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SOME REMARKS ON B.Y. CHEN'S INEQUALITY INVOLVING CLASSICAL INVARIANTS

BY

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Abstract. We provide an alternate proof of a basic inequality of B.Y. Chen established in [2] which relates some classical invariants. We use an approach close to the one used in [6]. We prove the inequality for submanifolds in the case of codimension one and of general codimension. We also provide a simple application.

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1. Introduction.

In [2] Chen stated the following:

Problem: *Find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold.*

He also mentioned in [2] that many famous results in differential geometry can be regarded as results in this respect. In this note we understand by classic curvature invariants the sectional curvature (denoted by *sec*), the scalar curvature (denoted by *scal*), and the Ricci curvature (denoted by *Ric*). On the other hand, by the new Riemannian curvature invariants we mean Chen's invariants defined as follows: Let M^n be a Riemannian n-manifold. For any orthonormal basis $e_1, ..., e_n$ of the tangent space T_pM, the scalar curvature at p is defined to be $scal(p) = \sum_{i<j} sec(e_i \wedge e_j)$. For any r-dimensional subspace of T_pM denoted L with orthonormal basis $e_1, ..., e_r$ one may define:

$$scal(L) = \sum_{1\leq i<j\leq r} sec(e_i \wedge e_j). \tag{1.1}$$

In fact, $scal(p) = scal(T_pM)$. In [3] Chen considered the finite set $S(n)$ of k-tuples $(n_1, ..., n_k)$ with $k \geq 0$ satisfying the conditions: $n_1 < n$, $n_i \geq 2$ and $n_1 + ... + n_k \leq n$. For $(n_1, ..., n_k) \in S(n)$ Chen introduced in [3] the following invariants:

$$\delta(n_1, ..., n_k)(p) = scal(p) - \inf\{scal(L_1) + ... + scal(L_k)\}(p), \tag{1.2}$$

where $L_1, ..., L_k$ run over all k mutually orthogonal subspaces of T_pM such that $dim\, L_j = n_j$, $j = 1, ..., k$.

Let us remark that the Chen's invariant for $k = 0$ is nothing else but the scalar curvature.

In the following we use the same notations as in [3]. Let

$$c(n_1, ..., n_k) = \frac{n^2(n + k - 1 - \sum n_j)}{2(n + k - \sum n_j)},$$
$$b(n_1, ..., n_k) = \frac{1}{2}\left\{ n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1) \right\}.$$

Chen's basic inequalities established in [3] can be stated as follows:

Theorem 1.1. *For any n-dimensional submanifold M of a Riemannian space form $R^{n+m}(\varepsilon)$ of constant sectional curvature ε and for any k-tuple $(n_1, ..., n_k) \in S(n)$, we have:*

$$\delta(n_1, ..., n_k) \leq c(n_1, ..., n_k)|H|^2 + b(n_1, ..., n_k)\varepsilon. \tag{1.3}$$

The equality case of the inequality above holds at a point $p \in M$ if and only if there exists an orthonormal basis $e_1, ..., e_{n+m}$ at p such that the shape operators of M in $R^{n+m}(\varepsilon)$ at p take the following forms: $S_r = diag\,(A_1^r, ..., A_k^r, \mu_r, ..., \mu_r)$ for $r = n + 1, ..., m$, where each A_j^r is a symmetric $n_j \times n_j$ submatrix such that $trace(A_1^r) = ... = trace(A_k^r) = \mu_r$.

Chen's basic inequality (1.3) has many nice applications; for example, one has the following important result as an immediate consequence:

Theorem 1.2. *Let M be a Riemannian n-manifold. If there exists a k-tuple $(n_1, ..., n_k) \in S(n)$ and a point $p \in M$ such that:*

$$\delta(n_1, ..., n_k)(p) > \frac{1}{2}\left\{n(n-1) - \sum n_j(n_j - 1)\right\}\varepsilon, \tag{1.4}$$

then M admits no minimal isometric immersion into any Riemannian space form $R^m(\varepsilon)$ with arbitrary codimension. In particular, if $\delta(n_1, ..., n_k)(p) > 0$ at a point for some k-tuple $(n_1, ..., n_k) \in S(n)$, then M admits no minimal isometric immersion into any Euclidean space for any codimension.

A special case of Chen's basic inequality is Lemma 1 of [2] which is stated as Corollary 3.1 below. The main purpose of this note is to give an alternate proof of this result.

2. Hypersurface Case. The main goal of this section is to prove the following:

Proposition 1. *Let M^n be a hypersurface in a Riemannian $(n+1)$-manifold $\overline{M}^{n+1}$. Then at every point $p \in M$ the following inequality holds:*

$$scal(p) \leq \frac{n(n-1)}{2}H^2 + \sum_{i<j} \overline{sec}(e_i \wedge e_j), \tag{2.1}$$

where scal is the scalar curvature of M at p, H is the mean curvature at p and $\overline{sec}(e_i \wedge e_j)$ is the sectional curvature on the plane generated by the vectors e_i and e_j in the ambient space $\overline{M}$. The equality holds at p if and only if p is an umbilical point.

