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EXCESSIVE ELEMENTS IN SEMI–DYNAMICAL SYSTEMS

BY

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Abstract. This paper continues the study of semi–dynamical systems on algebraic structures (s.d.s.a.s. for short), begun in [11]. The main result (§4, Th. 1) says that, under certain hypotheses, the supermedian (resp. excessive) points of a s.d.s.a.s. form a potential cone (in the sense of G. Mokobodzki) and a σ–H–cone. In order to establish this result, one studies the so called cancellable elements. We embed thus X into another s.d.s.a.s., in which we may consider, for some elements, a "difference". The necessary information about supermedian and excessive points, as well as an important example, are collected in §3.

1. Preliminaries.

Definition 1. We call *a semi–dynamical system on an algebraic structure* (in short s.d.s.a.s.) a set X endowed with:

1. A composition law $+ : X \times X \to X$, with the following properties:

1.1. associativity
1.2. commutativity
1.3. existence of the null element, denoted 0.

2. An external operation $\cdot : [0, +\infty) \times X \to X$, with the following properties:

2.1. $(\lambda + \mu)\cdot x = \lambda\cdot x + \mu\cdot x$
2.2. $\lambda(x + y) = \lambda\cdot x + \lambda\cdot y$
2.3. $\lambda(\mu\cdot x) = (\lambda\mu)x$
2.4. $1\cdot x = x,\ \forall x$
2.5. $0\cdot x = 0,\ \forall x$

3. An order relation $\mathcal{R} \subseteq X \times X$, with the following properties:

3.1. $x \leq y \Longrightarrow x + z \leq y + z$

3.2. $x \leq y \Longrightarrow \lambda \cdot x \leq \lambda \cdot y$

3.3. $0 \leq x, \forall x \in X$

3.4. $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \Longrightarrow \exists \bigvee_n x_n$ în X

In this situation, we will use the notation: $x_n \nearrow y$, where $y = \bigvee_n x_n$.

CL1. For any countable family (x_n) from X, there exists $\bigvee_n x_n$ and $\bigwedge_n x_n$ in X.

CL2. $$x + \bigvee_n x_n = \bigvee_n (x + x_n)$$
$$x + \bigwedge_n x_n = \bigwedge_n (x + x_n)$$
$$x \wedge \left(\bigvee_n x_n \right) = \bigvee_n (x \wedge x_n)$$
$$x \vee \left(\bigwedge_n x_n \right) = \bigwedge_n (x \vee x_n)$$

4. A map $\Phi : [0, +\infty) \times X \to X$ with the following properties:

4.1. $\Phi(s, \Phi(t, x)) = \Phi(s + t, x)$

4.2. $\Phi(t, x + x') = \Phi(t, x) + \Phi(t, x')$

4.3. $\Phi(t, \lambda \cdot x) = \lambda \cdot \Phi(t, x)$

4.4. $x \leq x' \Longrightarrow \Phi(t, x) \leq \Phi(t, x')$, $x_n \nearrow x \Longrightarrow \Phi(t, x_n) \nearrow \Phi(t, x)$, $\forall t > 0$

For each $x \in X$ we will denote:

$$\Gamma_x := \{ \Phi(t, x) \mid t \geq 0 \}$$

We accept temporarily an element $\omega \in X$ ([3]), with the following properties:

4.7. $\Phi(t, x) = \omega$, and $t' \geq t \Longrightarrow \Phi(t', x) = \omega$

4.8. $x_0 \neq \omega \Longrightarrow \exists \varepsilon_0 > 0$ such that $\Phi(t, x_0) \neq \omega$, $\forall t \in [0, \varepsilon)$

In this situation, one defines the "life time" $\zeta : X \to [0, +\infty]$ by:

$$\zeta(x) := \sup\{ t > 0 \mid \Phi(t, x) \neq \omega \} = \inf\{ t > 0 \mid \Phi(t, x) = \omega \}$$

Anyway, later on ω will be replaced by the exit boundary [9].

2. Cancellable elements.

Definition 1. $x \in X$ is called *cancellable* if:

$$y + x \leq y' + x \Longrightarrow y \leq y'$$

Remark. In the set of cancellable elements, $\prec$ (defined as: $x \prec y$ if there exists $z \in X$ such that $x + z = y$) is an order relation.

