non–negative, therefore

$$\begin{aligned} \left|\Delta w_{i+\frac{1}{2}}\right| \leq {} & C_1 \left|\Delta v_{i+\frac{5}{2}}\right| + (C_2 - C_1)\left|\Delta v_{i+\frac{3}{2}}\right| + \\ & + (1 - C_2 - C_3)\left|\Delta v_{i+\frac{1}{2}}\right| + (C_3 - C_4)\left|\Delta v_{i+\frac{1}{2}}\right| + C_4 \left|\Delta v_{i+\frac{3}{2}}\right|. \end{aligned}$$

Summing each side of this inquality for $i \in (\infty, \infty)$ we obtain the following result

$$TV(z \cdot v) \equiv \sum_{i=-\infty}^{\infty} \left|\Delta w_{i+\frac{1}{2}}\right| \leq \sum_{i=-\infty}^{\infty} C_1 \left|\Delta v_{i+\frac{5}{2}}\right| + \sum_{i=-\infty}^{\infty} (C_2 - C_1)\left|\Delta v_{i+\frac{3}{2}}\right| +$$
$$+ \sum_{i=-\infty}^{\infty} (1 - C_2 - C_3)\left|\Delta v_{i+\frac{1}{2}}\right| + \sum_{i=-\infty}^{\infty} (C_3 - C_4)\left|\Delta v_{i-\frac{1}{2}}\right| +$$
$$+ \sum_{i=-\infty}^{\infty} C_4 \left|\Delta v_{i-\frac{3}{2}}\right| = \sum_{i=-\infty}^{\infty} \left|\Delta v_{i+\frac{1}{2}}\right| \equiv TV(v).$$

This completes the proof of part (i).

To prove part (ii) we rewrite (14) as

$$\Delta w_{i+\frac{1}{2}} + (-C_1)\Delta v_{i+\frac{5}{2}} + (-C_2 + C_1)\Delta v_{i+\frac{3}{2}} + (-C_3 + C_4)\Delta v_{i-\frac{1}{2}} +$$
$$+(-C_4)\Delta v_{i-\frac{3}{2}} = (1 - C_2 - C_3)\Delta v_{i+\frac{1}{2}} \tag{15}$$

and observe that (13) is the condition for the coefficients of $\Delta v_{i+p+\frac{1}{2}}$, $p = 0, \pm, \pm 2$ in (15) to be non–negative. Therefore

$$\left|\Delta w_{i+\frac{1}{2}}\right| - C_1\left|\Delta v_{i+\frac{5}{2}}\right| - (C_2 - C_1)\left|\Delta v_{i+\frac{3}{2}}\right| - (C_3 - C_4)\left|\Delta v_{i-\frac{1}{2}}\right| -$$
$$-C_4\left|\Delta v_{i-\frac{3}{2}}\right| \geq (1 - C_2 - C_3)\left|\Delta v_{i+\frac{1}{2}}\right|.$$

Summing each side of this inequality for $i \in (-\infty, \infty)$, we obtain the following result

$$TV(z \cdot v) - \sum_{i=-\infty}^{\infty} (C_2 - C_3)\left|\Delta v_{i+\frac{1}{2}}\right| \equiv \sum_{i=-\infty}^{\infty} \left|\Delta w_{i+\frac{1}{2}}\right| - \sum_{i=-\infty}^{\infty} C_1\left|\Delta v_{i+\frac{5}{2}}\right| -$$
$$- \sum_{i=-\infty}^{\infty} (C_2 - C_1)\left|\Delta v_{i+\frac{3}{2}}\right| - \sum_{i=-\infty}^{\infty} (C_3 - C_4)\left|\Delta v_{i-\frac{1}{2}}\right| - \sum_{i=-\infty}^{\infty} C_4\left|\Delta v_{i-\frac{3}{2}}\right| \geq$$
$$\geq \sum_{i=-\infty}^{\infty} (1 - C_2 - C_3)\left|\Delta v_{i+\frac{1}{2}}\right| \equiv TV(v) - \sum_{i=-\infty}^{\infty} (C_2 + C_3)\left|\Delta v_{i+\frac{1}{2}}\right|.$$

This completes the proof of part (ii). ∎

Lemma 2. *Let z be a finite difference operator with 7–point in the form:*

$$(z\cdot v)_j = v_j + C_1\Delta v_{j+\frac{5}{2}} + C_2\Delta v_{j+\frac{3}{2}} + C_3\Delta v^n_{j-\frac{1}{2}} - \\ -C_4\Delta v_{j-\frac{1}{2}} - C_5\Delta v_{j-\frac{1}{2}} - C_6\Delta v_{j-\frac{5}{2}} \tag{16}$$

(i) *If for all j*

$$0 \le C_1 \le C_2 \le C_3,\ 0 \le C_6 \le C_5 \le C_4,\ C_3 \le C_4 \le 1, \tag{17}$$

then z is TVNI. *Note that the number of conditions is equal to seven.*

(ii) *If for all j*

$$-\infty < c \le C_3 \le C_2 \le C_1 \le 0,\ -\infty < c \le C_4 \le C_5 \le C_6 \le 0, \tag{18}$$

then z is TVI. *Note that the number of conditions is equal to six.*

Proof. Subtracting equation (16) at $j = i$ from (17) at $j = i+1$ to obtain

$$\Delta w_{i+\frac{1}{2}} = C_1\Delta v_{i+\frac{7}{2}} + (C_2 - C_1)\Delta v_{i+\frac{5}{2}} + (C_3 - C_2)\Delta v_{i+\frac{3}{2}} + \\ (1 - C_3 - C_4)\Delta v_{i+\frac{1}{2}} + (C_4 - C_5)\Delta v_{i-\frac{1}{2}} + \\ +(C_5 - C_6)\Delta v_{i-\frac{3}{2}} + C_6\Delta v_{i-\frac{5}{2}}. \tag{19}$$

