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SOME GLOBAL SOLVABILITY RESULTS

BY

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Abstract. The range values for a certain class of (graph) closed mappings are evaluated. As a by–product of these, some surjectivity criteria and global inversion theorems are deduced.

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1. Introduction. Let (X, d) and $(Y, ||.||)$ be a metric and, respectively, normed space. Given a mapping T from the domain D (of X) to Y and the point x_0 in D, a basic problem involving these elements is that of determining the "size" in Y for the image under T of different D–spheres around x_0 (of radius $\rho > 0$)

(1D1) $D(x_0, \rho) = \{x \in D; d(x_0, x) < \rho\},\ D[x_0, \rho] = \{x \in D; d(x_0, x) \leq \rho\},$

by comparing these subsets with different Y–spheres (around $y_0 = T(x_0)$)

(1D2) $Y(y_0, \sigma)$ and/or $Y[y_0, \sigma]$, for $\sigma > 0$.

Technically speaking, the ways to be followed may be of a very different nature; we quote in this direction the *metrical* one (cf. DUGUNDJI and GRANAS [9,ch.1,§1]) or the *topological degree* method (see DEIMLING [7,ch.2,§10]). But, in the following, the maximality techniques will be considered. Their prototype is the 1977 ALTMAN's statement [1,ch.5,§1] subsequently refined in 1981 by CRAMER and RAY [5]. To describe the actual state of the problem we note that, in order to solve an operator equation like

(OE) $Tx = y$ (where $y \in Y(Tx_0, \sigma)$)

the basic relations to be handled are

(1.1) $||\varepsilon^{-1}(Tx' - Tx) + Tx - y|| \leq \alpha||Tx - y||$

(1.2) $c(d(x_0, x))d(x, x') \leq \varepsilon b(||Tx - y||)$.

Here, α, ε are positive numbers and b, c are functions from $[0, \infty[$ to itself, subject to some specific conditions. For example, the above quoted contributions correspond to these functions being linear. Concerning the nonlinear (modulo c) case, some basic aspects were discussed in 1982 by RAY and WALKER [14] – with c being decreasing – and in 1986 by BAE and YIE [3], under a continuity assumption about c. It is our aim in the following to show that a further extension of these facts is possible, in the measurable context; details will be given in Section 4. The main tool of the investigations below is represented by a local existence result established in Section 3, under the line in TURINICI [16]. Section 2 in introductory. And, finally, in Section 5, some applications of the main results to surjectivity criteria will be provided; these, among others, comprise the (differential) Hadamard – Levy global inversion theorem [12], and the (accretive) Browder's global surjectivity result [4]. Some further enlargements of the obtained facts to multivalued maps (between D and Y) taken as in DOWNING and RAY [8], will be given elsewhere.

2. Preliminaries.

(A) Let $a : [0, \infty] \to [0, \infty[$ be a function with

(1H1) $a(0) = 0$, and $0 < a(\tau) < \tau$, for all $\tau > 0$.

For each positive integer n, put $a^n =$ the n–th *iterate* of a; in particular,

$$a^0 = i \text{ (the identical function) and } a^1 = a.$$

Given such an object, call the couple of functions b, g (from $[0, \infty[$ to itself), *admissible* (modulo a) when

$$\text{(2D1)} \qquad \frac{b(\tau)}{\tau - a(\tau)} \leq \frac{g(\tau) - g(\sigma)}{\tau - \sigma}, \quad \tau, \sigma > 0, \quad a(\tau) \leq \sigma < \tau.$$

The folowing simple fact will be useful in the sequel.

Proposition 1. *Let the couple b,g be admissible (modulo a); and, in addition*

(2H2) $\liminf_{\tau\to\rho+} b(\tau) > 0$, for each $\rho > 0$.

Then, the following conclusions are available

(a) $a^n(\tau) \to 0$ *as* $n \to \infty$, *and the series* $\sum_n b(a^n(\tau))$ *converges, for all* $\tau > 0$.

(b) g *is increasing on* $]0,\infty[$*; moreover, this property has a strict character, as soon as*

(2H3) $b(\tau) > 0$, for any $\tau > 0$.

Proof. Letting $\tau > 0$ be arbitrary fixed, put $\tau_n = a^n(\tau)$, $n = 0, 1, \ldots$. By (2H1), the sequence (τ_n) is descending (strictly) in $]0,\infty[$; hence, it converges to some $\rho \geq 0$. Assume $\rho > 0$. We have, by (2D1),

$$g(\tau_n) - g(\tau_{n+1}) \geq b(\tau_n), \quad \text{for all } n.$$

Therefore, adding these inequalities,

$$g(\tau_0) - g(\tau_n) \geq \sum_{m\leq n} b(\tau_m), \quad n = 0, 1, \ldots;$$

i.e., the series $\sum_n b(\tau_n)$ converges. In particular, $b(\tau_n) \to 0$ as $n \to \infty$, a contradiction to (2H2). Hence, $\rho = 0$ and **(a)** is proved. Conclusion **(b)** now follows directly by **(a)**, via

(c) $\begin{cases} g(\tau) \geq g(\sigma) \text{ (respectively, } g(\tau) > g(\sigma) \text{ under (2H3))} \\ \text{whenever } \tau, \sigma > 0 \text{ are such that } a(\tau) \leq \sigma < \tau. \end{cases}$

The proof is thereby complete. ∎

Condition (2H2) above, used for the deduction of **(a)** (the first part) is, e.g., holding when $b = i + a$. It becomes superfluous when the function $t \vdash a(t)$ fulfils a regularity condition like

(2H4) $\limsup_{\tau\to\rho+} a(\tau) < \rho$, for all $\rho > 0$.

In particular, the simplest choice for this function is the linear one

(2H5) $a(\tau) = \alpha\tau$, $\tau > 0$, for some α in $[0, 1[$.

Then, under an assumption like

(2H6) $\tau \vdash b(\tau)/\tau$ decreases on $]0,\infty[$ and $B(\tau) = \int_0^\tau \frac{b(\sigma)}{\sigma} d\sigma < \infty$, $\tau > 0$,

the couple (b, g) with

$$g(0) = 0, \quad g(\tau) = \frac{1}{1-\alpha} B(\tau), \quad \tau > 0,$$

is admisible (modulo a), because

$$\frac{g(\tau)-g(\sigma)}{\tau-\sigma} \geq \frac{1}{1-\alpha}\frac{b(\tau)}{\tau} = \frac{b(\tau)}{\tau-a(\tau)},$$

whenever $\tau, \sigma > 0$ are as in (2D1). Alternatively, under an assumption like

(2H7) $\tau \vdash b(\tau)$ increases on $]0,\infty[$ and $B(\tau) = \int_0^\tau \frac{b(\sigma)}{\sigma} d\sigma < \infty,\ \tau > 0,$

the couple (b, g) with

$$g(0) = 0, \quad g(\tau) = \frac{1}{1-\alpha} B(\frac{\tau}{\alpha}), \quad \tau > 0,$$

is again admissible, in view of

$$\frac{g(\tau)-g(\sigma)}{\tau-\sigma} \geq \frac{b(\sigma/\alpha)}{\tau} \geq \frac{b(\tau)}{\tau-a(\tau)}, \quad \tau, \sigma > 0,\ \alpha\tau \leq \sigma < \tau.$$

For concrete examples of such functions, see TURINICI [16].