Proof. We use for proof the following elementary:

Lemma 1. *Let g a real polynomial*

$$g(X) = a_0X^n + a_1X^{n-1} + a_2X^{n-2} + ... + a_n$$

with $a_0 \neq 0$ and $n > 1$. If all the roots of g are real, then

$$\Delta(g, n) \equiv \frac{2(n-1)}{n}a_1^2 - 4a_0a_2 \geq 0. \tag{2.2}$$

Proof of the lemma. If g has all the roots real, then g' has also all the roots real. After $(n-2)$ steps, $g^{(n-2)}$ has both roots real, and $\Delta \geq 0$. ∎

Now we can prove the proposition. Let $p \in M$ and $\eta \in T_p^{\perp}M$. Let $\{e_1, ..., e_n\}$ be an orthonormal basis of T_pM in which $S_\eta = S$ is in diagonal form, i.e. $S(e_i) = \lambda_i e_i$, $i = 1, ..., n$, where $\lambda_1, ..., \lambda_n$ are the shape operator's eigenvalues. Then, tacking g in the lemma to be the characteristic polynomial of S, we have:

$$nH = \lambda_1 + ... + \lambda_n = -a_1, \tag{2.3}$$

$$\sum_{i<j} \lambda_i \lambda_j = a_2, \tag{2.4}$$

$$a_0 = 1. \tag{2.5}$$

On the other hand (see for example [4, pg.131]) we have the following well-known fact:

$$sec(e_i \wedge e_j) - \overline{sec}(e_i \wedge e_j) = \lambda_i \lambda_j, \tag{2.6}$$

for any $i < j$. Therefore the inequality in the lemma becomes:

$$\frac{2(n-1)}{n}(nH)^2 - 4\sum_{i<j}\{sec(e_i \wedge e_j) - \overline{sec}(e_i \wedge e_j)\} \geq 0 \tag{2.7}$$

or:

$$n(n-1)H^2 - 2\sum_{i<j} sec(e_i \wedge e_j) + 2\sum_{i<j} \overline{sec}(e_i \wedge e_j) \geq 0 \tag{2.8}$$

and since the sum in the second term is nothing but $scal(p)$ we obtain, after a division by 2, the claimed inequality. The equality case holds for $\lambda_1 = ... = \lambda_n = H/n$, i.e. when the point p is an umbilical point. ∎

The following fact is Lemma 1 from [2] for the hypersurface case:

Corollary 2.1. *Let $R = 2\,scal(p)/n(n-1)$ denote the normalized scalar curvature of the hypersurface M^n isometrically immersed in a Riemannian space form $R^{n+1}(\varepsilon)$. Then we have the inequality:*

$$R \leq H^2 + \varepsilon, \tag{2.9}$$

at every point $p \in M$. The equality holds if and only if p is an umbilical point.

Proof. For any $i \neq j$, $i, j \in \{1, ..., n\}$ we use the fact that $\overline{sec}(e_i \wedge e_j) = \varepsilon$ in the inequality proved in the previous proposition. ∎

3. General Codimension Case. We have discussed in the previous section the hypersurface case. We present in this section an alternate proof of Lemma 1 of [2] in the general codimension case, i.e. as it was obtained in [2]. One of the main points of this result is that the codimension is arbitrary. We emphasize that the term in the left hand side is an intrinsic quantity and the terms in the right hand side term are extrinsic quantities.

Proposition 2. *Let M^n be isometrically immersed in a Riemannian manifold $\overline{M}^{n+m}$. Let sec, $\overline{sec}$ and scal(p) be the sectional curvature of M, the sectional curvature of $\overline{M}$ and the scalar curvature of M at p, respectively. Then the following inequality holds:*

$$scal(p) \leq \frac{n(n-1)}{2}|H|^2 + \sum_{i<j} \overline{sec}(e_i \wedge e_j) \tag{3.1}$$

Proof. The argument in the proof also uses Lemma 1. Let $\{\xi_1, ..., \xi_m\}$ be an orthonormal frame of $T_p^{\perp}M$ at p. Let us denote by $\lambda_1^r, ..., \lambda_n^r$ the eigenvalues of the shape operator $S_r = S_{\xi_r}$. Then $\lambda_1^r + ... + \lambda_n^r = trace(h^r)$ and if S_r is in diagonal form then the characteristic polynomial of S_r has the coefficient corresponding to λ^{n-2} equal to:

$$a_2 = \sum_{1 \leq i < j \leq n} \lambda_i^r \lambda_j^r. \tag{3.2}$$

If S_r is not in diagonal form, then the quantity $\sum_{i,j} \lambda_i^r \lambda_j^r$ is the sum of the 2×2 minors in the matrix $(h_{ij}^r)_{1 \leq i < j \leq n}$, since two equivalent matrices have the same characteristic polynomial. In fact, if $g = \det(S_r - \lambda I_n) = 0$, is the real polynomial in Lemma 1, then:

$$a_2 = \sum_{1 \leq i < j \leq n} (h_{ii}^r h_{jj}^r - (h_{ij}^r)^2), \tag{3.3}$$

