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ON A PRODUCT OF $\mathcal{D}$-COMPLETE σ-IDEALS

BY

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Abstract. In this paper we give sufficient conditions for the product $\mathcal{I}_1 \otimes \mathcal{I}_2$ of $\mathcal{D}$-complete σ-ideals be a complete σ-ideal relating to the family of Borel sets.

The study of the $\mathcal{D}$-complete σ-ideals products is important first of all because in this way we can build new $\mathcal{D}$-complete σ-ideals. It is also important in the study of the products of the topologies density type, topologies which appear, for example, in the connection between the approximate differentiability and the approximately partial derivability. In [7] we have shown that the product of the τ-complete σ-ideals is a τ-complete σ-ideal. If we "increase" the τ family and, instead, we take into consideration the classes of the borelian sets, the things become considerably complicated. Further on, we will settle sufficient conditions for a product of $\mathcal{D}$-complete σ-ideals to be complete, related to the family of borelian sets.

Let X be a linear space, $A \subset X$ and $\mathcal{I}, \mathcal{D}$ two families of subsets of X. We use the following notations: $x + I = \{x + u;\ u \in I\}$, $\lambda I = \{\lambda u;\ u \in I\}$, $x + \mathcal{I} = \{x + I;\ I \in \mathcal{I}\}$, $\lambda \mathcal{I} = \{\lambda I;\ I \in \mathcal{I}\}$. If (X, τ) is a topological space, let us denote $\mathcal{I}\dot{\Delta}\mathcal{D} = \{I \Delta D;\ I \in \mathcal{I},\ D \in \mathcal{D}\}$, $\mathcal{I}_\tau(A) = \{x \in X;\ \forall V \in \mathcal{V}_\tau(x),\ V \cap A \notin \mathcal{I}\}$. Also we denote by $int_\tau(A), cl_\tau(A), Fr_\tau(A)$, the interior, closure and boundary of the set A in the topology τ. Let $(X, \|\cdot\|)$ be a linear normed space and let $\mathcal{I}$ be a σ-ideal of subsets of X. We denote by $U(x, r)$ the open ball with center x and radius r, $\tau_{\|\cdot\|}$ the norm topology, $\mathcal{D}_\mathcal{I} = \left\{D \in \mathcal{D}; Fr_{\tau_{\|\cdot\|}}(D) \in \mathcal{I}\right\}$ and $\Phi_\mathcal{I}(A) = \{\ x \in X\ ; \forall (n_k)$ an increasing sequence of natural numbers there exists a subsequence (m_k) such that $\limsup_k [U(0,1) \setminus m_k (A - x)] \in \mathcal{I}\}$.

Consider the following assertions:

$(\mathcal{D})_1 : \mathcal{I}\dot{\triangle}\mathcal{D}$ is a σ-algebra of subsets of X

$(\mathcal{D})_2 : \forall A \subset X$, $\exists H_A \in \mathcal{I}\dot{\triangle}\mathcal{D}$ so that $A \subset H_A$ and $\forall P \in \mathcal{I}\dot{\triangle}\mathcal{D}$ with $A \subset P \Rightarrow H_A \backslash P \in \mathcal{I}$

$(i_1) : (\tau_{\|.\|})_{\mathcal{I}} = \tau_{\|.\|}$

$(i_2) : \mathcal{I}_{\tau_{\|.\|}}(X) = X$

$(i_3) : x + \mathcal{I} = \mathcal{I}$, $\lambda\mathcal{I} = \mathcal{I}$, $\forall x \in X$, $\forall \lambda \in \mathbf{R}_+^*$

(C_1) : If $x \in \Phi_{\mathcal{I}}(A)$, then $\forall (n_k)$ an increasing sequence of natural numbers there exists a subsequence (m_k) and there exists $(I_k) \subset \mathcal{I}$ so that $\liminf_k [n_k(A-x)\backslash I_k] \subset \bigcap_k [m_k(A-x)]$.