Proposition 1. *x is cancellable* $\Longleftrightarrow \bigwedge_n \frac{1}{n}x = 0$.

Proof. Let $u \leq \frac{1}{n}x, \forall n$. We verify by induction on n, that:

$$mu + \frac{2^n - 1}{2^n}x \leq x.$$

For $n = 0$ it is just the notation. In general, replacing m by $2m$ and adding x, we get the conclusion:

$$2mu + \frac{2^n - 1}{2^n}x + x \leq 2x.$$

Particularly $u + x \leq x$, whence $u \leq 0$.

Conversely, we verify by induction that

$$y + \frac{1}{n}x \leq y' + \frac{1}{n}x.$$

The case $n = 1$ is the hypothesis. In general:

$$(n+1)y + x \leq ny + y' + x \leq ny' + x + y' = (n+1)y' + x.$$

Hence, using the hypothesis $y \leq y'$.

Remarks. 0 is cancellable. If x is cancellable, and $x' \leq x$, then x' is cancellable. If x and x' are cancellables, then λx and $x + x'$ are cancellables. Particularly, $\lambda 0 = 0$, $\forall \lambda \geq 0$. If x is cancellable, then it clearly follows that $0 \cdot x = 0$.

In what follows, we construct a "completion" of the set X, in order to be able to consider, in certain situations "$x - y$".

Let us denote by X_- the set of all elements $x \in X$, for which $\Phi(t, x)$ is cancellable $\forall t \geq 0$. On the set of all pairs (x, y), with $y \leq x$ and $y \in X_-$

we define the relation: $(x,y) \sim (x',y')$ if $x+y'=y+x'$. It is easy to prove that this is an equivalence relation. For the transitivity, from:

$$\begin{aligned} x+y' &= y+x' \\ x'+y'' &= y'+x'' \end{aligned}$$

one gets: $x+y'+y''=y+x'+y''=y+y'+x''$. The conclusion follows, since y' is a cancellable element.

Let us denote by $\overline{(x,y)}$ the equivalence class of the pair (x,y) and by $\overline{X}$ the set of all equivalence classes. We have a canonical embedding $X \hookrightarrow \overline{X}$ defined as $x \mapsto \overline{(x,0)}$.

On the set $\overline{X}$ we define the operations:

$$\begin{aligned} \overline{(x,y)}+\overline{(x',y')} &= \overline{(x+x',y+y')} \\ \alpha\cdot\overline{(x,0)} &= \overline{(\alpha x,\alpha y)} \end{aligned}$$

It is easy to prove that the definitions are correct and moreover that the requested properties hold.

Let us remark that

$$\begin{aligned} \overline{(x,0)}+\overline{(y,0)} &= \overline{(x+y,0)} \\ \alpha\overline{(x,0)} &= \overline{(\alpha x,0)} \end{aligned}$$

Next, let us define a relation by: $\overline{(x,y)} \leq \overline{(x',y')}$ if $x+y' \leq y+x'$. It is an order relation, compatible with the previously defined operations. Moreover:

$$x \leq y \Longleftrightarrow \overline{(x,0)} \leq \overline{(y,0)}$$

Finally, let us define $\overline{\Phi}$ as:

$$\overline{\Phi}(t,\overline{(x,y)}) = \overline{(\Phi(t,x),\Phi(t,y))}$$

The definition is correct by the choice of X_-. In this way, $(\overline{X},\overline{\Phi})$ becomes a s.d.s.a.s.

We have $\overline{\Phi}(t,\overline{(x,0)}) = \overline{(\Phi(t,x),0)}$.

In $\overline{X}$ the following characterization holds:

$$\overline{(x,y)} \text{ is cancellable in } \overline{X} \Longleftrightarrow x \text{ is cancellable in } X.$$

Indeed:

$$\overline{(u,v)}+\overline{(x,y)} \leq \overline{(u',v')}+\overline{(x,y)} \Longleftrightarrow u+x+v'+y \leq v+y+u'+x$$

As y is cancellable by definition, it remains $u + v' + x \leq v + u' + x$, which gives $u + v' \leq v + u'$ if and only if x is cancellable.