Using (18) we find all the coefficients in equation (19) are non–negative, therefore

$$\left|\Delta w_{i+\frac{1}{2}}\right| \le C_1\left|\Delta v_{i+\frac{7}{2}}\right| + (C_2 - C_1)\left|\Delta v_{i+\frac{5}{2}}\right| + (C_3 - C_2)\left|\Delta v_{i+\frac{3}{2}}\right| + \\ +(1 - C_3 - C_4)\left|\Delta v_{i+\frac{1}{2}}\right| + \\ +(C_4 - C_5)\left|\Delta v_{i-\frac{1}{2}}\right| + (C_5 - C_6)\left|\Delta v_{i-\frac{3}{2}}\right| + C_6\left|\Delta v_{i-\frac{5}{2}}\right|.$$

Summing each side of this inequality for $i \in (-\infty, \infty)$, we obtain the following result

$$\begin{aligned} TV(z \cdot v) \equiv \sum_{i=-\infty}^{\infty} \left|\Delta w_{i+\frac{1}{2}}\right| \le \sum_{i=-\infty}^{\infty} C_1 \left|\Delta v_{i+\frac{7}{2}}\right| + \sum_{i=-\infty}^{\infty} (C_2 - C_1)\left|\Delta v_{i+\frac{5}{2}}\right| + \\ + \sum_{i=-\infty}^{\infty} (C_3 - C_2)\left|\Delta v_{i+\frac{3}{2}}\right| + \sum_{i=-\infty}^{\infty} (1 - C_3 - C_4)\left|\Delta v_{i+\frac{1}{2}}\right| + \\ + \sum_{i=-\infty}^{\infty} (C_4 - C_5)\left|\Delta v_{i-\frac{1}{2}}\right| + \sum_{i=-\infty}^{\infty} (C_5 - C_6)\left|\Delta v_{i-\frac{3}{2}}\right| + \\ + \sum_{i=-\infty}^{\infty} C_6 \left|\Delta v_{i-\frac{5}{2}}\right| = \sum_{i=-\infty}^{\infty} \left|\Delta v_{i+\frac{1}{2}}\right| \equiv TV(v) \end{aligned}$$

which shows that (8) is satisfied, and we can prove the part (ii) as similar to part (ii) of Lemma 1.

Lemma 3. *In general, if we consider a finite difference with* $2m+1$*–point in the form:*

$$\begin{aligned} (z \cdot v)_j = v_j^n + C_1 \Delta v_{j+\frac{2m-1}{2}} + C_2 \Delta v_{j+\frac{2m-3}{2}} + \cdots + \\ + C_m \Delta v_{j+\frac{1}{2m+1}} - C_{m+1} \Delta v_{j-\frac{1}{2}} - C_{m+2} \Delta v_{j-\frac{3}{2}} - \cdots - \\ - C_{2m} \Delta v_{j-\frac{2m-1}{2}}. \end{aligned} \tag{20}$$

(i) *If for all* j

$$\begin{gathered} 0 \le C_1 \le C_2 \le \cdots \le C_m, \ 0 \le C_{2m} \le C_{2m-1} \le \cdots \le C_{m+1}, \\ C_m + C_{m+1} \le 1, \end{gathered} \tag{21}$$

then z *is* TVNI. *Note that the number of conditions is equal to* $2m+1$.

(ii) *If for all* j

$$\begin{gathered} -\infty \le c \le C_m \le C_{m-1} \le \cdots \le C_1 \le 0, \\ -\infty \le c \le C_{m+1} \le C_{m+2} \le \cdots \le C_{2m} \le 0, \end{gathered} \tag{22}$$

then z *is* TVI. *Note that the number of conditions is equal to* $2m$.

3. High resolution explicit schemes. We know that a monotone numerical method is consistent of at most order 1, therefore VAN LEER [13], ROE [10], CHAKRAVARTHY and OSHER [4] and SWEBY [11] derived a scheme using flux limiter. They present a higher resolution TVNI scheme in the same way as the flux corrected transport (FCT) of BORIS and BOOK [3] and ZALESAK [14], they derive a high resolution *second order accurate* schemes by means of adding *a limited of antidiffusive flux* to a gene4ral entropy inequality which satisfy a first order scheme. The constraints on the limiter, as functions of gradient ratios, have been obtained so that the resulting scheme is TVNI. Aslo SWEBY [11] gives several examples of limiter and presents some numerical comparisons. Hybrid method of this form were also introduced by HARTEN and ZWAS [8] and HARTEN [5].

In this section we shall derive a high resolution TVNI for new fourth, sixth and eighth order, explicit schemes by using the flux limiter method. First, we consider the linear scalar equation

$$u_t + au_x = 0,\ a > 0 \tag{23}$$

with the initial value

$$u(x,0) = \begin{cases} u_L & x < 0, \\ u_R & x > 0, \end{cases} \tag{24}$$

where $u(x,0)$ is assumed to be a function of bounded total variation in the interval $(-\infty, \infty)$.