Definition 1. A σ-ideal $\mathcal{I}$ is called $\mathcal{D}$-complete if it satisfies the conditions $(\mathcal{D})_1, (\mathcal{D})_2, (i_2), (i_3)$ and $D\backslash\Phi_{\mathcal{I}}(D) \in \mathcal{I}, \forall\ D \in \mathcal{D}$.

Theorem 1. ([6,Theorem 6]) *Let $(X, \|.\|)$ be a separable space and let $\mathcal{I}$ be a σ-ideal of subsets of X which satisfies the conditions $(i_2), (i_3)$. Then $\mathcal{I}$ is $\tau_{\|.\|}$-complete if and only if $\mathcal{I}$ satisfies (i_1).*

Let $(X_1, \|.\|_1)$, $(X_2, \|\cdot\|_2)$ be two linear normed spaces, $\tau_1 = \tau_{\|\cdot\|_1}$, $\tau_2 = \tau_{\|\cdot\|_2}$, $X = X_1 \times X_2$ and $\|\cdot\| : X \to \mathbf{R}_+^*$, $\|(x_1, x_2)\| = \max\{\|x_1\|_1, \|x_2\|_2\}$. If $A \subset X$ and $(x_1, x_2) \in X$ we denote by A_{x_1} the set $\{x_2 \in X_2;\ (x_1, x_2) \in A\}$ and by A_{x_2} the set $\{x_1 \in X_1;\ (x_1, x_2) \in A\}$. Let $\mathcal{I}_1 \subset \mathcal{P}(X_1)$, $\mathcal{I}_2 \subset \mathcal{P}(X_2)$ be the two non-empty families and $\mathcal{I}_1 \otimes \mathcal{I}_2$ the family of subsets $I \subset X_1 \times X_2$ with property $\{x_1 \in X_1;\ I_{x_1} \notin \mathcal{I}_2\} \in \mathcal{I}_1$.

Theorem 2. ([7, Theorem 7]) *If X_1 and X_2 are separable spaces, $\mathcal{I}_1$ is a τ_1-complete σ-ideal and $\mathcal{I}_2$ is a τ_2-complete σ-ideal, then $\mathcal{I}_1 \otimes \mathcal{I}_2$ is a $\tau_1 \otimes \tau_2$-complete σ-ideal.*

Proposition 1. *If $\mathcal{I}_1$ and $\mathcal{I}_2$ are σ-ideals then $\mathcal{I}_1 \otimes \mathcal{I}_2$ is a σ-ideal.*

Proposition 2. *If $\mathcal{I}_1$ and $\mathcal{I}_2$ satisfy the condition (i_2) (respectively (i_3)) then $\mathcal{I}_1 \otimes \mathcal{I}_2$ satisfies the condition (i_2) (respectively (i_3)).*

Proof. We suppose that $\mathcal{I}_1$ and $\mathcal{I}_2$ satisfy the condition (i_2). Let D be a non empty open subset of $X_1 \times X_2$. Since $D_{x_1} \in \tau_2, \forall x_1 \in X_1$, $\{x_1 \in X_1;\ D_{x_1} \notin \mathcal{I}_2\} = \{x_1 \in X_1;\ D_{x_1} \neq \emptyset\}$ and hence for $x_1 \in \{x_1 \in X_1;\ D_{x_1} \notin \mathcal{I}_2\}$ there exists $x_2 \in X_2$ and $\varepsilon > 0$ such that $U((x_1, x_2), \varepsilon) \subset D$. Then $U(x_1, \varepsilon) \subset \{x_1 \in X_1;\ D_{x_1} \notin \mathcal{I}_2\}$ and $\{x_1 \in X_1;\ D_{x_1} \notin \mathcal{I}_2\} \notin \mathcal{I}_1$. It result that $(\tau_1 \otimes \tau_2) \cap (\mathcal{I}_1 \otimes \mathcal{I}_2) = \{\emptyset\}$. Hence $\mathcal{I}_1 \otimes \mathcal{I}_2$ satisfies the condition $(\mathcal{I}_1 \otimes \mathcal{I}_2)_{\tau_1 \otimes \tau_2}(X) = X$. Now we suppose that $\mathcal{I}_1$ and $\mathcal{I}_2$ satisfy the condition (i_3). Let $E \in \mathcal{I}_1 \otimes \mathcal{I}_2$, $(y_1, y_2) \in X$ and $\lambda > 0$. We have: $\{x_1 \in X_1;$