In $\overline{X}$ the following property holds:

$$\text{if } x \in X_- \text{ and } \overline{(x,y)} \leq \overline{(x',y')} \text{ then there exists a unique } \overline{(u,v)} \in \overline{X} \text{ such that } \overline{(x,y)} + \overline{(u,v)} = \overline{(x',y')}$$

Indeed, the fact that $\overline{(x,y)}$ is cancellable assures the unicity; as for the existence, one can choose $u = y + x'$, $v = x + y'$.

Particularly, if $x \in X_-$, $y \in X$ and $x \leq y$, then there exists a unique $u \in \overline{X}$ such that $x + u = y$ (in $\overline{X}$). Hence, we may speak of $x - y$, on condition that $y \in X_-$.

Comments. In [6], the condition

$$(!) \qquad y \leq z \Longrightarrow \exists x \text{ with } x + y = z$$

is imposed from the beginning. Adopting such a condition reduces essentially the flexibility; for instance, the case when X contains only excessive elements is excluded.

The construction is similar to that from [6], but the hypotheses are different.

3. Supermedian and excessive elements.

Definition 1. $x \in X$ is called *supermedian* if it is cancellable and $\Phi(t,x) \leq x,\ \forall t \geq 0$.

We denote by X_s the set of all supermedian elements.

If $x \in X_s$, then $\Phi(t,x)$ is cancellable, for each $t \geq 0$.

Let us remark that, if $x \in X_s$, then the map $t \mapsto \Phi(t,x)$ is decreasing. Indeed, if $t \leq t'$, then $t' = t + h$, with $h \geq 0$. We get next:

$$\Phi(t',x) = \Phi(t, \Phi(h,x)) \leq \Phi(t,x)$$

Hence, for each decreasing sequence $t_n \searrow 0$, the family $(\Phi(t_n,x))_n$ is increasing and dominated by x; hence there exists $\bigvee_n \Phi(t_n,x)$. Let $s_n \searrow 0$ be another decreasing sequence; then $\bigvee_n \Phi(s_n,x)$ is the same, since $\forall n\ \exists m$ such that $s_m \leq t_n$ (and conversely). We may use the notation:

$$\tilde{x} := \bigvee_{t \geq 0} \Phi(t,x)$$

Clearly $\tilde{x} \leq x$. We have $\tilde{x} \in X_s$: indeed, it follows that it is a cancellable element, while:

$$\Phi(s, \tilde{x}) = \bigvee_n \Phi(s, \Phi(t_n, x)) = \bigvee_n \Phi(t_n, \Phi(s, x)) \leq \bigvee_n \Phi(t_n, x) = \tilde{x},\ \forall s \geq 0$$

One can thus consider $\tilde{\tilde{x}}$:

$$\tilde{\tilde{x}} = \bigvee_n \Phi(t_n, \tilde{x}) = \bigvee_n \Phi(t_n, \bigvee_m \Phi(s_m, x)) = \bigvee_{n,m} \Phi(t_n + s_m, x) = \tilde{x}$$

since $t_n + s_m \searrow 0$.

The next properties hold:

$$\widetilde{x + y} = \tilde{x} + \tilde{y}$$
$$x \leq y \Longrightarrow \tilde{x} \leq \tilde{y}$$
$$x_n \nearrow x \Longrightarrow \tilde{x}_n \nearrow \tilde{x}$$
$$\Phi(t, \tilde{x}) = \widetilde{\Phi(t, x)}$$

Theorem 2. *For each $x \in X_s$: either $t \mapsto \Phi(t, x)$ is a bijection between $[0, \zeta(x))$ and Γ_x; or there exists t_0 for which $\Phi(t_1, x) = \Phi(t_2, x)$, $\forall t_1, t_2 > t_0$.*

Proof. The correspondence $t \mapsto \Phi(t, x)$ is surjective, by the definition of Γ_x. If it is not injective, there exists $t_1 < t_2$ such that $\Phi(t_1, x) = \Phi(t_2, x)$. But, for each $t \in (t_1, t_2)$ we have $\Phi(t_2, x) \leq \Phi(t, x) \leq \Phi(t_1, x)$, hence the correspondence is constant on (t_1, t_2). Analogously, the property holds for each interval $(t_2 + n \cdot h, t_2 + (n+1) \cdot h)$, where $h := t_2 - t_1$.