(B) We are now passing to another circle of problems. Let σ be arbitrary fixed in $]0,\infty[$. We say that $e : [0,\sigma] \to [0,\infty[$ is a *step function* (on $[0,\sigma]$) when,there is a division $\{\tau_0, ..., \tau_n\}$ of $[0,\sigma]$ with $0 = \tau_0 < \tau_1 < ... < \tau_n = \sigma$, as well as a set of numbers $\{\gamma_0, ..., \gamma_{n-1}\}$ with

(2D2) $e(\tau) = \gamma_i,$ when $\tau \in]\tau_i, \tau_{i+1}[, \quad 0 \leq i \leq n-1.$

The set of all such function will be denoted $\mathcal{E}(\sigma)$.For any function e, of this type, define its *Lebesgue integral* by the convention

$$\int_0^\sigma e(\xi)d\xi = \gamma_0(\tau_1-\tau_0) + ... + \gamma_{n-1}(\tau_n - \tau_{n-1}). \tag{2D3}$$

The following approximation formula will be useful. For each $\eta > 0$ fulfiling

(2H8) $\eta < \mu/6,$ where $\mu = \min\{\tau_{i+1} - \tau_i\}; 0 \leq i \leq n-1\},$

let $e_\eta : [0,\sigma] \to]0,\infty[$ be a certain function of the class $\mathcal{E}(\sigma)$ with

(2H9) $e_\eta(\tau) = e(\tau),$ whenever $\tau \in]\tau_i + \eta, \tau_{i+1} - \eta[, \quad 0 \leq i \leq n-1$

(2H10) $e_\eta(\tau) \leq \min\{e(\xi); \xi \in J_\eta(\tau)\}, 0 \leq \tau \leq \sigma.$

Here, by convention,

(2D4) $J_\eta(\tau) = [0,\sigma] \cap]\tau - \eta, \tau + \eta[, \quad 0 \leq \tau \leq \sigma.$

Note that the class of all these objects is nonempty. In fact, an immediate example of such a function is the one suggested by (2H10); that is

$$e_\eta(\tau) = \min\{e(\xi); \xi \in J_\eta(\tau)\}, \quad 0 \leq \tau \leq \sigma.$$

The relationships between the initial function e and its associated functions $\{e_\eta\}$ are precised in the statement below. Denote for simplicity

$$k = \text{the integral part of } \ \sigma/\eta \ \ (\text{hence } k\eta \leq \sigma < (k+1)\eta).$$

Proposition 2. *Let the number ω be arbitrary fixed in $]0, \int_0^\sigma e(\xi)d\xi[$, and the number $\eta > 0$ be such that (in addition to* (2H8))

(2H11) $\eta < \omega/5\eta\lambda$, where $\lambda = \max\{e(\xi); 0 < \xi < \sigma\}$.

If the finite subset M of $[0, \sigma]$ fulfils

(2H12) $M \cap [j\eta, (j+1)\eta[$ is nonempty for $0 \leq j \leq k-1$

then

$$\eta \cdot \sum_{\theta \in M} e_\eta(\theta) > \int_0^\sigma e(\xi)d\xi - \omega. \tag{2.1}$$

And consequently, we necessarily have

$$\int_0^\sigma e(\xi)d\xi \geq \int_0^\sigma e_\eta(\xi)d\xi > \int_0^\sigma e(\xi)d\xi - \omega. \tag{2.2}$$

Proof. By (2H8), there may be determined the ranks $0 = k(0) < < k(1) < ... < k(n) = k$, in such a way that

$$k(i)\eta \leq \tau_i < (k(i)+1)\eta, \quad 0 \leq i \leq n.$$

On the other hand, by the choice (2H12) of M, one may assume without any loss that $M = \{\theta_0, \theta_1, ..., \theta_{k-1}\}$, where

$$j\eta \leq \theta_j < (j+1)\eta, \quad 0 \leq j \leq k-1.$$

Now, by (2H9), we have the evaluation

$$\sum_{j=0}^{k-1} e_\eta(\theta_j) \geq \gamma_0 \text{card}\,(I_0) + ... + \gamma_{n-1}\text{card}\,(I_{n-1}), \tag{2.3}$$

where, by convention,

$I_0 = \{k(0)+3, ..., k(1)-2\}, ..., I_{n-1} = \{k(n-1)+3, ..., k(n)-2\}$. (These rank sets are nonempty, by (2H8)). Hence

$$\sum_{j=0}^{k-1} e_\eta(\theta_j) \geq \gamma_0(k(1)-k(0)-4) + ... + \gamma_{n-1}(k(n)-k(n-1)-4). \tag{2.4}$$

On the other hand, by definition,

$$\begin{cases} \int_0^\sigma e(\xi)d\xi = \eta \cdot \sum_{i=0}^{n-1} \gamma_i(k(i+1)-k(i)-1) \\ + \sum_{i=0}^{n-1} \gamma_i[(k(i)+1)\eta - \tau_i + \tau_{i+1} - k(i+1)\eta] \end{cases} \tag{2.5}$$

This, combined with (2H1), yields

$$\begin{cases} \int_0^\sigma e(\xi)d\xi - \eta \cdot \sum_{j=0}^{k-1} e_\eta(\theta_j) \leq 3\eta(\gamma_0 + ... + \gamma_{n-1}) + \\ 2\eta(\gamma_0 + ... + \gamma_{n-1}) \leq 5\eta\lambda n < \omega, \end{cases}$$

and (2.1) follows. Now, (2.2) is (via (2H10)) an immediate consequence of

$$\int_0^\sigma e_\eta(\xi)d\xi \geq \eta\ [\gamma_0 \mathrm{card}\,(I_0) + ... + \gamma_{n-1}\mathrm{card}\,(I_{n-1})] \tag{2.6}$$

and the argument we already sketched. Hence the result. ∎

Let $\sigma > 0$ be arbitrary fixed, We term the function $e : [0,\sigma] :\to]0,\infty[$, *standard measurable* (over $[0,\sigma]$) when there exists a sequence $(e_n : [0,\sigma] \to]0,\infty[)$ of functions in $\mathcal{E}(\sigma)$ with

$$\begin{cases} e_1(\tau) \leq e_2(\tau) \leq ... \leq e(\tau), \text{ for all } \tau \in [0,\sigma] \\ e_n(\tau) \uparrow e(\tau) \text{ as } n \to \infty, \text{ for almost all } \tau \in [0,\sigma]. \end{cases} \tag{2D4}$$

The class of all such functions will be designated by $\mathcal{M}_S(\sigma)$. We note the invariance property

$$e_1, e_2 \in \mathcal{M}_S(\sigma) \Longrightarrow e_1 \cdot e_2 \in \mathcal{M}_S(\sigma). \tag{2.7}$$

Clearly, any function e in $\mathcal{M}_S(\sigma)$ is *Lebesgue measurable* (and positive) on $[0,\sigma]$. In this case, by the Lebesgue–Beppo–Levi theorem (see, e.g.,

KOLMOGOROV and FOMIN [11,ch.5,§5]) its (finite or not) *Lebesgue integral* exists and is obtainable as

$$\int_0^\sigma e(\xi)d\xi = \lim_{n\to\infty} \int_0^\sigma e_n(\xi)d\xi. \tag{2.8}$$

In particular, any function $e : [0, \sigma] \to]0, \infty[$ with

(2H13) e is continuous or decreasing on $[0, \sigma]$,

is, necessarily in the class $\mathcal{M}_S(\sigma)$. Moreover, it is Lebesgue integrable on $[0, \sigma]$; and, its (finite) Lebesgue integral is nothing but the Riemann integral of e over $[0, \sigma]$. Finally, given ρ in $]0, \infty]$, call the function $e : [0, \rho[\to]0, \infty[$, *standard measurable* over $[0, \rho[$ when

(2D5) the restriction of e to $[0, \sigma]$ is in $\mathcal{M}_S(\sigma)$ for each σ in $]0, \rho[$.

The family of all such functions will be denoted $\mathcal{M}_S(\rho)$. As before, one has the invariance property

(2.7') $e_1, e_2 \in \mathcal{M}_S(\rho) \Longrightarrow e_1 \cdot e_2 \in \mathcal{M}_S(\rho)$.

Now, any function e in $\mathcal{M}_S(\rho)$ is Lebesgue measurable (and positive) on $[0, \rho[$; so, its (finite or not) Lebesgue integral exists and may be obtained as

$$\int_0^\rho e(\xi)d\xi = \lim_{\sigma\to\rho} \int_0^\sigma e(\xi)d\xi. \tag{2.9}$$

In particular, when e is taken as in (2H13), its associated Lebesgue integral is nothing but the generalized Riemann integral of e over $[0, \rho[$.