$((y_1,y_2)+E)_{x_1} \notin \mathcal{I}_2\} = \{x_1 \in X_1;\ E_{x_1-y_1} \notin \mathcal{I}_2\} = y_1 + \{u_1 \in X_1;\ E_{u_1} \notin \mathcal{I}_2\} \in \mathcal{I}_1 \Rightarrow (y_1,y_2)+E \in \mathcal{I}_1 \otimes \mathcal{I}_2$ and hence $\mathcal{I}_1 \otimes \mathcal{I}_2 \subset (y_1,y_2)+\mathcal{I}_1 \otimes \mathcal{I}_2$. It follows that $(y_1,y_2)+\mathcal{I}_1 \otimes \mathcal{I}_2 \subset (y_1,y_2)+(-(y_1,y_2)+\mathcal{I}_1 \otimes \mathcal{I}_2) = \mathcal{I}_1 \otimes \mathcal{I}_2$. Since $(\lambda E)_{x_1} = \lambda E_{\frac{1}{\lambda}x_1}$, $\{x_1 \in X_1;\ (\lambda E)_{x_1} \notin \mathcal{I}_2\} = \lambda\{x_1 \in X_1;\ E_{x_1} \notin \mathcal{I}_2\} \in \mathcal{I}_1$. It result that $\lambda E \in \mathcal{I}_1 \otimes \mathcal{I}_2$. Hence $\lambda(\mathcal{I}_1 \otimes \mathcal{I}_2) = \mathcal{I}_1 \otimes \mathcal{I}_2$. ∎

Proposition 3. *Let $\mathcal{I}_1$ be a σ-ideal of subset of X_1, $\mathcal{I}_2$ be a σ-ideal of subset of X_2 and $x_1 \in X_1$. Then $y \in (\Phi_{\mathcal{I}_1 \otimes \mathcal{I}_2}(A))_{x_1}$ if and only if for every increasing sequence of natural numbers (m_k) there exists a subsequence (m'_k) and a set $D \in \mathcal{I}_1$ so that*

$$\forall x \in U_1(0,1)\setminus D,\ U_2(0,1) \setminus \liminf_k m'_k(A_{x_1+\frac{1}{m'_k}x} - y) \in \mathcal{I}_2.$$

Proof. If $y \in (\Phi_{\mathcal{I}_1 \otimes \mathcal{I}_2}(A))_{x_1}$ then $(x_1,y) \in \Phi_{\mathcal{I}_1 \otimes \mathcal{I}_2}(A)$. Hence for every increasing sequence of natural numbers (m_k) there exists a subsequence (m'_k) so that $U((0,0),1) \setminus \liminf_k (m'_k(A-(x_1,y)) \in \mathcal{I}_1 \otimes \mathcal{I}_2$. It result that $\{x \in U_1(0,1);\ U_2(0,1) \setminus [\liminf_k (m'_k(A-(x_1,y))]_x \notin \mathcal{I}_2\} \in \mathcal{I}_1$. Let $D = \{x \in U_1(0,1);\ U_2(0,1) \setminus [\liminf_k (m'_k(A-(x_1,y))]_x \notin \mathcal{I}_2\}$ and $x \in U_1(0,1)\setminus D$. It follows that $U_2(0,1) \setminus [\liminf_k (m'_k(A-(x_1,y))]_x \in \mathcal{I}_2$. But

$$[\liminf_k (m'_k(A-(x_1,y))]_x = \bigcup_l \bigcap_{k\geq l} (m'_k(A-(x_1,y))_x =$$

$$= \bigcup_l \bigcap_{k\geq l} m'_k(A_{x_1+\frac{1}{m'_k}x} - y) = \liminf_k m'_k(A_{x_1+\frac{1}{m'_k}x} - y) \in \mathcal{I}_2.$$

Hence $U_2(0,1) \setminus \liminf_k m'_k(A_{x_1+\frac{1}{m'_k}x} - y) \in \mathcal{I}_2$.