Corollary 3. *The relation $\leq_\Phi$, defined by: $x \leq_\Phi y$ if $\exists t \geq 0$ such that $y = \Phi(t, x)$, is an order relation on X_s.*

Remark. Constant trajectories do exist: for example, the decreasing functions, constant from a point on, for the translation semi-group.

Definition 4. $x \in X$ is called *excessive* if $x \in X_s$ and $x \leq \tilde{x}$.

We denote by X_e the set of all excessive elements.

X_s and X_e are convex sub–cones, in which there exists increasing, countable and *dominated* supremum. Moreover, these are *absorbent* parts for the relation $\leq_\Phi$; that is: $x \in X_s$ resp. $X_e \Longrightarrow \Phi(t, x) \in X_s$ resp. X_e.

Remark. If $x \in X_s$ and $t_n \searrow t$, then $\Phi(t_n, x) \nearrow \Phi(t, x)$. Indeed:

$$\Phi(t_n, x) = \Phi(t_{n+1} + h, x) = \Phi(t_{n+1}, \Phi(h, x)) \leq \Phi(t_{n+1}, x)$$

Then:

$$\Phi(t_n, x) = \Phi(t + h_n, x) = \Phi(t, \Phi(h_n, x))$$

But $h_n \searrow 0$, hence $\Phi(h_n, x) \nearrow x$ (since x is supermedian), hence

$$\bigvee_n \Phi(t_n, x) = \bigvee_n \Phi(t, \Phi(h_n, x)) = \Phi\left(t, \bigvee_n \Phi(h_n, x)\right) = \Phi(t, x)$$

Particularly: $x \in X_e \Longrightarrow \Phi(t, x) \in X_e,\ \forall t \geq 0$.

Example. Let (X, Φ) be a semi-dynamical system with pole (but without any measurability assumption) [3].

For $s : X \to [0, +\infty]$ one considers the following two properties:

a) $s[\Phi(t, x)] \leq s(x),\ \forall x \in X, \forall t \geq 0$. For such functions, there exists $\lim_{t \to 0} s[\Phi(t, x)]$, which will be denoted by $\tilde{s}(x)$ (in accordance with the previous notation $\tilde{x}$).

b) $\lim_{t \to 0} s[\Phi(t, x)] = s(x), \forall x \in X$.

Let us denote by $\mathcal{E}_\Phi$ the set of all finite functions, which satisfy the properties a) and b). Clearly, $\mathcal{E}_\Phi$ is a linearizable convex cone.

Let $(s_i)_i$ be a increasing directed and dominated family from $\mathcal{E}_\Phi$; then $\sup_i s_i \in \mathcal{E}_\Phi$, hence $\bigvee_i s_i$ exists; clearly:

$$s + \bigvee_i s_i = \bigvee_i (s + s_i).$$

For any family $(s_i)_i$ from $\mathcal{E}_\Phi$, the function $\inf_i s_i = f$ has the property a):

$$f(\Phi(t, x)) = \inf_i s_i[\Phi(t, x)] \leq s_i[\Phi(t, x)] \leq s_i(x),\ \forall i$$

hence $f(\Phi(t, x)) \leq f(x)$. Thus, one can consider $s = \tilde{f}$ and we have $s \in \mathcal{E}_\Phi$. Clearly $s \leq s_i,\ \forall i$.

Conversely, if $s' \in \mathcal{E}_\Phi$ and $s' \leq s_i, \forall i$, then $s'(x) \leq f(x), \forall x \in X$. Hence: $s' = \tilde{s}' \leq \tilde{f} = s$, which means that $\bigwedge_i s_i$ exists and equals s. Next:

$$\begin{aligned}
\bigwedge_i (s + s_i)(x) &= \widetilde{\inf_i}(s + s_i)(x) = \lim_{t \to 0} \inf_i (s + s_i)[\Phi(t, x)] = \\
&= \lim_{t \to 0} \left[s[\Phi(t, x)] + + \inf_i s_i[\Phi(t, x)] \right] = \\
&= s(x) + \lim_{t \to 0} \inf_i s_i[\Phi(t, x)] = \left(s + \bigwedge_i s_i \right)(x)
\end{aligned}$$

Finally, in order to prove the Riesz' splitting property, let us denote by $\mathcal{X}$ the σ–algebra generated by all the functions from $\mathcal{E}_\Phi$. In this way, for each fixed $t > 0$ and $s \in \mathcal{E}_\Phi$, the map: $x \mapsto s[\Phi(t,x)]$ is $\mathcal{X}$–measurable, since it belongs also to $\mathcal{E}_\Phi$.