3. A local existence principle. Let (X, d) be a metric space and $(Y, ||.||)$, a Banach space. Further, take a map T from the domain D (of X) to Y, and a point x_0 in D. To solve the question posed in Section 1, we firstly need conditions under which the operator equation $Tx = 0$ has a solution in an open D–sphere around x_0. Call the underlying map T, *graph–closed*, provided

(3D1) $(x_n) \subseteq D,\ x_n \to x,\ Tx_n \to y \Longrightarrow x \in D$ and $Tx = y$.

Now, assume in the following

(3H1) T is graph closed.

Further, letting $a : [0, \infty[\to [0, \infty[$ be as in (2H1), take the function $f : [0, \infty[\to [0, \infty[$ in such a way that

(3H2) $(i+a, f)$ is admissible (modulo a).

Let also the couple of functions b, g from $[0, \infty[$ to itself be taken in accordance with

(3H3) (b, g) is admissible (modulo a).

Finally, let the decreasing function $c : [0, \infty[\to]0, \infty[$ satisfy

(3H4) $h(\infty) > g(||Tx_0||)$,

where $h : [0, \infty[\to [0, \infty[$ stands for its *indefinite integral*:

(3D2) $h(\tau) = \int_0^\tau c(\xi) d\xi, \quad \tau \geq 0$.

The following answer to the posed question is basic to considerations below.

Theorem 1. *Suppose that for each x in*

$$D_0(h) = \{u \in D; ||Tx_0 - Tu|| \leq f(||Tx_0||),\ h(d(x_0, u)) \leq g(||Tx_0||)\}$$

there exist x' in D and ε in $]0, 1]$ with

(3H5) $||Tx' - (1-\varepsilon)Tx|| \leq \varepsilon a(||Tx||),\ c(d(x_0, x))d(x, x') \leq \varepsilon b(||Tx||)$.

Then, $Tx = 0$ for at least one x in $D_0(h)$.

Proof. Suppose by contradiction that

(3H6) $Tx \neq 0,$ for all x in $D_0(h)$.

Denote

$$\begin{cases} D_0^*(h) = \{u \in D_0(h); ||Tx_0 - Tu|| \leq f(||Tx_0||) - f(||Tu||), \\ h(d(x_0, u)) \leq g(||Tx_0||) - g(||Tu||)\}. \end{cases}$$

Let x be arbitrary fixed in $D_0^*(h)$. We have, by (3H5) (the first half) that

(3.1) $||Tx'|| \leq (1-\varepsilon)||Tx|| + \varepsilon a(||Tx||)$.

Denote $\tau = ||Tx||, \sigma = (1-\varepsilon)\tau + \varepsilon a(\tau)$. These are strictly positive numbers; and (cf. (3.1))

$$a(\tau) \leq \sigma < \tau,\ \varepsilon = \frac{\tau - \sigma}{\tau - a(\tau)}, \quad ||Tx'|| \leq \sigma. \tag{3.2}$$

So, by the admissibility conditions (3H2)+(3H3),

$$\begin{cases} ||Tx - Tx'|| \leq \dfrac{\tau + a(\tau)}{\tau - a(\tau)} \cdot (\tau - \sigma) \leq f(\tau) - f(\sigma) \\ c(d(x_0, x))d(x, x') \leq \dfrac{b(\tau)}{\tau - a(\tau)} \cdot (\tau - \sigma) \leq g(\tau) - g(\sigma). \end{cases}$$

This firstly gives

$$\begin{cases} ||Tx_0 - Tx'|| \le ||Tx_0 - Tx|| + ||Tx - Tx'|| \le f(||Tx_0||) - f(\tau) + \\ f(\tau) - f(\sigma) \le f(||Tx_0||) - f(\sigma) \le f(||Tx_0||). \end{cases}$$

Secondly, by the decreasing property of c,

(3.4) $h(\lambda + \mu) - h(\lambda) \le \mu c(\lambda)$, for each $\lambda, \mu \ge 0$.

And this, combined with h being increasing, yields

$$\text{(3.5)} \qquad \begin{cases} h(d(x_0, x')) \le h(d(x_0, x) + d(x, x')) \le h(d(x_0, x)) + \\ c(d(x_0, x))d(x, x') \le g(||Tx_0||) - g(\tau) + g(\tau) - g(\sigma) \\ \le g(||Tx_0||) - g(\sigma) \le g(||Tx_0||). \end{cases}$$

The obtained relations tell us x' is an element of $D_0(h)$; hence $Tx' \neq 0$. This, combined with (3.1)+(3.2), and (cf. Proposition 1) the increasing properties of f and g, yields

$$\begin{cases} ||Tx_0 - Tx'|| \le f(||Tx_0|) - f(\sigma) \le f(||Tx_0) - f(||Tx'||) \\ h(d(x_0, x')) \le g(|Tx_0||) - g(\sigma) \le g(||Tx_0||) - g(||Tx'||); \end{cases}$$

hence, x' is an element of $D_0^*(h)$. Finally, noting that

$$\delta = h^{-1}(g(||Tx_0||)) \text{ exists (by (3H4))}$$

one clearly has

$$d(x_0, x) \le \delta, \quad \text{for all } x \in D_0(h).$$

And this, combined with (3.3), gives

$$\text{(3.6)} \qquad \begin{cases} ||Tx - Tx'|| \le f(||Tx||) - f(||Tx'||) \\ d(x, x') \le (1/c(\delta))(g(||Tx||) - g(||Tx'||)). \end{cases}$$

Now, let us introduce over D the metric structure

(MS) $e(x, y) = \max(||Tx - Ty||, d(x, y))$

and the ordering

(OS) $x \le y$ iff (3.6) holds (with y in place of x').

Suppose (x_n) is an ascending sequence in $D_0^*(h)$; namely, for $n \le m$,

$$\text{(3.7)} \qquad \begin{cases} ||Tx_n - Tx_m|| \le f(||Tx_n|) - f(||Tx_m||) \\ d(x_n, x_m) \le (1/c(\delta))(g(||Tx_n||) - g(||Tx_m||)) \end{cases}$$

As $(f(||Tx_n||))$ and $(g(||Tx_n||))$ are decreasing (hence Cauchy) sequences in $[0,\infty[$,

$$x_n \to x \text{ and } Tx_n \to Tx, \text{ for some } x \in D.$$

if we take he closedness condition (3H1) into account. Moreover, as $D_0(h)$ is closed (modulo e), x is necessarily an element of it; and therefore, $Tx \neq 0$. But (again via Proposition 1) f is increasing strictly over $]0,\infty[$; hence the sequence $(||Tx_n||)$ is decreasing in $]0,\infty[$ and

$$||Tx_n|| \geq ||Tx||, \text{ for all } n.$$

This, combined with (x_n) being a part of $D_0^*(h)$ gives, for all n

$$\begin{cases} |Tx_0 - Tx_n|| \leq f(||Tx_0||) - f(||Tx||), \\ h(d(x_0,x_n)) \leq g(||Tx_n||) - g(||Tx||). \end{cases} \tag{3.8}$$

Hence, passing to limit as $n \to oo$, we deduce $x \in D_0^*(h)$. On the other hand, if we take the limit as $m \to \infty$ in (3.7), one gets, for each n

$$\begin{cases} ||Tx_n - Tx|| \leq f(||Tx_n||) - f(||Tx||) \\ d(x_n,x) \leq (1/c(\delta))(g(||Tx_n||) - g(||Tx||)); \end{cases} \tag{3.9}$$

that is,

$$x_n \leq x, \quad \text{for all} \quad n.$$

Summing up, each ascending sequence in $(D_0^*(h), e, \leq)$ is a Cauchy one, bounded from above. In this case, the maximality principle in TURINICI [16] gives us, for the starting point x_0 in $D_0^*(h)$, a maximal element x in $D_0^*(h)$ with $x_0 \leq x$. By hypothesis,there must be a point x' in D and a number ε in $]0,1]$ so that (3H5) be valid. Note that,necessarily, $x \neq x'$ (because $Tx \neq 0$); and, by the developments above one also has $x \leq x'$. This, however,contradicts the maximality of x in $D_0^*(h)$. Hence, our initial assumption (3H6) imposed upon T cannot be retained, and the conclusion follows. ∎