Now we suppose that $\forall (m_k), \exists (m'_k)$, a subsequence of (m_k) and $\exists D \in \mathcal{I}_1$ so that $\forall x \in U_1(0,1)\setminus D$, $U_2(0,1) \setminus \liminf_k m'_k(A_{x_1+\frac{1}{m'_k}x} - y) \in \mathcal{I}_2$. Since $U_2(0,1) \setminus \liminf_k m'_k(A_{x_1+\frac{1}{m'_k}x} - y) = U_2(0,1) \setminus [\liminf_k (m'_k(A-(x_1,y))]_x$, it results that $\{x \in U_1(0,1);\ U_2(0,1) \setminus [\liminf_k (m'_k(A-(x_1,y))]_x \notin \mathcal{I}_2\} \subset D$. Hence $U((0,0),1) \setminus \liminf_k (m'_k(A-(x_1,y)) \in \mathcal{I}_1 \otimes \mathcal{I}_2$.

Theorem 3. *Let $\mathcal{D} \subset \mathcal{P}(X_1 \times X_2)$ and let $\mathcal{I} \subset P(X_1 \times X_2)$ be a σ-ideal. If $\mathcal{I}\dot{\Delta}\ \mathcal{D}$ is a σ-algebra and $\exists (D_n) \subset \mathcal{D}\setminus\mathcal{I}$ so that $\forall D \in \mathcal{D}\setminus\mathcal{I}, \exists n$ so that $D_n \setminus D \in \mathcal{I}$, then $\forall A \subset X_1 \times X_2, \exists H_A \in \mathcal{I}\dot{\Delta}\ \mathcal{D}$ so that $A \subset H_A$ and $\forall P \in \mathcal{I}\dot{\Delta}\ \mathcal{D}$ with $A \subset P$ it results that $H_A \setminus P \in \mathcal{I}$.*

Proof. Let $(D_n) \subset \mathcal{D}\setminus\mathcal{I}$ so that $\forall D \in \mathcal{D}\setminus\mathcal{I}, \exists n$ so that $D_n \setminus D \in \mathcal{I}$ and let $A \subset X_1 \times X_2$. If $A \in \mathcal{I}\dot{\Delta}\ \mathcal{D}$ then let $H_A = A$. Now we suppose

that $A \notin \mathcal{I}\dot{\Delta}\ \mathcal{D}$. Let $N_0 = \{n \in \mathbf{N}; A \cap D_n \in \mathcal{I}\}$, $I = \underset{n\in N_0}{\cup} (A \cap D_n)$ and $H_A = \left(\underset{n\in N_0}{\cap} CD_n\right) \cup I$. It results that $I \in \mathcal{I}$ and $H_A \in \mathcal{I}\dot{\Delta}\ \mathcal{D}$. We suppose that $\exists P \in \mathcal{I}\dot{\Delta}\ \mathcal{D}$ such that $A \subset P$ and $H_A \backslash P \notin \mathcal{I}$. Since $H_A \backslash P \in \mathcal{I}\dot{\Delta}\ \mathcal{D}, \exists D \in \mathcal{D}$ and $I_1, I_2 \in \mathcal{I}$ so that $H_A \backslash P = (D \backslash I_1) \cup I_2$. By $H_A \backslash P \notin \mathcal{I}$, $D \notin \mathcal{I}$. It follows that there exists $n_0 \in \mathbf{N}$ such that $D_{n_0} \backslash D \in \mathcal{I}$. We have: $D \backslash I_1 \subset H_A \backslash P \subset H_A \backslash A \Rightarrow (D \backslash I_1) \cap A = \emptyset \Rightarrow D \cap A \subset I_1 \Rightarrow D \cap A \in \mathcal{I}$; $D_{n_0} \cap A \subset (D_{n_0} \backslash D) \cup (D \cap A) \in \mathcal{I}_1 \Rightarrow D_{n_0} \cap A \in \mathcal{I} \Rightarrow n_0 \in N_0 \Rightarrow D_{n_0} \cap H_A = (D_{n_0} \cap \underset{n\in N_0}{\cap} CD_n) \cup (D_{n_0} \cap I_1) = D_{n_0} \cap I_1 \in \mathcal{I} \Rightarrow D_{n_0} \cap D \in \mathcal{I}$ therefore $D_{n_0} \in \mathcal{I}$. ∎