We get thus a sub–markovian semi–group of kernels $(P_t)_{t>0}$ on $(X, \mathcal{X})$ if we define:

$$P_t f(x) = f[\Phi(t,x)]$$

Clearly, the set of all $\mathcal{P}$–excessive functions is exactly $\mathcal{E}_\Phi$. We deduce that $\mathcal{E}_\Phi$ is a H–cone of functions ([5]).

Let us recall ([11]) that we defined, for the elements $\mu \in X_s^*$ of the dual (equivalently: a decreasing function for the relation $\leq_\Phi$) a regularization, denoted by:

$$\hat{\mu}(x) = \lim_{t \searrow 0} \mu[\Phi(t,x)]$$

The definition clearly coincides with the preceding one.

Passing to the bidual, we find the same regularization. Indeed, for each $x \in X$ and $\mu \in X^*$ we have:

$$\widehat{\varepsilon(x)}(\mu) = \lim_{t \searrow 0} \varepsilon(x)[\Phi^*(t,x)] = \lim_{t \searrow 0} \mu[\Phi(t,x)] = \hat{\mu}(x) = \varepsilon(x)(\hat{\mu})$$

4. A Cone of Potentials.

Theorem 1. *Under each one of the following hypotheses:*

(a) $t_n \searrow 0$ *and* $\Phi(t_n, x) \leq y, \forall n \Longrightarrow x \leq y$

(b) *For each increasingly directed family* $F \subseteq X$, $\bigvee F \in X$ *exists, and the following properties hold:*

$$a + \bigvee F = \bigvee_{x \in F} (a + x), \forall a \in X \text{ and } \Phi(t, \bigvee F) = \bigvee_{x \in F} \Phi(t,x)$$

it is true that, for each $x \in X$ *there exists the smallest element, denoted* $Rx \in X_s$, *for which* $Rx \geq x$.

Proof. (G. Mokobodzki). Let $G \subseteq [0, +\infty)$ be a countable part, for which the following hold: $t, t' \in G \Longrightarrow t + t' \in G$; $\forall t \in [0, +\infty)$ $\exists (t_n)_n$ in G such that $t_n \searrow t$.

For the proof under the assumption (a), one can consider as $G = \mathbf{Q}_+$. Let us denote the elements as: $G = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\}$.

We define inductively the sequence $(x_n)_n$ as follows:

$$x_1 = x;\ \ x_{n+1} = x_n \vee \Phi(\alpha_1, x_n) \vee \ldots \vee \Phi(\alpha_n, x_n)$$

The sequence $(x_n)_n$ is clearly increasing. Let us denote $u^G := \bigvee_n x_n$ (u, if no confusion may arise). For each fixed $k \in \mathbf{N}$ and each $n \geq k$, we have:

$$\Phi(\alpha_k, u) = \Phi(\alpha_k, \bigvee_n x_n) = \bigvee_n \Phi(\alpha_k, x_n) \leq \bigvee_n x_{n+1} = u$$

which means that $\Phi(t, u) \leq u, \forall t \in G$. Let now $t \in [0, +\infty)$ be arbitrary, and $(t_n)_n$ be a sequence from G, such that $t_n \searrow t$. Using (a), it follows that $\Phi(t, u) \leq u$, hence $u \in X_s$. Clearly $u \geq x$.

Finally, let $v \in X_s$ be such that $v \geq x$. By induction, it follows that $x_n \leq v$. Indeed:

$$\Phi(\alpha_k, x_n) \leq \Phi(\alpha_k, v) \leq v, \forall k$$

hence $x_{n+1} \leq v$, which proves that $u \leq v$.