3.1. Fourth order scheme. ABARBANEL, GOTTLIEBB and TURKEL [1] present explicit method of fourth order accurate (in space and time, i.e. a $4-4$ scheme), two–level, 4–steps for the solution of (13). A modified for this method is written in compact form as:

$$\begin{aligned} v_j^{n+1} = v_j^n - \lambda[&A_1(v_{j+2}^n - v_{j+1}^n) + A_2(v_{j+1}^n - v_j^n) + \\ &+A_3(v_j^n - v_{j-1}^n) + A_4(v_{j-1}^n - v_{j-2}^n)] \end{aligned} \tag{25}$$

where

$$\begin{aligned} A_1 &= \frac{1}{12}\left(-1 + \frac{1}{2}\lambda + \lambda^2 - \frac{1}{2}\lambda^3\right), A_2 = \frac{1}{12}\left(7 - \frac{15}{2}\lambda - \lambda^2 + \frac{3}{2}\lambda^3\right), \\ A_3 &= \frac{1}{12}\left(7 + \frac{15}{2}\lambda - \lambda^2 - \frac{3}{2}\lambda^3\right), A_4 = \frac{1}{12}\left(-1 - \frac{1}{2}\lambda + \lambda^2 + \frac{1}{2}\lambda^3\right). \end{aligned} \tag{26}$$

By numerical experiments, we found that the scheme (25) with Riemann problem (24) gives oscillations around the discontinuity, as shown in fig. 2a.

Therefore we use the flux limiter approach. For clarity of this approach, the sechem (25) may be written as:

$$v_j^{n+1} = v_j^n - \lambda(v_j^n - v_{j-1}^n) - \frac{\lambda}{12}\Delta_- \left[\sum_{i=1}^{3} B_i \Delta v^n_{j+\frac{5}{2}-i}\right] \tag{27}$$

where Δ and Δ_- are given from (7), and

$$\begin{aligned} B_1 &= \left(-1 + \frac{1}{2}\lambda + \lambda^2 - \frac{1}{2}\lambda^3\right), \ B_2 = (6 - 7\lambda + \lambda^3), \\ B_3 &= \left(1 + \frac{1}{2}\lambda - \lambda^2 - \frac{1}{2}\lambda^3\right). \end{aligned} \tag{28}$$

It is seen that (27) is in fact the result of a first order upwind scheme

$$v_j^{n+1} = v_j^n - \lambda(v_j^n - v_{j-1}^n) \tag{29}$$

with an additional term

$$-\frac{\lambda}{12}\Delta_- \left[\sum_{i=1}^{3} B_i \Delta v^n_{j+\frac{5}{2}-i}\right]. \tag{30}$$

Also, we found that the numerical flux of the $4-4$ scheme is the numerical flux of the first order scheme (29) plus an additional flux

$$\frac{1}{12}\left[\sum_{i=1}^{3} B_i \Delta v^n_{j+\frac{5}{2}-i}\right] \tag{31}$$

which is called an *antidiffusive flux*.

Now we try to treat osscilation in this method by adding *only a limited* amount of antidiffusive flux (31) to the first order scheme i.e.

$$v_j^{n+1} = v_j^n - \lambda(v_j^n - v_{j-1}^n) - \frac{\lambda}{12}\Delta_- \left[\sum_{i=1}^{3} B_i \Delta v^n_{j+\frac{5}{2}-i}\right] \phi_j \tag{32}$$

where ϕ_j is some nonnegative form limiter. Like ROE [10], VAN LEER [13], and CHAKRAVARTHY and OSHER [4] we take the limiter to be a function of consecutive gradients (in the liner case), i.e., $\phi_j = \phi(\theta_j)$ where

$$\theta_j = \frac{\Delta v_{j-\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}}. \tag{33}$$

We found that the scheme (32) is not enough to give TVNI i.e. it does not cancel totally the oscillations. Therefore we add *three limiters* amount of the antidiffusive flux (31) to the first order scheme i.e.

$$v_j^{n+1} = v_j^n - \lambda(v_j^n - v_{j-1}^n) - \frac{\lambda}{12}\left[\sum_{i=1}^{3}\phi_{ij}B_{ij}\Delta v^n_{j+\frac{5}{2}-i}\right] \tag{34}$$

where $\phi_{ij},\ i = 1, 2, 3$ are some non–negative form limiters. We take each of these limiters to be a function of the corresponding consecutive gradients (in the linear case), i.e.,

$$\phi_{ij} = \phi(\theta_{ij}),\ i = 1, 2, 3, \tag{35}$$

where

$$\theta_{ij} = \frac{\Delta v_{j+\frac{3}{2}-i}}{\Delta v_{j+\frac{5}{2}-i}} \quad i = 1, 2, 3. \tag{36}$$

An easy calculation using Taylor series expansions shows that this numerical method is fourth–order consistent if $\phi_{ij} = 1,\ i = 1, 2, 3$.

We now seek to choose the functions $\phi_{ij},\ i = 1, 2, 3$ in such a way that the limited antidiffusive flux (31) is maximized in amplitude subject to the constraint of the resulting scheme being TVNI. Viewing the scheme (5.18), we choose the coefficients C_1, C_2, C_3 and C_4 in (12) as:

$$\begin{aligned} &C_1 = 0,\ C_2 = -\frac{\lambda}{12}\Delta_-\left[\phi_{1j}B_1\Delta v_{j+\frac{3}{2}}\right]/\Delta v_{j+\frac{1}{2}}\ \forall i = 3, 4,\\ &C_i = \frac{\lambda}{12}D_-\left[\phi_{(i-1)j}B_{(i-1)}\Delta v_{j+\frac{7}{2}-i}\right]/\Delta v_{j+\frac{5}{2}-i} + \lambda\delta_{i3}. \end{aligned} \tag{40}$$

We can write (40) as

$$\begin{aligned} &C_1 = 0,\ C_2 = -\frac{\lambda}{12}B_1(\phi(\theta_{1j})/\theta_{1j} - \phi(\theta_{1(j-1)})\ \forall i = 3, 4,\\ &C_i = \frac{\lambda}{12}B_{i-1}\left(\phi(\theta_{(i-1)j})/\theta_{(i-1)j} - \phi(\theta_{(i-1)(j-1)})\right) + \lambda\delta_{i3} \end{aligned} \tag{41}$$

and the limitations on C_2, C_2, C_4 of the form

$$C_2 \in \left[0, -\frac{\lambda}{12}B_2\Phi_{2j}\right],\ |C_3 - \lambda| \le \frac{\lambda}{12}B_2\Phi_2,\ C_4 \in \left[0, \frac{\lambda}{12}B_3\Phi_{3j}\right] \tag{42}$$

where

$$\Phi_i = \left|\phi(\theta_{ij})/\theta_{ij} - \phi(\theta_{i(j-1)})\right|,\ i = 1, 2, 3. \tag{43}$$