Concerning different aspects of the admissibility conditions (3H2)+(3H3), the discussion in Section 2 may now be applicable. In particular, the simplest form of the function $t \vdash a(t)$ is – as we already precised – the linear one

$$a(\tau) = \alpha\tau \text{ (hence } f(\tau) = \frac{1+\alpha}{1-\alpha}\tau), \quad \tau \geq 0,$$

for some α in $[0,1[$. On the other hand, according to the same remarks, the simplest choice of the function b is

$$b(\tau) = \tau \text{ (hence } g(\tau) = \frac{1}{1-\alpha}\tau), \quad \tau \geq 0.$$

Finally, note that the function c may be taken constant on its existence domain

$$c(\tau) = \gamma \text{ (hence } h(\tau) = \gamma\tau) \text{ for some } \gamma > 0.$$

In this case, as a practical application of Theorem 1, we have (under the closedness assumption (3H1)) the following solvability result (comparable with the one in ALTMAN [1,ch.5,§1]):

Corollary 1. *Let the constants α in $[0,1[$ and $\gamma > 0$ be such that: for each x in*

$$D_0 = \{u \in D; ||Tx_0 - Tu|| \leq \tfrac{1+\alpha}{1-\alpha}||Tx_0||,\ \gamma d(x_0,u) \leq \tfrac{1}{1-\alpha}||Tx_0||\}$$

there exist x' in D and ε in $]0,1]$ with

(3H7) $||\varepsilon^{-1}(Tx' - Tx) + Tx|| \leq \alpha||Tx||,\ \gamma d(x,x') \leq \varepsilon||Tx||.$

Then, $Tx = 0$ for at least one x in D_0.

The obtained fact may be now used in establishing the following local–global stattement (cf. also CRAMER and RAY [5]):

Corollary 2. *Suppose the constants α in $[0,1[$, $\gamma > 0$ and $\eta > 0$ are such that: for each x in $D(x_0,\eta)$ and y in $Y(Tx_0,(1-\alpha)\gamma\eta)$,there exist x' in D and ε in $]0,1]$ with*

(3H8) $||\varepsilon^{-1}(Tx' - Tx) + Tx - y|| \leq \alpha||Tx - y||,\ \gamma d(x,x') \leq \varepsilon||Tx - y||.$

Then, necessarily

$$Y(Tx_0,(1-\alpha)\gamma\eta) \subseteq T(D(x_0,\eta)).$$

Proof. Let y be arbitrary fixed in $Y(Tx_0,(1-\alpha)\gamma\eta)$ and put, for simplicity

$$T_y(x) = Tx - y, \quad \text{for all } x \in D.$$

The mapping T_y fulfils evidently (3H1). Further, note that the assumption about y may be written as

$$\frac{1}{1-\alpha}||T_y(x_0)|| < \gamma\eta; \text{ and consequently, it is the case that}$$

$$\gamma d(x_0,u) \leq \frac{1}{1-\alpha}||T_y(x_0)|| \Longrightarrow d(x_0,u) < \eta.$$

In other words, for each x in

$$D_0^{(y)} = \{u \in D; \gamma d(x_0,u) \leq \tfrac{1}{1-\alpha}||T_y(x_0)||\}$$

there exist x' in D and ε in $]0,1]$ such that (3H7) be fulfilled with T_y in place of T. By Corollary 1, $Tx = y$ for at least one x in $D_0^{(y)}$; hence – by the above implication – for at least one x in $D(x_0,\eta)$. The proof is complete. ∎

4. Main results. By the developments in the preceding sections it is now possible to derive an appropriate answer to the problem of the introductory part. Let (X,d) be a complete metric space and $(Y,||.||)$, a Banach space. Take a mapping T from a domain D (of X) to Y which satisfies (3H1). Also, letting σ be arbitrary fixed in $]0,\infty[$, take a couple of functions $c : [0,\sigma] \to]0,\infty[$ and $a : [0,\sigma] \to]0,1[$ with

(4H1) c and a belong to the class $\mathcal{E}(\sigma)$.

This means that, for a certain division $\{\tau_0,...,\tau_n\}$ of the interval $[0,\sigma]$ with $0 = \tau_0 < \tau_1, ... < \tau_{n-1} < \tau_n = \sigma$, it is the case that

(4.1) c and a are constants on $]\tau_i,\tau_{i+1}[$, $0 \leq i \leq n-1$.
Note that the function $e : [0,\sigma] \to]0,\infty[$ defined by

$$e(\tau) = (1-a(\tau))c(\tau), \quad \tau \geq 0$$

also belongs to the class $\mathcal{E}(\sigma)$. The first main result of the present exposition may now be stated as follows.

Theorem 2. *Let the point $x_0 \in D$ be such that: for each x in $D(x_0,\sigma)$ and y in $Y(Tx_0, \int_0^\sigma (1-a(\xi))c(\xi)d\xi)$ there exist x' in D and ε in $]0,1]$ with*

$$\text{(4H2)} \qquad \begin{cases} ||\varepsilon^{-1}(Tx'-Tx)+Tx-y|| \leq a(d(x_0,x))||Tx-y||, \\ c(d(x_0,x))d(x,x') \leq \varepsilon||Tx-y||. \end{cases}$$

Then, necessarily

(4.2) $Y(Tx_0, \int_0^\sigma (1-a(\xi))c(\xi)d\xi) \subseteq T(D(x_0,\sigma))$.

Proof. Without any loss, one may assume $Tx_0 = 0$. Then, conclusion in the statement may be rephrased as: for each w in Y with $||w|| = 1$,

(4.3) $\tau w \in T(D(x_0,\sigma))$, whenever $0 \leq \tau < \int_0^\sigma (1-a(\xi))c(\xi)d\xi$.

To do this, let ω, θ in $]0,1[$ be arbitrary fixed. Denote

$$\lambda = \max\{e(\xi); 0 < \xi < \sigma\},\ \mu = \min\{\tau_{i+1} - \tau_i; 0 \le i \le n-1\}.$$

Further, let η be taken in accordance with $0 < \eta < (1/5)\min(\mu, \omega/n\lambda)$ and put $k=$ the integral part of σ/η. Denote (cf. the conventions in Section 2)

$$\gamma_0 = \min\{c(\xi); \xi \in J_\eta(0)\},\ \alpha_0 = \max\{a(\xi); \xi \in J_\eta(0)\}.$$

Clearly,

$$c(d(x_0, x)) \ge \gamma_0,\ a(d(x_0, x)) \le \alpha_0,\ \text{whenever } x \in D(x_0, \eta).$$

As $D(x_0, \eta) \subseteq D(x_0, \sigma)$, Corollary 2 applies to (α_0, γ_0) and x_0. So, for

$$y_1 = \theta(1-\alpha_0)\gamma_0\eta w \in Y(0, (1-\alpha_0)\gamma_0\eta)$$

there exists x_1 in $D(x_0, \eta)$ with $Tx_1 = y_1$. Moreover, the same argument shows

$$\tau w \in T(D(x_0, \eta)),\ \text{if}\ \ 0 \le \tau \le \theta(1-\alpha_0)\gamma_0\eta,$$

which, in turn, gives

(4.4_1) $\tau w \in T(D(x_0, \sigma)),\ \ 0 \le \tau \le \theta(1-\alpha_0)\gamma_0\eta.$

Further, denote

$$\gamma_1 = \min\{c(\xi); \xi \in J_\eta(d(x_0, x_1))\},\ \alpha_1 = \max\{a(\xi); \xi \in J_\eta(d(x_0, x_1))\}.$$

As before, one clearly has

$$c(d(x_0, x)) \ge \gamma_1,\ a(d(x_0, x)) \le \alpha_1,\ \text{whenever}\ \ x \in D(x_1, \eta).$$