Under the hypothesis (b), we have $\Phi(t, u^G) \leq u^G, \forall t \in G$. Using the order given by inclusion, the family $(u^G)_G$ is increasingly directed and dominated. There exists $u := \bigvee_G u^G$. Let $t \in [0, +\infty)$. We remark that

$$G' := \{s + r.t |\ s \in G,\ r \in \mathbf{Q}, r \geq 0\}$$

has the same properties as G, $G \subseteq G'$ and $t \in G'$. Hence:

$$\Phi(t, u) = \bigvee_G \Phi(t, u^G) \leq \bigvee_G \Phi(t, u^{G'}) \leq \bigvee_G u^{G'} = u$$

Finally, let $v \in X_s$ be such that $v \geq x$. As above, we get $v \geq u^G$ by induction, for each G, hence $v \geq u$.

In what follows, we suppose that one of the above hypotheses (a) or (b) hold; and moreover that:

$$(s)\ \ x \leq y \text{ and } x \text{ cancellable} \Longrightarrow \exists z \text{ such that } x + z = y$$

In this situation, we shall use the notation $y - x := z$ (cf. §2).

Theorem 2. *If $x \leq y$, $x, y \in X_s$, then there exists $x' \in X_s$ for which:*

$$R(y - x) + x' = y$$

Proof. We shall use the same notations as in the proof of Th. 1. for the case $x_1 := y - x$.

We prove first, under the hypothesis (a), that for a fixed part G:

$$\Phi(t,y) + x_n \leq y + \Phi(t,u),\ \forall t \in G \tag{1}$$

Indeed: $u + x \geq x_1 + x = y$, hence

$$\Phi(t,u) + \Phi(t,x) \geq \Phi(t,y)$$

Since $\Phi(t,x) \leq x$, we obtain: $\Phi(t,y) \leq \Phi(t,u) + x$; next:

$$\Phi(t,y) + x_1 \leq \Phi(t,u) + x + x_1 = \Phi(t,u) + y$$

which proves the relation (1), in the case $n = 1$.

Then:

$$\Phi(t,y) + x_{n+1} = \Phi(t,y) + x_n \bigvee_{1\leq k\leq n} \Phi(\alpha_k, x_n) =$$
$$= [\Phi(t,y) + x_n] \bigvee_{1\leq k\leq n} [\Phi(t,y) + \Phi(\alpha_k, x_n)]$$

By induction: $\Phi(t,y) + x_n \leq y + \Phi(t,u)$. Let us prove next that:

$$\Phi(t,y) + \Phi(\alpha_k, x_n) \leq y + \Phi(t,u),\ \forall 1 \leq k \leq n$$

We consider the cases:

I. $t \leq \alpha_k$: we write $\alpha_k = t + h$, with $h \in G$ and:

$$\Phi(t,y) + \Phi(\alpha_k, x_n) = \Phi(t,y) + \Phi(t, \Phi(h, x_n)) = \Phi(t, y + \Phi(h, x_n)) \leq$$
$$\leq \Phi(t, y + u) \leq y + \Phi(t,u)$$

II. $t > \alpha_k$: now we put $t = \alpha_k + h$, with $h \in G$; it follows that:

$$\Phi(t,y) + \Phi(\alpha_k, x_n) = \Phi(\alpha_k, \Phi(h,y) + x_n) \leq \Phi(\alpha_k, y + \Phi(h,u)) \leq y + \Phi(t,u)$$

In both cases, the induction is finished; from (1) we get:

$$\Phi(t,y) + u \leq y + \Phi(t,u) \tag{2}$$

Since $R(y - x) \leq y$, it follows that there exists $x' \in X$ such that

$$x' + R(y - x) = y.$$

The relation (2) becomes (taking into account that $u = R(y-x) \in X_s$): $\Phi(t, x') \leq x'$, $\forall t \in G$. Under (a), we obtain $x' \in X_s$.