Hence the first four inequalities of the set (12) are satisfied under CFL condition

$$\lambda \leq 1. \tag{44}$$

Using the conditions (12) and (42) we get

$$\lambda + \frac{\lambda}{12}(6 - 7\lambda + \lambda^3)\Phi_2 + \frac{\lambda}{12}\left(1 - \frac{1}{2}\lambda - \lambda^2 + \frac{1}{2}\lambda^3\right)\Phi_1 \leq 1, \tag{44a}$$

$$0 \leq \lambda - \frac{\lambda}{12}(6 - 7\lambda + \lambda^3)\Phi_2 \tag{44b}$$

For simplification, choosing $\Phi_1 \approx \Phi_2 \approx \Phi_3$, the condition (44) will take the form

$$\Phi \leq \frac{24}{\lambda(2-\lambda)(3-\lambda)}, \tag{45a}$$

$$\Phi \leq \frac{12}{(1-\lambda)(2-\lambda)(3+\lambda)}, \tag{45b}$$

respectively, choosing $\Phi \leq 2$ which satisfies the above two inequalities. If, in addition the requirement of ϕ_{ij}, $i = 1, 2, 3$ to be non–negative we also insist on

$$\phi(\theta_{ij}) = 0,\ \theta_{ij} \leq 0, \tag{46a}$$

then the limitation (43) is reduced to the condition

$$0 \leq (\phi(\theta_{ij})/\theta_{ij}, \phi(\theta_{ij})) \leq 2,\ i = 1, 2, 3. \tag{46b}$$

Hence for the scheme (32) to be TVNI the limiter functions ϕ_{ij}, $i = 1, 2, 3$ must lie in the shaded region of figure 1a. From our experiments, we found that the left oscillations are free under the condition (46) as shown in figure 3a.

3.2. Sixth order scheme. In this section we present a new finite difference in the form

$$
\begin{aligned}
v_j^{n+1} &= v_j^n - \lambda[A_1(v_{j+3}^n - v_{j+2}^n) + A_2(v_{j+2}^n - v_{j+1}^n) + \\
&+A_3(v_{j+1}^n - v_j^n) + A_4 1(v_j^n - v_{j-1}^n) + \\
&+A_5(v_{j-1}^n - v_{j-2}^n) + A_6(v_{j-2}^n - v_{j-3}^n)]
\end{aligned} \tag{47}
$$

where

$$
\begin{aligned}
A_1 &= \frac{1}{60}\left(1 - \frac{1}{3}\lambda - \frac{5}{4}\lambda^2 + \frac{5}{12}\lambda^3 + \frac{1}{4}\lambda^4 - \frac{1}{12}\lambda^5\right), \\
A_2 &= \frac{1}{60}\left(8 - \frac{25}{6}\lambda - \frac{35}{4}\lambda^2 - \frac{55}{12}\lambda^3 - \frac{3}{4}\lambda^4 - \frac{5}{12}\lambda^5\right), \\
A_3 &= \frac{1}{60}\left(37 - \frac{245}{6}\lambda - \frac{15}{2}\lambda^2 + \frac{35}{3}\lambda^3 + \frac{1}{2}\lambda^4 + \frac{5}{6}\lambda^5\right), \\
A_4 &= \frac{1}{60}\left(37 + \frac{245}{6}\lambda - \frac{15}{2}\lambda^2 - \frac{35}{3}\lambda^3 + \frac{1}{2}\lambda^4 - \frac{5}{6}\lambda^5\right), \\
A_5 &= \frac{1}{60}\left(-8 - \frac{25}{6}\lambda + \frac{35}{4}\lambda^2 + \frac{55}{12}\lambda^3 - \frac{3}{4}\lambda^4 - \frac{5}{12}\lambda^5\right), \\
A_6 &= \frac{1}{60}\left(1 + \frac{1}{3}\lambda - \frac{5}{4}\lambda^2 - \frac{5}{12}\lambda^3 + \frac{1}{4}\lambda^4 + \frac{1}{12}\lambda^5\right).
\end{aligned}
$$

The explicit scheme (47) is one step, two level, seven points in level n and one point in level $n+1$. An easy calculation using Taylor–series expansions show that the scheme (47) is consistent of sixth order (in space and time, i.e., $6-6$ scheme). Furthermore it is stable under the CFL condition (44). Also, this numerical method with problem (23) gives oscillations around the discontinuity, as shown in fig. 2b. Therefore we use the flux limiter method by the same technique of flux limiters fourth order by writing the scheme (47) in the form:

$$
v_j^{n+1} = v_j^n - \lambda(v_j^n - v_{j-1}^n) - \frac{\lambda}{60}\Delta_-\left[\sum_{i=1}^{5} B_u \Delta v_{j+\frac{7}{2}-i}^n\right] \tag{48}
$$

where Δ_- and Δ are defined in (7) and

$$
\begin{aligned}
B_1 &= \left(1 - \frac{1}{2}\lambda - \frac{5}{4}\lambda^2 + \frac{5}{12}\lambda^3 + \frac{1}{4}\lambda^4 - \frac{1}{12}\lambda^5\right), \\
B_2 &= \left(-7 + \frac{23}{6}\lambda + \frac{15}{2}\lambda^2 - \frac{50}{12}\lambda^3 - \frac{1}{2}\lambda^4 + \frac{1}{3}\lambda^5\right), \\
B_3 &= \left(30 - \frac{222}{6}\lambda + \frac{45}{6}\lambda^3 - \frac{1}{2}\lambda^5\right), \\
B_4 &= \left(7 + \frac{23}{6}\lambda - \frac{15}{2}\lambda^2 - \frac{50}{12}\lambda^3 + \frac{1}{2}\lambda^4 + \frac{1}{3}\lambda^5\right), \\
B_5 &= \left(-1 - \frac{1}{3}\lambda + \frac{5}{4}\lambda^2 + \frac{5}{12}\lambda^3 - \frac{1}{4}\lambda^4 - \frac{1}{12}\lambda^5\right).
\end{aligned} \tag{49}
$$