In view of $x_1 \in D(x_0, (k-1)\eta)$, the relation $D(x_1, \eta) \subseteq D(x_0, \sigma)$ holds; and this shows Corollary 2 is applicable to (α_1, γ_1) and x_1. So, for

$$y_2 = \theta[(1-\alpha_0)\gamma_0 + (1-\alpha_1)\gamma_1]\eta w \in Y(Tx_1, (1-\alpha_1)\gamma_1\eta)$$

there exists x_2 in $D(x_1, \eta)$ with $Tx_2 = y_2$. Moreover, the same argument shows

$$\tau w \in T(D(x_1, \eta)),\ \text{if}\ \theta(1-\alpha_0)\gamma_0\eta \le \tau \le \theta[(1-\alpha_0)\gamma_0 + (1-\alpha_1)\gamma_1]\eta;$$

hence, combining with a previous fact,

(4.4_2) $\tau w \in T(D(x_0, \sigma)),\ 0 \le \tau \le \theta[(1-\alpha_0)\gamma_0 + (1-\alpha_1)\gamma_1]\eta.$

This procedure may be continued in the precised way. Generally, assume that, for some rank $p \ge k-1$,

$$x_q \in D(x_0, (k-1)\eta),\ \ q = 0, 1, ..., p.$$

With the notation

$$\gamma_p = \min\{c(\xi); \xi \in J_\eta(d(x_0, x_p))\},\ \alpha_p = \max\{a(\xi); \xi \in J_\eta(d(x_0, x_p))\}$$

one has at once

$$c(d(x_0, x)) \geq \gamma_p,\ a(d(x_0, x)) \leq \alpha_p,\ \text{whenever } x \in D(x_p, \eta).$$

From the hypothesis above, $D(x_p, \eta) \subseteq D(x_0, \sigma)$; and this shows Corollary 2 is applicable to (α_p, γ_p) and x_p. So, for

$$y_{p+1} = \theta[(1-\alpha_0)\gamma_0 + ... + (1-\alpha_p)\gamma_p]\eta w \in Y(Tx_p, (1-\alpha_p)\gamma_p\eta)$$

there exists x_{p+1} in $D(x_p, \eta)$ with $Tx_{p+1} = y_{p+1}$. Moreover, the same argument shows

$$\begin{cases} \tau w \in T(D(x_p, \eta)),\ \text{if } \theta[(1-\alpha_0)\gamma_0 + ... + (1-\alpha_{p-1})\gamma_{p-1}]\eta \\ \leq \tau \leq \theta[(1-\alpha_0)\gamma_0 + ... + (1-\alpha_p)\gamma_p]\eta; \end{cases}$$

hence, combining with the preceding ones

(4.4_p) $\tau w \in T(D(x_0, \sigma)),\ 0 \leq \tau \leq \theta[(1-\alpha_0)\gamma_0 + ... + (1-\alpha_p)\gamma_p]\eta.$

Now, two alternatives are open before us.

Case 1. The obtained point x_{p+1} belongs to $D(x_0, (k-1)\eta)$, for all such ranks p. Then, in view of

(4.5) $(1-\alpha_n)\gamma_n \geq \min\{e(\xi); 0 \leq \xi \leq \sigma\} > 0,\quad n = 0, 1, ...,$

the series $\sum_n (1-\alpha_n)\gamma_n$ diverges. Hence, by (4.4_p),

$$\tau w \in T(D(x_0, \sigma)),\quad \text{whenever } 0 \leq \tau < \infty$$

and the conclusion is clear.

Case 2. The point x_{p+1} does not belong to $D(x_0, (k-1)\eta)$. By the evident fact

(4.6) $\quad d(x_q, x_{q+1}) < \eta,\quad q = 0, ..., p$

and the choice of p, we thus have

(4.7) $(k-1)\eta \leq d(x_0, x_{p+1}) < k\eta.$

Denote $n(0) = 0,\ m(0) = n(0) + 1 = 1$. By (4.6),one gets

(4.8_0) $0 \leq d(x_0, x_{m(0)}) < \eta.$

Now, define

$$n(1) = \max\{q \in [m(0), p]; d(x_0, x_r) < \eta,\ r = m(0), ..., q\}.$$

Clearly, $n(1) \leq p - 1$; since, otherwise, (4.7) will be contradicted. In this case, putting $m(1) = n(1) + 1$, and making again use of (4.6), we derive

(4.8_1) $\eta \leq d(x_0, x_{m(1)}) < 2\eta$.

This process may continuate; one obtains in this way a sequence of ranks $1 = m(0) < m(1) < ... < m(k-1) \leq p + 1$, so that

$$j\eta \leq d(x_0, x_{m(j)}) < (j+1)\eta, \quad j = 0, 1, ..., k-1.$$

In other words, the subset $M = \{d(x_0, x_q); 0 \leq q \leq p + 1\}$ of $[0, \sigma]$ fulfils condition (2H12) of Proposition 2. But then, the conclusion (2.1) of that statement gives

$$[(1 - \alpha_0)\gamma_0 + ... + (1 - \alpha_p)\gamma_p]\eta > \int_0^\sigma (1 - a(\xi))c(\xi)d\xi - \omega.$$

And so, combining with (4.4_p),

$$\tau w \in T(D(x_0, \sigma)), \quad \text{whenever } 0 \leq \tau \leq \theta\ [\int_0^\sigma (1 - a(\xi))c(\xi)d\xi - \omega].$$

It suffices now taking the limit as $\omega \to 0$ and $\eta \to 1$ in this relation, to get the desired conclusion. ∎

A further enlargement of these facts may be obtained under the lines below. Take a couple of functions $c : [0, \sigma] \to]0, \infty[$ and $a : [0, \sigma] \to]0, 1[$ in accordance with

(4H3) c and $1 - a$ belong to $\mathcal{M}_S(\sigma)$.

This shows that, there exist a sequence $(c_n : [0, \infty] \to]0, \infty[)$ of functions in $\mathcal{E}(\sigma)$ with

$$\text{(4.10)} \qquad \begin{cases} c_1(\tau) \leq c_2(\tau) \leq ... \leq c(\tau),\ \tau \in [0, \sigma] \\ c_n(\tau) \uparrow c(\tau) \text{ as } n \to \infty,\ \text{a.e. } \tau \in [0, \sigma] \end{cases}$$

and a sequence $(a_n : [0, \sigma] \to]0, 1[)$ of functions in $\mathcal{E}(\sigma)$ with

$$\text{(4.10}') \qquad \begin{cases} a_1(\tau) \geq a_2(\tau) \geq ... \geq a(\tau),\ \tau \in [0, \sigma] \\ a_n(\tau) \downarrow a(\tau) \text{ as } n \to \infty,\ \text{a.e. } \tau \in [0, \sigma]. \end{cases}$$

Note that, by the remarks in Section 2, the function $e = (1-a)c$ is Lebesgue measurable (and positive) on $[0, \sigma]$; moreover, its (finite or not) Lebesgue integral existts and is representable as

$$\int_0^\sigma (1 - a_n(\xi))c_n(\xi)d\xi \uparrow \int_0^\sigma (1 - a(\xi))c(\xi)d\xi, \quad \text{as } n \to \infty. \tag{4.11}$$

Theorem 3. *Let the point x_0 in D be such that: for each x in $D(x_0, \sigma)$ and y in $Y(Tx_0, \int_0^\sigma (1 - a(\xi))c(\xi)d\xi)$, there exist x' in D and ε in $]0, 1]$ so that* (4H2) *be valid. Then, conclusion* (4.2) *of Theorem* 2 *is retainable* (*under this new setting*).

Proof. Let n be an arbitrary fixed positive integer. From the monotone portion of (4.10)+(4.10') and (4.11), it follows that, for each x in $D(x_0, \sigma)$ and y in $Y(Tx_0, \int_0^\sigma (1 - a(\xi))c(\xi)d\xi)$ there exist x' in D and ε in $]0, 1]$ so that (4.1) be accepted, with (c_n, a_n) in place of (c, a). But then, Theorem 2 gives

$$Y(Tx_0, \int_0^\sigma (1 - a_n(\xi))c_n(\xi)d\xi) \subseteq T(D(x_0, \sigma)). \tag{4.2$_n$}$$

This, by the arbitrariness of n and the convergence part of (4.1) ends the argument. ∎

The final step of these developments is now the following. Take a number ρ in $]0, \infty]$ and a couple of functions $c : [0, \rho[\to]0, \infty[$, $a : [0, \rho[\to]0, 1[$, in accordance with

(4H4) c and $1 - a$ belong to $\mathcal{M}_S(\rho)$.