Under (b), we get an element x^G, for which the next properties hold: $x^G + u^G = y$ and $\Phi(t, x^G) \leq x^G$, $\forall t \in G$. $(x^G)_G$ is a decreasing family. Since $u = \bigvee_G u^G \in X_s$ and $u \leq y$, it follows that there exists $x' \in X$ for which $x' + u = y$. From $x' + u = y = x^G + u^G, \forall G$ and the fact that u is cancellable, it follows that $x' \leq x^G, \forall G$. Conversely, if $z \in X$ satisfies $z \leq x^G, \forall G$, then $z + u^G \leq x^G + u^G = x' + u$, hence $z + u \leq x' + u$, which means that $x' = \bigwedge_G x^G$.

Now: $\Phi(t, x') \leq \Phi(t, x^{G'}) \leq x^{G'}$, for $t \in G'$. Since $\bigwedge_G x^{G'} = \bigwedge_G x^G$ (where $t \in G'$), we deduce that $\Phi(t, x') \leq x'$, $\forall t$, hence $x' \in X_s$.

Corollary 3. *Under each one of the hypotheses* (a) *or* (b) *and with* (s)*, the following property holds:*

$$\text{(R)} \qquad \begin{array}{c} s \leq s_1 + s_2 \text{ with } s, s_1, s_2 \in X_s \Longrightarrow \\ \exists s_1', s_2' \in X_s \text{ such that } s = s_1' + s_2',\ s_1' \leq s_1, s_2' \leq s_2 \end{array}$$

The proof is well–known.

Remarks. The hypothesis (a) implies that $X_s \equiv X_e$. Indeed, let $x \in X_s$; then, for each sequence $t_n \searrow 0$ we have, by definition $\Phi(t_n, x) \leq \tilde{x}$; using (a), we obtain $x \leq \tilde{x}$, hence $x \in X_e$.

The fact that X_e^* separates X (meaning that:

$$\mu(x) \leq \mu(y),\ \forall \mu \in X_e^* \Longrightarrow x \leq y)$$

implies (a). Indeed:

$$\mu\left[\Phi(t_n, x)\right] = \Phi^*(t_n, \mu)(x) \nearrow \mu(x)$$

hence $\mu(x) \leq \mu(y)$, $\forall \mu \in X_e^*$.

The hypothesis (b) is fulfilled, for instance, in the case of the translations semi–group on $\mathbf{R}$ and also for $(X^*)_s$.

The inequalities $R(u-v) \geq \tilde{R}(u-v) \geq R(\tilde{u}-\tilde{v})$ are immediate. Let us remark that a reduced element exists also in the cone of excessives elements. Denoting this reduced operation by $R^e(u-v)$ (for $u, v \in X_e$), we have $\tilde{R}(u-v) \geq R^e(\tilde{u}-\tilde{v})$. If $u, v \in X_e$, then clearly $R^e(u-v) \geq R(u-v)$; in this case: $R^e(u-v) = R(u-v) = \tilde{R}(u-v)$. The inequality $R(u-v) \geq \tilde{R}(u-v)$ may be strict: for example, we may take $u \in X_s \setminus X_e$, $v = 0$.

The example of the translations semi–group on $\mathbf{R}$ shows the necessity of a hypothesis in Th. 1. Indeed, the Dirichlet function u is invariant for

any rational translation, hence $u^G = u$. However, the reduced function is identically 1. Hence, under one of the hypotheses (a) or (b), X_s and X_e are σ–H–cones [1].

The proof above holds also in the cone $\mathcal{S}_{\mathcal{P}}$ of the supermedian functions with respect with a semi–group $\mathcal{P}$. This cone may be not even specifically solid in the cone $\mathcal{S}_{\mathcal{V}}$ of the supermedian functions with respect to the associated resolvent. For example, for the translations semi–group on $\mathbf{R}$, let us consider the functions:

$$s_1 = \begin{cases} 1, & x < 0 \\ 2, & x = 0 \\ 0, & x > 0 \end{cases} \qquad s_2 = \begin{cases} 2, & x < 0 \\ 0, & x \geq 0 \end{cases}$$

Then: $s_1 \in \mathcal{S}_{\mathcal{V}} \setminus \mathcal{S}_{\mathcal{P}}$, $s_2 \in \mathcal{S}_{\mathcal{P}} \subset \mathcal{S}_{\mathcal{V}}$, while $s_1 + s_2 \in \mathcal{S}_{\mathcal{P}}$.

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