It is clear that the scheme (48) is in fact the result of first order scheme (29) with an additional term

$$
-\frac{\lambda}{60}\Delta_- \left[\sum_{i=1}^{5} B_i \Delta v^n_{j+\frac{7}{2}-i}\right]. \tag{50}
$$

Also, the antidiffusive flux of the scheme (48) takes the form

$$
\frac{1}{60}\left[\sum_{i=1}^{5} B_i \Delta v^n_{j+\frac{7}{2}-i}\right]. \tag{51}
$$

Now we shall add *five limiters* amount of antidiffusive flux (51) to the first order scheme as follows:

$$
v_j^{n+1} = v_j^n - \lambda\left(v_j^n - v_{j-1}^n\right) - \frac{\lambda}{60}\Delta_- \left[\sum_{i=1}^{5} \phi_{ij} B_i \Delta v^n_{j+\frac{7}{2}-i}\right] \tag{52}
$$

where $\phi_{ij},\ i = 1, 2, 3, 4, 5$ in the liner case take the form

$$
\phi_{ij} = \phi(\theta_{ij}),\ i = 1, 2, 3, 4, 5 \tag{53}
$$

and

$$
\phi_{ij} = \frac{\Delta v_{j+\frac{5}{2}-i}}{\Delta v_{j+\frac{7}{2}-i}},\ i = 1, 2, 3, 4, 5. \tag{54}
$$

Using Taylor–series expansions shows that this numerical method is sixth order consistent if $\phi_{ij} = 1$, $i = 1, 2, 3, 4, 5$. We now choose these functions such that the finite difference scheme (5.31) is TVNI. If we atry applying Lemma 5.2, then the numerical method (48) has to be written in the same form as (16). One possibility is to choose C_1, C_2, C_3, C_4, C_5 and C_6 to have the values

$$\begin{aligned} &C_1 = 0, \\ &C_i = -\frac{\lambda}{60} B_{i-1} \left(\frac{\phi_{(i-1)j}}{\theta_{(i-1)j}} - \phi_{(i-1)(j_1)} \right), \ \forall i = 2, 3, \\ &C_i = \frac{\lambda}{60} B_{i-1} \left(\frac{\phi_{(i-1)j}}{\theta_{(i-1)j}} - \phi_{(i-1)(j-1)} \right) + \lambda \delta_{i4}, \ \forall i = 4, 5, 6 \end{aligned}$$

and the limitation on C_2, C_3, C_4, C_5 and C_6 are such that

$$\begin{aligned} &\forall i = 2, 3, \\ &C_i \in \left[0, -\frac{\lambda}{60} B_{i-1} \Phi_{(i-1)j} \right], \ |C_4 - \lambda| \leq \frac{\lambda}{60} B_3 \Phi_3, \\ &\forall i = 5, 6 \ C_i \in \left[0, \frac{\lambda}{60} B_{i-1} \Phi_{(i-1)j} \right] \end{aligned} \tag{55}$$

where

$$\Phi_i = \left| \phi(\theta_{ij})/\theta_{ij} - \phi(\theta_{i(j-1)}) \right|, \ i = 1, 2, 3, 4, 5. \tag{56}$$

Hence the first six inequalities of the set (17) are satisfied under CFL condition (29). Using the conditions (17) and (55) we get

$$\begin{aligned} &\frac{\lambda}{60} \left(-7 + \lambda + \frac{30}{4}\lambda^2 - \frac{50}{12}\lambda^3 - \frac{1}{2}\lambda^4 + \frac{1}{3}\lambda^4 \right) \Phi_2 + \lambda + \\ &\quad + \frac{\lambda}{60} \left(30 - \frac{222}{6}\lambda + \frac{45}{6}\lambda^3 - \frac{1}{2}\lambda^5 \right) \Phi_3 \leq 1 \end{aligned} \tag{57a}$$

$$0 \leq \lambda - \lambda 60 \left(30 - \frac{222}{6}\lambda + \frac{45}{6}\lambda^3 - \frac{1}{2}\lambda^5 \right) \Phi_3. \tag{57b}$$

For simplification, choosing $\Phi_1 \approx \Phi_2 \approx \Phi_3 \approx \Phi_4 \approx \Phi_5 \approx \Phi$, then we have

$$\Phi \leq \frac{360}{\lambda(2-\lambda)(3-\lambda)((1+\lambda)(2\lambda+7) + 3(\lambda^2 + 6\lambda + 10))}, \tag{58a}$$

$$\Phi \leq \frac{120}{(2-\lambda)(3-\lambda)(1-\lambda)(\lambda^2 + 6\lambda + 10)}, \tag{58b}$$

respectively. Therefore we choose $\Phi \leq 2$ which satisfies the above two inequalities. Also, we consider the following condition on $\phi_{ij},\ i = 1, 2, 3, 4, 5$ to be non–negative

$$\phi(\theta_{ij}) = 0,\ \theta_{ij} \leq 0, \tag{59a}$$

then the constraints on $\phi_{ij},\ i = 1, 2, 3, 4, 5$ take the form

$$0 \leq \left(\frac{\phi(\theta_{ij})}{\theta_{ij}}, \phi(\theta_{ij})\right) \leq 2,\ i = 1, 2, 3, 4, 5. \tag{59b}$$

Hence for the scheme (48) to be TVNI the limiter functions $\phi_{ij},\ i = 1, 2, 3, 4, 5$ must lie in the shaded region of figure 1b. From our experiments, we found that the left oscillations are free under the condition (59) as shown in figure 3b.