As before, the function $e = (1 - a)c$ is measurable (and positive) on $[0, \rho[$; moreover, its (finite or not) Lebesgue integral exists and is representable via

$$\int_0^\sigma (1 - a(\xi))c(\xi)d\xi \uparrow \int_0^\rho (1 - a(\xi))c(\xi)d\xi \quad \text{as } \sigma \to \rho. \tag{4.11$'$}$$

Theorem 4. *Let the point x_0 in D be such that: for each x in $D(x_0, \rho)$ and y in $Y(Tx_0, \int_0^\rho (1 - a(\xi))c(\xi)d\xi)$ there exist x' in D and ε in $]0, 1]$ so that* (4.2) *be accepted. Then, for each σ in $]0, \rho]$, conclusion of Theorem* 3 *is retainable.*

Proof. Let σ be arbitrary fixed in $]0, \rho[$. By the increasing part of (4.11'), conditions of Theorem 3 are satisfied; and hence, the claim follows for all such σ. This, in combination with the convergence part of (4.11') gives the conclusion we need for the case $\sigma = \rho$. Hence the result. ∎

In particular, condition (4H3) will be surely fulfilled when,either

(4H5) c and a are continuous on $[0, \rho[$

or

(4H6) c is decreasing and a is increasing on $[0, \rho[$.

Note that, in such a case, the statement above may be viewed as a non–differential countepart of the one in BAE and YIE [3]; see also BAE and SUNG [2].

5. Some applications. The information contained in the results above is particularly relevant in the presence of a normed linear structure for the domain space of the underlying map. So, assume in the following X is a linear space and its metric d is induced by a norm $||.||$, with respect to which X becomes a Banach space. A natural choice for the point x' in (4H2) is

$$x' = x + \varepsilon h, \quad \text{for some } h \text{ in } X.$$

Precisely, let D be an open part of X and $T : D \to Y$, a mapping fulfiling (3H1). Further, letting ρ be arbitrary fixed in $]0, \infty]$, take the functions $c : [0, \rho[\to]0, \infty[$, $a : [0, \rho[\to]0, 1[$ in accordance with (4H4). As an immediate consequence of Theorem 4, we have

Corollary 3. *Let the point x_0 in D be such that: for each x in $D(x_0, \rho)$ and y in $Y(Tx_0, \int_0^\rho (1 - a(\xi))c(\xi)d\xi)$ there exist h in X and ε in $]0, 1]$ with*

$$\text{(5H1)} \qquad \begin{cases} ||\varepsilon^{-1}(Tx + \varepsilon h) - Tx) + Tx - y|| \le a(||x - x_0||)||Tx - y|| \\ c(||x - x_0||)||h|| \le ||Tx - y||. \end{cases}$$

Then, necessarily,

$$\text{(5.1)} \quad Y(Tx_0, \int_0^\sigma (1 - a(\xi))c(\xi)d\xi) \subseteq T(D(x_0, \rho)), \quad 0 < \sigma \le \rho.$$

Now, to get suficient conditions under which (5H1) be satisfied, the differential settting is the most appropriate one. Let $L(X, Y)$ stand for the space of all bounded linear operators from X to Y, endowed with the usual norm

$$||A|| = \sup\{||Ax||; ||x|| \le 1\}, \quad A \in L(X, Y).$$

We say that the mapping T is *Gateaux differentiable* at the point $x \in D$, provided

(5D1) $T'(x)(h) = \lim_{\tau \to 0+} \tau^{-1}(T(x+\tau h) - Tx),\ h \in X$

exists and $h \vdash T'(x)(h)$ is an element of $L(X,Y)$. Now, suppose in the following

(5H2) T is Gateaux differentiable at any point of D.

Then, (5H1) will be satisfied as soon as

$$\text{(5H3)} \qquad \begin{cases} T'(x)(h) = -(Tx - y), \text{ for some } h \in X \\ \text{with } c(||x - x_0||)||h|| \leq ||Tx - y||. \end{cases}$$

The first part of this relation may be handled in a standard way; namely, it requires a condition like

(5H4) $T'(x)$ is surjective, for all $x \in D$.

Moreover, under such a hypothesis, the relation in question is fulfilled with any function $a : [0, \rho[\to]0, 1[$; hence, in particular, for any constant function

$$a(\tau) = \alpha, \quad 0 \leq \tau < \rho,$$

where α is arbitrary fixed in $]0, 1[$. But, the treatment of the second part in (5H3) necessitates some preliminary facts. Let $L^s(X,Y)$ stand for the class of all bounded linear operators from X to Y which, in addition, are surjective. For an arbitrary fixed element A of this class, denote

(5D2) $X(A) = \{x \in X; Ax = 0\}$ (the nullspace of A).

The mapping $\hat{A}$ from the quotient space $\hat{X}_A = X|X(A)$ to Y defined as
(5D3) $\hat{A}(\hat{x}) = \{Ax; x \in \hat{x}\}, \quad \hat{x} \in \hat{X}_A,$

is clearly shown to be continuous and bijective. So, by the well known Banach inversion result (see, e.g., KOLMOGOROV and FOMIN [11,ch.4, §5]), the operator $(\hat{A})^{-1}$ exists and belongs to $L(Y, \hat{X}_A)$. Denote $\delta(A) = 1/||(\hat{A})^{-1}||$; it is not hard to see that

$$\text{(5.2)} \qquad \begin{cases} \delta(A) = \sup\{\lambda > 0; \text{ for each } y \in Y \text{ there exists } x \in X \\ \text{with } Ax = y \text{ and } \lambda||x|| \leq ||y||\}. \end{cases}$$

Having these precised, the following differential version of the result above may be formulated. (The assumptions (5H2)+(5H4) prevail).

Theorem 5. *Let the point x_0 in D be such that*
(5H5) $c(||x - x_0||) \leq \delta(T'(x))$, for all $x \in D(x_0, \rho)$.

Then, necessarily

(5.3) $Y(Yx_0, \int_0^\sigma c(\xi)d\xi) \subseteq T(D(x_0, \sigma)), \quad 0 < \sigma \leq \rho.$

Hence, if in addition,

(5H6) $\int_0^\rho c(\xi)d\xi = \infty,$

then, $T(D(x_0, \rho))$ *is all of* Y.

Proof. Let θ be arbitrary fixed in $]0, 1[$. By (5H5),

$$\theta c(|x - x_0||) < \delta(T'(x)), \quad x \in D(x_0, \rho).$$

So, by (5.2), for each x in $D(x_0, \rho)$ and $y \in Y$ there must be a direction $h = h(x, y)$ in X with

(5.4) $T'(x)(h) = -(Tx - y), \quad \theta c(||x - x_0||)||h|| \leq ||Tx - y||.$

This, combined with a remark above shows that the inclusion below is true, for each α in $]0, 1[$

(5.5) $Y(Tx_0, (1 - \alpha)\theta \int_0^\sigma c(\xi)d\xi) \subseteq T(D(x_0, \sigma)), \ 0 < \sigma \leq \rho.$

It suffices now performing the limit as $\alpha \to 0, \ \theta \to 1$, in the obtained relation to end the argument. ∎

In particular, when $c : [0, \infty[\to]0, \infty[$ is continuous on its domain, this statement reduces to the one in BAE and YIE [3] proved by a different method; see also RAY and WALKER [14].

The following aspect of Theorem 5 is technically useful. Remember that, by a result in GRAVES [10], the class $L^s(X, Y)$ of all surjective (bounded) linear operators from X to Y is open in $L(X, Y)$ (endowed with the usual operator norm); precisely

$$(5.6) \quad \begin{cases} \text{whenever } A \in L^s(X, Y), \text{ and } B \in L(X, Y) \text{ fulfils } ||B|| < \delta(A) \\ \text{then } A + B \in L^s(X, Y) \text{ and } \delta(A + B) \geq \delta(A) - ||B||. \end{cases}$$

Now, assume the mapping T is to be represented as

(5H7) $T(x) = T_1(x) + T_2(x), \quad x \in D,$

where the component maps T_1, T_2 satisfy

(5H8) T_1, T_2 are Gateaux differentiable on D

(5H9) ${T_1}'(x)$ is surjective, for all $x \in D$.