$$-a_1 = \sum_{i=1}^{n} h_{ii}^r, \tag{3.4}$$

$$a_0 = 1, \tag{3.5}$$

$$\sum_{i,j} \lambda_i^r \lambda_j^r = \sum (h_{ii}^r h_{jj}^r - (h_{ij}^r)^2) \tag{3.6}$$

and the inequality $\Delta(g, n) \geq 0$ is, in fact, for our choice of g :

$$\frac{2(n-1)}{n}\left(\sum_{i=1}^{n} h_{ii}^{r}\right)^{2} - 4\sum_{1\le i<j\le n}(h_{ii}^{r}h_{jj}^{r} - (h_{ij}^{r})^{2}) \ge 0 \tag{3.7}$$

or, summing for $r = n+1, ..., n+m$:

$$n(n-1)|H|^{2} - 2\sum_{i<j}\sum_{r=n+1}^{n+m}(h_{ii}^{r}h_{jj}^{r} - (h_{ij}^{r})^{2}) \ge 0. \tag{3.8}$$

To express the last sum we need the Gauss' equation (and this would be the major difference with respect to the hypersurface case). Let us denote by R the curvature tensor of M and by $\bar{R}$ the curvature tensor of $\overline{M}$. Then, for any $X, Y \in T_pM$:

$$\begin{aligned} &< \bar{R}(X,Y)X, Y > = \\ &= < R(X,Y)X, Y > - < h(X,X), h(Y,Y) > + |h(X,Y)|^{2} \end{aligned} \tag{3.9}$$

or, if $\{e_1, ..., e_n\}$ is an orthonormal frame at $p \in M$:

$$sec(e_i \wedge e_j) - \overline{sec}(e_i \wedge e_j) = \sum_{r=n+1}^{n+m}(h_{ii}^{r}h_{jj}^{r} - (h_{ij}^{r})^{2}) \tag{3.10}$$

and with this substitution the inequality becomes:

$$2\sum_{i<j} sec(e_i \wedge e_j) \le n(n-1)|H|^{2} + 2\sum_{i<j}\overline{sec}(e_i \wedge e_j) \tag{3.11}$$

which is the inequality we had to prove, after a division by 2. ∎

Lemma 1 of [2] is the following:

Corollary 3.1. *Let $f : M^n \to R^{n+m}(\varepsilon)$ be an isometric immersion of a Riemannian n-manifold M^n with normalized scalar curvature ρ into an $(n+m)$-dimensional Riemannian space form $R^{n+m}(\varepsilon)$ of sectional curvature ε. Then*

$$\rho \le |H|^{2} + \varepsilon. \tag{3.12}$$

The equality holds at a point $p \in M$ if and only if p is totally umbilical point.

Proof. For any $i \ne j$, $i, j \in \{1, ..., n\}$ we use that $\overline{sec}(e_i \wedge e_j) = \varepsilon$ in the inequality proved in the proposition. ∎

One can state the following immediate consequence which is more or less in the same spirit as the obstruction results obtained in [3]:

Corollary 3.2. *Let M^n be a Riemannian n-manifold and $\overline{M}^{n+m}$ a Riemannian $(n+m)$-manifold. If the scalar curvature of M is greater than the scalar curvature of every n-plane section L of $\bar{M}^{n+m}$, then M admits no minimal immersion into $\overline{M}$.* ∎

We recall the fact that the inequality $\Delta(g,n) \geq 0$ has also been used in [6]. Let us denote by

$$\sigma_i = 2||S_i||^2 - \frac{2}{n}(trace(S_i))^2, \tag{3.13}$$

where S_i is the shape operator in the normal direction ξ_i and $||S_i||^2 = (\lambda_1^i)^2 + ... + (\lambda_n^i)^2$. Let us denote by L_i the length of shape operator's spectrum in the direction of the normal vector ξ_i, i.e. the distance on the real axis between the greatest and the smallest of the shape operator's eigenvalues in the normal direction ξ_i.

Then the following result was proved in [6]:

Theorem 3.1. *Let M^n be a submanifold in a Riemannian manifold $\overline{M}^{n+m}$. For any $p \in M$ and for any normal basis $\xi_1, ..., \xi_m$ we have:*

$$\sqrt{\frac{2}{n(n-1)}\sigma_i(p)} \leq L_i(p) \leq \sqrt{\sigma_i(p)}. \tag{3.14}$$

The equality holds if and only if p is a totally umbilical point of M in $\overline{M}$.

One may get from the previous inequalities the following.

Corollary 3.3. *Let M^n be a submanifold of a Riemannian manifold $\overline{M}^{n+m}$. If for some $p \in M$ there exists a normal direction ξ such that $\sigma_\xi(p) > 0$, then the point p can't be a totally umbilical point.*

Since the double inequality (3.14) was obtained by the same procedure as Chen's basic inequality involving the classic invariants, they have in common the proof based on the idea $\Delta(g,n) \geq 0$, as it is presented in this note. In fact, the main idea used in both cases is that the shape operator's characteristic polynomial has all its roots real.

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