3.3. Eighth order scheme. In this section we present a new finite difference scheme in the form

$$\begin{aligned} v_j^{n+1} &= v_j^n - \lambda[A_1(v_{j+4}^n - v_{j+3}^n) + A_2(v_{j+3}^n - v_{j+2}^n) + \\ &+ A_3(v_{j+2}^n - v_{j+1}^n) + A_4(v_{j+1}^n - v_j^n) + A_5(v_j^n - v_{j-1}^n) + \\ &+ A_6(v_{j-1}^n - v_{j-2}^n) + A_7(v_{j-2}^n - v_{j-3}^n) + A_8(v_{j-3}^n - v_{j-4}^n)] \end{aligned} \tag{60}$$

where

$$\begin{aligned} A_1 &= \frac{1}{240}\left(-\frac{6}{7} + 3\frac{\lambda}{14} + \frac{7\lambda^2}{6} - \frac{7\lambda^3}{24} - \frac{\lambda^4}{3} + \frac{\lambda^5}{12} + \frac{\lambda^6}{42} - \frac{\lambda^7}{168}\right), \\ A_2 &= \frac{1}{240}\left(\frac{58}{7} - \frac{17\lambda}{6} + \frac{65\lambda^2}{6} + \frac{89\lambda^3}{24} + \frac{8\lambda^4}{3} - \frac{11\lambda^5}{12} - \frac{5\lambda^6}{42} + \frac{\lambda^7}{24}\right), \\ A_3 &= \frac{1}{240}\left(-\frac{278}{7} + \frac{12\lambda}{6} + \frac{91\lambda^2}{2} - \frac{587\lambda^3}{24} - 6\lambda^4 + \frac{41\lambda^5}{12} + \frac{3\lambda^6}{14} - \frac{\lambda^7}{8}\right), \\ A_4 &= \frac{1}{240}\left(\frac{1066}{7} - \frac{1025\lambda}{6} - \frac{215\lambda^2}{6} + \frac{455\lambda^3}{8} + \frac{11\lambda^4}{3} - \frac{25\lambda^5}{4} - \frac{5\lambda^6}{42} + \frac{5\lambda^7}{24}\right), \\ A_5 &= \frac{1}{240}\left(\frac{1066}{7} - \frac{1025\lambda}{6} - \frac{215\lambda^2}{6} - \frac{455\lambda^3}{8} + \frac{11\lambda^4}{3} + \frac{25\lambda^5}{4} - \frac{5\lambda^6}{42} - \frac{5\lambda^7}{24}\right), \\ A_6 &= \frac{1}{240}\left(-\frac{278}{7} - \frac{127\lambda}{6} + \frac{91\lambda^2}{2} + \frac{587\lambda^3}{24} - 6\lambda^4 - \frac{41\lambda^5}{12} + \frac{3\lambda^6}{14} + \frac{\lambda^7}{8}\right), \\ A_7 &= \frac{1}{240}\left(\frac{58}{7} + \frac{17\lambda}{6} - \frac{65\lambda^2}{6} - \frac{89\lambda^3}{24} + \frac{8\lambda^4}{3} + \frac{11\lambda^5}{12} - \frac{5\lambda^6}{42} - \frac{\lambda^7}{24}\right), \\ A_8 &= \frac{1}{240}\left(-\frac{6}{7} - \frac{3\lambda}{14} + \frac{7\lambda^2}{6} + \frac{7\lambda^3}{24} - \frac{\lambda^4}{3} - \frac{\lambda^5}{12} + \frac{\lambda^6}{42} + \frac{\lambda^7}{168}\right). \end{aligned}$$

The explicit scheme (60) is one step, two level, nine points in level n and one point in level $n+1$. An easy calculation using Taylor–series expansions shows that the scheme (60) is consistent of eighth order (in space and time, i.e., $8-8$ scheme). Furthermore it is stable under the CFL condition (44). Also, this numerical method with problem (23) gives oscillations around the discontinuity, as shown in fig. 2c. Therefore we use the flux limiter method by the same technique of flux limiters fourth order by writing the scheme (60) in the form:

$$\begin{aligned} v_j^{n+1} = v_j^n - \lambda(v_j^n - v_{j-1}^n) - \frac{\lambda}{240}[A_1(v_{j+4}^n - v_{j+3}^n) + A_2(v_{j+3}^n - v_{j+2}^n) + \\ + A_3(v_{j+2}^n - v_{j+1}^n) + A_4(v_{j+1}^n - v_j^n) + \hat{A}_5(v_j^n - v_{j-1}^n) + \\ + A_6(v_{j-1}^n - v_{j-2}^n) + A_7(v_{j-2}^n - v_{j-3}^n) + A_8(v_{j-3}^n - v_{j-4}^n)] \end{aligned} \tag{61}$$

where $\hat{A}_5$ is defined by

$$\hat{A}_5 = \frac{1000}{14} + \frac{1025}{6}\lambda - \frac{215}{6}\lambda^2 - \frac{455}{8}\lambda^3 + \frac{11}{3}\lambda^4 + \frac{25}{4}\lambda^5 - \frac{5}{42}\lambda^6 - \frac{5}{24}\lambda^7.$$