Further, letting ρ be a number in $]0, \infty]$, suppose that the function $c : [0, \rho[\to]0, \infty[$ (in the class $\mathcal{M}_S(\rho)$) is written as a difference

(5H10) $c(\tau) = c_1(\tau) - c_2(\tau), \quad 0 \leq \tau < \rho$

between a couple of functions $c_1, c_2 : [0, \rho[\to [0, \infty[$. As a direct consequence of the above discussion, we have

Corollary 4. *Let the point x_0 in D be such that*

(5H11) $||T_2{}'(x)|| \leq c_2(||x-x_0||) < c_1(||x-x_0||) \leq \delta(T_1{}'(x)), \quad x \in D(x_0,\rho)$.

Then, conclusions of Theorem 5 *are retainable.*

Note that, if c_1 and c_2 are continuous over $[0,\rho[$, Corollary 4 reduces to the main result in BAE and SUNG [2]. This, mutatis mutandis, shows their result is but a consequence of the one due to BAE and YIE [op.cit.]. On the other hand, it is clear that – in the same continuity setting – the reverse implication is also valid (under the choice $T_2 = 0,\ c_2 = 0$). Hence, the quoted statements are in fact equivalent each other.

Returning to the general setting, call the underlying map T, *open*, when

(5D4) the image under T of each part of D is open.

The following openess result is an immediate consequence of Theorem 5 above.

Corollary 5. *Suppose that*

$$\text{(5H12)} \qquad \begin{cases} x \vdash \delta(T'(x)) \text{ is locally bounded from below} \\ \text{(by strictly positive constants) over } D. \end{cases}$$

Then, T is an open mapping.

Proof. Let x_0 be arbitrary fixed in D and take $\rho > 0$ in acordance with $D(x_0,\rho) \subseteq D$ and

$$\delta(T'(x)) \geq \gamma > 0, \quad x \in D(x_0,\rho).$$

(Note that this is always possible, in view of (5H12) plus the openess of D). This shows that Theorem 5 applies with (x_0 as before and)

$$c(\tau) = \gamma, \quad 0 \leq \tau < \rho.$$

Hence, by the conclusion of that result,

$$Y(Tx_0,\gamma\sigma) \subseteq T(D(x_0,\sigma)), \quad 0 < \sigma \leq \rho.$$

This, combined with the openess of $D(x_0,\sigma)$ (for all such σ) ends the argument. ∎

In particular, (5H12) will be fulfilled as soon as

(5H13) $\quad c(||x-x_0||) \leq \delta(T'(x)), \quad x \in D,$

for some x_0 in D and some function $c : [0,\infty[\to]0,\infty[$ in the class $\mathcal{M}_S(\infty)$. On the other hand, sufficient conditions for (3H1) and (5H2) (with $X = Y$) are

(5H14) $\ T$ is continuous on D

and, respectively,

(5H2') $T'(x)$ is bijective, for all $x \in D$.

Note that, under these stronger hypotheses, plus (5H6) (with $\rho = \infty$) and

(5H15) $x \vdash T'(x)$ is continuous over D

the statements above give (under the choice $D = X$) the global inversion theorem in CRISTEA [6] (which, in turn, extends the Hadamard–Levy statement [12]). The proof consists in establishing that, under such requirements, the global inversion theorem in S. RADULESCU and M. RADULESCU [15] is available; we do not give further details.

The openess property above is not only interesting by itself. Because, as we shall see, it may act as a substitute for the Gateaux differentiability condition upon the underlying map to get co–domain evaluations like in Theorem 5. Precisely, letting X, Y be a couple of Banach spaces, take a mapping T from the open subset D (of X) to Y, in accordance with (3H1) and

(5H16) T is an open map.

Fix $x_0 \in D,\ \rho \in]0, \infty]$, and take a function $c : [0, \rho[\to]0, \infty[$ so as (4H4) be valid. Let also $\varphi : [0, \infty[\times [0, \infty[\to [0, \infty[$ be a function which is supposed to be *normal* in the sense

(5D5) $\varphi(r, s)$ is in the interval generated by r and s for all $r, s \geq 0$.

We call the mapping T, *locally* (x_0, c, φ)*–expansive* over $D(x_0, \rho)$, when

$$(5D6)\quad \begin{cases} \text{for each } x \in D(x_0, \rho) \text{ there exists } \beta = \beta(x) \text{ in }]0, \rho - ||x - x_0|| \,[\\ \text{such that } ||Tx - Tx'|| \geq c(\varphi(||x - x_0||, ||x' - x_0||))||x - x'||, \\ \text{for all } x' \text{ in } D(x, \beta). \end{cases}$$

Now, as a non–differential counterpart of Theorem 5, we have

Theorem 6. *Let the above data be taken in such a way that*

(5H17) T is localy $(x_0, c, \varphi) -$ expansive on $D(x_0, \rho)$.

Then, conclusions of Theorem 5 *are retainable.*

Proof. It will suffice establishing the desired property when σ is arbitrary fixed in $]0, \rho[$ and the function c is an element of $\mathcal{E}(\sigma)$; because, the remaining part of the argument will follow by a limit process. Let ω be arbitrary fixed in $]0, \int_0^\sigma c(\xi)d\xi[$. For each $\eta > 0$ fulfiling (2H8)+(2H11), let $c_\eta : [0, \sigma] \to]0, \infty[$ be a function (in the class $\mathcal{E}(\sigma)$) taken in accordance with (2H9)+(2H10). Note that, by Proposition 2, one has an evaluation like

$$\int_0^\sigma c_\eta(\xi)d\xi > \int_0^\sigma c(\xi)d\xi - \omega.$$

Let also α be arbitrary fixed in $]0,1[$. We now intend to verify conditions of Theorem 2 over $D(x_0,\sigma)$, with respect to the couple of functions

$$c^*_\eta(\tau)=\frac{1}{1-\alpha}c_\eta(\tau),\ a_\eta(\tau)=\alpha,\quad 0\le\tau\le\sigma.$$

Let x in $D(x_0,\sigma)$ and y in Y be arbitrary. There exists, by (5H16), a $\eta^*=\eta^*(x)$ in $]0,\min(\eta,\beta(x))[$ so that $D(x,\eta^*)\subseteq D(x_0,\sigma)$ and $T(D(x,\eta^*))$ is open in Y. Further, in view of

$$Tx-\varepsilon(1-\alpha)(Tx-y)\to Tx,\quad \text{as}\ \ \varepsilon\to 0,$$

there must be an x' in $D(x,\eta^*)$ and an $\varepsilon>0$ small enough so that

$$Tx'=Tx-\varepsilon(1-\alpha)(Tx-y).$$

This immediately gives the first part of (4H2). For the second one, note that by the same relation, coupled with (5H17), as well as

$$||x-x_0||-\eta<\varphi(||x-x_0||,||x'-x_0||)<||x-x_0||+\eta$$

one gets

$$\begin{cases}\varepsilon(1-\alpha)||Tx-y||=||Tx-Tx'||\ge\\ c(\varphi(||x-x_0||,||x'-x_0||))||x-x'||\ge(1-\alpha)c^*_\eta(||x-x_0||)||x-x'||.\end{cases}$$

This proves our assertion. And then, by the conclusion of that result,

$$Y(Tx_0,\textstyle\int_0^\sigma c_\eta(\xi)d\xi)\subseteq T(D(x_0,\sigma));$$

wherefrom (combining with a previous relation)

$$Y(Tx_0,\textstyle\int_0^\sigma c(\xi)d\xi-\omega)\subseteq T(D(x_0,\sigma)).$$

This inclusion holds for any $\omega>0$ taken as before. It suffices now performing the limit as $\omega\to 0$ to get the desired conclusion. The proof is complete. ∎

A particular case of these considerations may be described as follows. Let Δ be a multivalued mapping from X to Y^* (=the topological dual of Y), which is supposed to be *regular*, in the sense

(5D7) $\sup\{||y^*||;y^*\in\Delta(x)\}\le 1$, for all $x\in X$.