The scheme (61) can be written in the form

$$v_j^{n+1} = v_j^n - \lambda(v_j^n - v_{j-1}^n) - \frac{\lambda}{240}\Delta_- \left[\sum_{i=1}^{7} B_i \Delta v_{j+\frac{9}{2}-i}^n\right] \tag{62}$$

where

$$B_i = \sum_{k=1}^{i} A_k, \ i = 1(1)7.$$

The equation (61) is the first order upwind scheme (29) with an additional term

$$-\frac{\lambda}{240}\Delta_- \left[\sum_{i=1}^{7} B_i \Delta v_{j+\frac{9}{2}-i}^n\right] \tag{63}$$

where an antidiffusive flux is given by

$$\frac{1}{240}\left[\sum_{i=1}^{7} B_i \Delta v_{j+\frac{9}{2}-i}^n\right]. \tag{64}$$

Now we try to treat oscillations in this method by adding *seven limiters* amount of the atidiffusive flux (64) to the first order scheme i.e.

$$v_j^{n+1} = v_j^n - \lambda(v_j^n - v_{j-1}^n) - \frac{\lambda}{240}\Delta_- \left[\sum_{i=1}^{7} \phi_{ij} B_i \Delta v^n_{j+\frac{9}{2}-i}\right] \tag{65}$$

where the limiters $\phi_{ij},\ i = 1, 2, 3, 4, 5, 6, 7$ are taken in the form

$$\phi_{ij} = \phi(\theta_{ij}),\ i = 1, 2, 3, 4, 5, 6, 7 \tag{66}$$

and

$$\theta_{1j} = \frac{\Delta v_{j+\frac{7}{2}} - i}{\Delta v_{j+\frac{9}{2}-i}},\ i = 1(1)7. \tag{67}$$

Also, this scheme is eighth–order consistent if $\phi_{ij} = 1,\ 1, 2, 3, 4, 5, 6, 7$. Substituting by $m = 4$ in the general finite difference equation (20) and choose thecoefficients $C_1, C_2, C_3, C_4, C_5, C_6, C_7$ and C_8 according to the scheme (65) as:

$$\begin{aligned} &C_1 = 0,\ \forall i = 2(1)4, \\ &C_i = -\frac{\lambda}{240}\ \frac{\Delta_- \left[\phi_{(i-1)j} B_{i-1} \Delta v_{j+\frac{22}{2}-1}\right]}{\Delta v_{j+\frac{9}{2}-1}},\ \forall i = 5(1)8, \\ &C_i = \frac{\lambda}{240}\ \frac{\Delta_- \left[\phi_{(i-1)j} B_{i-1} \Delta v_{j+\frac{11}{2}-1}\right]}{\Delta v_{j+\frac{9}{2}-1}} + \lambda\delta_{i5}, \end{aligned} \tag{68}$$

we can write (68) as

$$\begin{aligned} &C_1 = 0,\ \forall i = 2(1)4 \\ &C_i = -\frac{\lambda}{60} B_{i-1} \left(\frac{\phi_{(i-1)j}}{\theta_{(i-1)j}} - \phi_{(i-1)(j_1)}\right),\ \forall i = 5(1)8 \\ &C_i = \frac{\lambda}{60} B_{i-1} \left(\frac{\phi_{(i-1)j}}{\theta_{(i-1)j}} - \phi_{(i-1)(j-1)}\right) + \lambda\delta_{i5} \end{aligned} \tag{69}$$

and the limitation on $C_2, C_3, C_4, C_5, C_6, C_7$ and C_8 of the form

$$\begin{aligned} &\forall i = 2, 3, 4 \quad C_i \in \left[0, -\frac{\lambda}{240} B_{i-1} \Phi_{(i-1)j}\right],\ |C_5 - \lambda| \le \frac{\lambda}{240} B_4 \Phi_{4j}, \\ &\forall i = 6, 7, 8 \quad C_i \in \left[0, \frac{\lambda}{240} B_{i-1} \Phi_{(i-1)j}\right] \end{aligned} \tag{70}$$

where

$$\Phi_i = \left| \frac{\phi_{ij}}{\theta_{ij}} - \phi_{i(j-1)} \right|, \ i = 1, 2, 3, 4, 5, 6, 7. \tag{71}$$

Hence the first six inequalities of the set (21) are satisfied by multiplication of the second and sixth terms antidiffusive flux by $-\phi_{2j}$ and $-\phi_{6j}$ respectively instead of ϕ_{2j} and ϕ_{6j}. Using the conditions (21) and (70) we get

$$-\frac{\lambda}{240} B_2 \Phi_{3j} + \frac{\lambda}{240} B_3 \Phi_{4j} + \lambda \leq 1, \tag{72a}$$

$$0 \leq -\frac{\lambda}{240} B_3 \Phi_{4j} + \lambda.$$

For simplification, choosing $\Phi_1 \approx \Phi_2 \approx \Phi_3 \approx \Phi_4 \approx \Phi_5 \approx \Phi_6 \approx \Phi_7 \approx \Phi$, the condition (72) will take the form

$$\Phi \leq \frac{240(1-\lambda)}{B_3 - B_2}, \tag{73a}$$

$$\Phi \leq \frac{240}{B_3}, \tag{73b}$$

respectively. Then the finite difference scheme (65) is TVNI if the limiters $\phi_{ij}, \ i = 1, 2, 3, 4, 5, 6, 7$ 1c and satisfy the following conditions:

$$0 \leq (\phi(\theta_{ij})\theta_{ij}, \phi(\theta_{ij})) \leq 2, \ i = 1, 2, 3, 4, 5, 6, 7. \tag{73c}$$

From our experiments, we found that the left oscillations are free under the condition (73) as shown in figure 3c.

4. Treatment of the right oscillations. By numerical experiments, we found that the schemes (32), (48) and (59) under the conditions (46), (59) and (73) respectively give TVNI only before the discontinuity. For this reason, we treat these schemes to derive methods suitable in each domain before and after the discontinuity by the two methods:

4.1. The first proposed derived method. It is done by a multiplication of the antidiffusive flux of (32), (48) and (59) respectively by the factor δ as follows:

$$v_j^{n+1} = v_j^n - \lambda \Delta v_{i-\frac{1}{2}}^n - \lambda \delta \Delta_- \sum_{p=1}^{2m-1} B_p \phi_{pj} \Delta v_{j+\frac{1}{2}((2m+1)-2p)}, \ m = 1, 2, 3, 4 \tag{74a}$$

where

$$\delta = \begin{cases} 1 & \text{if the variation is increasing,} \\ -1 & \text{if the variation is decreasing.} \end{cases} \tag{74b}$$

These schemes give TVNI of high–order in smooth regions and low–order near the discontinuity region.

4.2. The second proposed derived method. It is done by reversing the chosen conditions in (32), (48) and (59) for the limiter function to the form:

$$-\infty < (\phi(\theta_{pj})/\theta_{pj}, \phi(\theta_{pj})) \leq 0, \quad i = 1, 2, 3, ... \tag{75a}$$

$$\phi(\theta_{pj}) = 0, \ \theta_{pj} > 0, \qquad i = 1, 2, 3, ... \tag{75b}$$

when the variation is decreasing, and the conditions (32), (48) and (59) when it is increasing. The considered methods with the second proposed give also TVNI of high–order in smooth regions, but with higher accuracy than the same methods with (74) as shown in table 1 and the figures 4 a, b, c.

We present comparison between the TVNI explicit and implicit flux limiters methods which are studied in this chapter in the following table:

5. Accurate flux limiter methods. In light of the above discussion, we observe that, the flux limiter method is used to give TVNI methods of high

order accuracy in the smooth regions and of low order near the discontinuity and so the result method satisfied the convergent theorems. Therefore we will make the following modification. Firstly, we calculate the first two levels by using the flux limiters schemes. Secondly, we calculate the solutions at the higher levels by the same schemes, except the case of three points which are the point after, the point before and the point at discontinuity. These points are calculated by the following relations:

$$\begin{aligned} v_j^{n+1} &= v_{j-1}^n \quad j = J-1, J, \\ v_j^{n+1} &= v_{j+1}^n \quad j = J+1 \end{aligned} \tag{76}$$

where $J-1$, J and $J+1$ represent the points before, at and after discontinuity respectively as shown in the figure:

This technique will give the exact solution because the region of the discontinuity contains few points in the 2^{nd} and 3^{rd} level. Hence we can discuss the accuracy for the modification for levels higher than three. The procedure is as to calculate the flux limiter technique for the first four levels to get non–zero results which we will apply the modification at the 5^{th} level and so on; the same process can be continued beginning the flux limiter technique up to the 19^{th} level and the modification for the 20^{th} level and so on.

Since we know that the solution has a discontinuity at a point, we try to distincuish between the points near it. Calculating the solution at the successive points, we can choose the maximum difference between the solution for any successive points for which we can consider the biggest one to lie between x_j and x_{j+1}, hence we discuss the solution for the ten points near the discontinuity defined as x_{j-4}, x_{j-3}, x_{j-2}, x_{j-1}, x_j, x_{j+1}, x_{j+2},x_{j+3},x_{j+4} and x_{j+5}.

These results can be summarized in the tables 2 and 3 concerning the L_2–norm for ten points around the discontinuity. It is clear that starting calculation from 20^{th} level up to 100^{th} level gives bad results than that for the starting from 5^{th} level wile the best one appears in the starting from the 3^{th} level. The idea which shows that the exact solution for the given problem can be estimated by using this precedes for the nearest first few levels

6. Numerical and comparative results. All the numerical experiments reported here deal with numerical solution of the differential equation

$$u_t + au_x = 0,\ a > 0, \tag{77a}$$

with the initial value

$$u(x,0) = \begin{cases} 1 & x < 0, \\ 0 & x > 0. \end{cases} \tag{77b}$$

The exact solution is given by

$$u(x,t) = \begin{cases} 1 & x < at, \\ 0 & x > at. \end{cases} \tag{77c}$$

In this section, we describe the computational Global error for the results produced using: the high accuracy schemes {original schemes (25), (47) and (60), TVNI schemes (32), (48) and (73) under the conditions (46), (59) and (73), the schemes (32), (48) and (73) under the conditions (46), (59) and (73) with (75), and the modification scheme (76)} which are listed in tables 1–4 and are shown in figures 2–4.

There are different limiters which can serve our purpose such as Roe's limiter [10] which is defined by

$$\phi(\theta_{ij}) = \max(0, \min(2\theta_{ij}, 1), \min(\theta_{ij}, 2)), \tag{78a}$$

which satisfy the conditions (46), (59) and (73) and we choose our limiter in the form

$$\phi(\theta_{ij}) = \min(0, \min(2\theta_{ij}, 1), \min(\theta_{ij}, 2)), \tag{78a}$$

which satisfy the conditions (75) and Van Leer limiter [13] shich is defined by

$$\phi(\theta_{ij}) = (|\theta_{ij}| + \theta_{ij})/(1 + |\theta_{ij}|) \tag{79a}$$

which satisfy the conditions (5.61) and we choose our limiter in the form

$$\phi(\theta_{ij}) = (|\theta_{ij}| - \theta_{ij})/(1 + |\theta_{ij}|) \tag{79b}$$

which satisfy the conditions (75), $\forall i = 1, 2, 3, \ldots$.

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* *Benha Higher Institute of Technology*

** *Faculty of Education, Ain Shams University*
EGYPT