We associate it a (nonlinear) functional $[.,.]_\Delta$ over $X\times Y$ by the convention

(5D8) $[x,y]_\Delta=\sup\{y^*(y);y^*\in\Delta(x)\},\ x\in X,y\in Y$.

Now, with the same data, call the mapping $T:D\to Y$, *locally strongly* (x_0,c,φ,Δ)– *accretive* over $D(x_0,\rho)$ provided

$$\text{(5D9)}\quad\begin{cases}\text{for each } x\in D(x_0,\rho) \text{ there exists } \beta=\beta(x)\in\,]0,\rho-||x-x_0||\,[\\ \text{in such a way that } [x-x',Tx-Tx']_\Delta\ge\\ c(\varphi(||x-x_0||,||x'-x_0||))||x-x'||,\ \text{for all } x'\in D(x,\beta).\end{cases}$$

It is clear, by this very definition that

(5H18) T is locally strongly $(x_0, c, \varphi, \Delta)-$accretive over $D(x_0, \rho)$

is a sufficient condition for (5H17). So, if we substitute (5H17) by (5H18) in Theorem 6, its conclusion is retainable. This, in the particular case of c being continuous on its domain, yields Theorem 3.2 in BAE and YIE [3], provided

(5H19) $\varphi(r, s) = \max(r, s), \quad r, s \geq 0;$

and, respectively, Theorem 3.3 in RAY and WALKER [14], provided

(5H19') $\varphi(r, s) = r, \quad r, s \geq 0.$

For concrete aspects of the problem we refer to the 1977 monograph by ALTMAN [1].

Concerning the basic openess condition (5H16), the most natural setting for its deduction is – in the differential case (5H2)+(5H4) – the one precised by Corollary 5. It remains to study what happens in he non–differentiable case. An appropriate answer of this is to be given in the case $X = Y$ and Δ identical with the standard *duality map* J of the couple (X, X^*), introduced as

$$\text{(5D10)} \qquad \begin{cases} J(x) = \{x^* \in X^*; ||x^*|| = 1, x^*(x) = ||x||\}, & x \neq 0, \\ J(0) = \{x^* \in X^*; ||x^*|| = 1\} \end{cases}$$

Precisely, let $T : X \to Y$ be a mapping fulfiling (5H14). Fix $x_0 \in X$ and take some function $c : [0, \infty[\to]0, \infty[$ in the class $\mathcal{M}_S(\infty)$. Also, let $\varphi : [0, \infty[\times [0, \infty[\to [0, \infty[$ be a normal function. We shall term the underlying map, *(globally) strongly* $(x_0, c, \varphi)-$ *accretive* over X, provided

(5D11) $[x - x', Tx - Tx']_J \geq c(\varphi(||x - x_0||, ||x' - x_0||))||x - x'||, \quad x, x' \in X.$

Note that, in such a case, T is necessarily injective on X. We are now in position to state the announced answer to the question above.

Corollary 6. *Let the mapping $T : X \to X$ and the triplet (x_0, c, φ) be such that*

(5H20) T is (globally) strongly $(x_0, c, \varphi)-$accretive over X.

Then, the following conclusions are valid

- **(a)** *T is an open map; and therefore (by the remark above) it is a homeomorphism of X onto $T(X)$*
- **(b)** *the conclusions in Theorem 5 are retainable; so that, if* (5H6) *holds (with $\rho = \infty$) then T is a homeomorphism of X onto X.*

Proof. Evidently, **(b)** is an immediate consequence of **(a)** plus Theorem 6. So, all we have to do is to prove the openess property **(a)** (for T). Let x_1 be arbitrary fixed in X and put

$$\gamma = \inf\{c(\xi); 0 \leq \xi \leq 1 + ||x_1 - x_0||\}.$$

(Note that $\gamma > 0$, by the choice of c). We show that the following inclusion is retainable

(5.7) $X(Tx_1, \gamma\sigma) \subseteq T(X(x_1, 2\sigma))$, for all σ in $]0, 1[$.

(And this will complete the argument in the standard way). Let y_2 be arbitrary fixed in $X(Tx_1, \gamma\sigma)$; our goal is to determine a $x_2 \in X(x_1, 2\sigma)$ with $y_2 = Tx_2$. Without any loss one may assume $y_2 = 0$; for, otherwise, replacing T by its translate

$$T_2(x) = T(x) - y_2, \quad x \in X,$$

this property will be fulfilled (as well as the remaining ones). So, assume our data fulfil

(5H21) $||Tx_1|| < \gamma\sigma$, for some σ in $]0, 1[$.

We shall be interested to determine a point x_2 in $X(x_1, 2\sigma)$ with $Tx_2 = 0$. To this end, we start by noting that, in view of (5H20), the mapping $-T$ is 0–dissipative. So, by a standard result (see, e.g., DEIMLING [7,ch.3,§13]), it generates a *contractive semigroup* $(t, x) \vdash S(t)x$ over X. In particular, the function

$$u(t) = S(t)x_1, \quad t \geq 0$$

is the unique solution to the Cauchy problem

(CP) $u'(t) = -Tu(t), \quad t \geq 0; \; u(0) = x_1$.

We now claim that the following invariance property holds

(5.8) $u(t) \in X(x_1, \sigma)$, for all $t \geq 0$.

To this end, denote

$$\varphi(t) = ||u(t) - x_1||, \quad t \geq 0.$$

One has $\varphi(0) = ||u(0) - x_1|| = 0 < \sigma$. Suppose $t > 0$ is such that $\varphi(t) = \sigma$. This gives at once

$$||u(t) - x_0|| \leq \sigma + ||x_1 - x_0|| < 1 + ||x_1 - x_0||.$$

And then, by some standard properties of the semi–inner product (cf. the above reference)

$$(5.9) \quad \begin{cases} \varphi'_{(-)}(t) = -[u(t) - x_1, Tu(t)]_J \leq -[u(t) - x_1, Tu(t) - Tx_1]_J \\ +||Tx_1|| \leq -c(\varphi(||u(t) - x_0||, ||x_1 - x_0||))||u(t) - x_1|| \\ +||Tx_1|| \leq -\gamma\sigma + ||Tx_1|| \leq -\gamma\sigma + \gamma\sigma = 0. \end{cases}$$

(Here, $(-)$ stands for the left derivative). And, from this, (5.7) is proved, via differential inequalities techniques; see, for instance, BROWDER [4]. Further, we claim that the following contractive property is valid

(5.10) $||u(t+\tau)-u(t)|| \leq \sigma e^{-\gamma t}, \quad t \geq 0, \tau \geq 0,$

where σ and γ are as before. In fact, let us denote (for a certain $\tau \geq 0$)

$$\psi(t) = ||u(t+\tau)-u(t)||, \qquad t \geq 0.$$

One has

$$\psi(0) = ||u(\tau)-u(0)|| = ||u(\tau)-x_1|| < \sigma,$$

if we take (5.8) into account. This, combined with

$$\psi'_{(-)}(t) = -[u(t+\tau)-u(t), Tu(t+\tau)-Tu(t)]_J \leq -\gamma\psi(t)$$

yields he desired relation (again via differential inequalities techniques). Now, it is easy to complete the argument. By (5.8)+(5.10) above

$$x_2 = \lim_{t\to\infty} u(t) \text{ exists and belongs to } X(x_1, 2\sigma).$$

Moreover, combining with

$$u(t+s) = S(t)u(s), \qquad t, s \geq 0,$$

one gets (passing to limit as $s \to \infty$)

$$S(t)x_2 = x_2, \quad \text{for all } t \geq 0.$$

In other words, $t \vdash S(t)x_2$ is a constant function. Hence, its derivative is identical zero. This, added to the choice of our semigroup, yields $Tx_2 = 0$, and conclusion follows. ∎

In particular, when the function c is continuous on its existence domain, the statement above reduces to Theorem 4.1 in RAY and WALKER [14], proved by a slightly different method; see also BROWDER [op.cit.]. For other aspects we refer to the 1983 RAY's survey paper [13] and the 1985 DEIMLING's monograph [7].

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