Theorem 4. *Let X_2 be a separable space, $\mathcal{I}_2$ be a τ_2-complete σ-ideal and $\mathcal{I}_1$ be a $\mathcal{D}_1$-complete σ-ideal which satisfies the conditions C_1 and $\tau_1 \subset \mathcal{I}_1\dot{\Delta}\mathcal{D}_1$. Then $A \backslash \Phi_{\mathcal{I}_1 \otimes \mathcal{I}_2}(A) \in \mathcal{I}_1 \otimes \mathcal{I}_2, \forall A$ a F_σ subset of $X_1 \times X_2$.*

Proof. Let (y_n) be a sequence of X_2 such that $cl_{\tau_2}\{y_n\} = X_2$ and let A be a closed subset of $X_1 \times X_2$. Since A_{y_n} is a closed set in X_1 and $\tau_1 \subset \mathcal{I}_1\dot{\Delta}\mathcal{D}_1$ it results that $A_{y_n} \in \mathcal{I}_1\dot{\Delta}\mathcal{D}_1$ and $A_{y_n} \backslash \Phi_{\mathcal{I}_1}(A_{y_n}) \in \mathcal{I}_1, \forall n$.

Let $E = \underset{n}{\cup} (A_{y_n} \backslash \Phi_{\mathcal{I}_1}(A_{y_n}))$. Hence $E \in \mathcal{I}_1$. Let $x_1 \notin E$. The set A_{x_1} is a closed subset of X_2, therefore $A_{x_1} \in \mathcal{I}_2\dot{\Delta}\mathcal{D}_2$. If $A_{x_1} \in \mathcal{I}_2$, then $\Phi_{\mathcal{I}_2}(A_{x_1}) = \emptyset \subset (\Phi_{\mathcal{I}_1 \otimes \mathcal{I}_2}(A))_{x_1}$. Now we suppose that $A_{x_1} \notin \mathcal{I}_2$. Since $\mathcal{I}_2$ is τ_2-complete, by Theorem 1, it results that $\mathcal{I}_2$ satisfies the condition (i_1). Hence $A_{x_1} \setminus int_{\tau_2} A_{x_1} \in \mathcal{I}_2$, $\Phi_{\mathcal{I}_2}(A_{x_1}) = \Phi_{\mathcal{I}_2}(int_{\tau_2} A_{x_1}) \subset \Phi_{\mathcal{I}_2}(cl_{\tau_2}(int_{\tau_2} A_{x_1}))$ and $int_{\tau_2} A_{x_1} \neq \emptyset$. Since $cl_{\tau_2}\{y_n\} = X_2$ there exists $\{y_{n_k}\} \subset int_{\tau_2} A_{x_1}$ a subsequence of $\{y_n\}$ such that $cl_{\tau_2}\{y_{n_k}\} = cl_{\tau_2}(int_{\tau_2} A_{x_1})$. Since $\{y_{n_k}\} \subset A_{x_1}$ we have that $x_1 \in \underset{k}{\cap} A_{y_{n_k}}$, and by $x_1 \notin E$, $x_1 \in \cap_k \Phi_{\mathcal{I}_1}(A_{y_{n_k}})$. Let $y \in \Phi_{\mathcal{I}_2}(A_{x_1})$ and let (m_k) be an increasing sequence of natural numbers. By $y \in \Phi_{\mathcal{I}_2}(cl_{\tau_2}(int_{\tau_2} A_{x_1}))$, there exists (m'_k) a subsequence of (m_k) such that $U_2(0,1) \setminus \liminf_k [m'_k(cl_{\tau_2}(int_{\tau_2} A_{x_1}) - y)] \in \mathcal{I}_2$. On the other hand, by $x_1 \in \Phi_{\mathcal{I}_1}(A_{y_{n_1}})$, there exists (m''_k) a subsequence of (m'_k) such that $U_1(0,1) \setminus \liminf_k [m''_k(A_{y_{n_1}} - x_1)] \in \mathcal{I}_1$. By the condition (C_1), there exists (m^1_k) a subsequence of (m''_k) and there exists $\left(I^1_k\right) \subset \mathcal{I}_1$ such that $\underset{k}{\cap} [m^1_k(A_{y_{n_1}} - x_1)] \supset \liminf_k [m''_k(A_{y_{n_1}} - x_1) \backslash I^1_k]$. We have:

$$\underset{k}{\cap} [m^1_k(A_{y_{n_1}} - x_1)] \supset \liminf_k [m''_k(A_{y_{n_1}} - x_1)] \setminus \underset{k}{\cup} I^1_k$$

and hence

$$U_1(0,1) \setminus \underset{k}{\cap} [m^1_k(A_{y_{n_1}} - x_1)] \subset [U_1(0,1) \setminus \liminf_k (m''_k(A_{y_{n_1}} - x_1))] \cup \cup_k I^1_k \in \mathcal{I}_1.$$

It results that $U_1(0,1)\setminus \bigcap\limits_k [m_k^1(A_{y_{n_1}}-x_1)]\in\mathcal{I}_1$. Analogous, for every $i\in\mathbf{N}\setminus\{0,1,2\}$ there exists (m_k^i) a subsequence of (m_k^{i-1}) so that

$$U_1(0,1)\setminus\bigcap_k [m_k^i(A_{y_{n_i}}-x_1)]\in\mathcal{I}_1.$$

Let $m_k'''=m_k^k,\forall k$. It results that $U_1(0,1)\setminus\bigcap\limits_k [m_k'''(A_{y_{n_i}}-x_1)]\in\mathcal{I}_1,\forall i$. Let $D=\bigcup\limits_i [U_1(0,1)\setminus\bigcap\limits_k m_k'''(A_{y_{n_i}}-x_1)]$. We have $D\in\mathcal{I}_1$ and $U_1(0,1)\setminus D=$ $=\bigcap\limits_{i,k} m_k'''(A_{y_{n_i}}-x_1)$. If $x\in U_1(0,1)\setminus D$, then $\{y_{n_i}\}\subset\cap_k A_{x_1+\frac{1}{m_k'''}x}$. Therefore $cl_{\tau_2}(int_{\tau_2}A_{x_1})=cl_{\tau_2}\{y_{n_k}\}\subset cl_{\tau_2}(\bigcap\limits_k A_{x_1+\frac{1}{m_k'''}x})=\cap_k A_{x_1+\frac{1}{m_k'''}x}$. Since

$$\begin{aligned}\liminf_k m_k'''(A_{x_1+\frac{1}{m_k'''}x}-y)\supset&\liminf_k m_k'''(\bigcap_i A_{x_1+\frac{1}{m_i'''}x}-y)\supset\\ &\supset\liminf_k m_k'''(cl_{\tau_2}(int_{\tau_2}A_{x_1})-y)\ \text{ and}\\ U_2(0,1)\setminus\liminf_k&[m_k'''(cl_{\tau_2}(int_{\tau_2}A_{x_1})-y)]\in\mathcal{I}_2\end{aligned}$$

it results that $U_2(0,1)\setminus\liminf\limits_k m_k'''(A_{x_1+\frac{1}{m_k'''}x}-y)\in\mathcal{I}_2$. As a conclusion for every increasing sequence of natural numbers (m_k) there exists a subsequence (m_k''') and a set $D\in\mathcal{I}_1$ so that $\forall x\in U_1(0,1)\setminus D$,

$$U_2(0,1)\setminus\liminf_k m_k'''(A_{x_1+\frac{1}{m_k'''}x}-y)\in\mathcal{I}_2.$$

By Proposition 3, $y\in(\Phi_{\mathcal{I}_1\otimes\mathcal{I}_2}(A))_{x_1}$. Hence $\Phi_{\mathcal{I}_2}(A_{x_1})\subset(\Phi_{\mathcal{I}_1\otimes\mathcal{I}_2}(A))_{x_1}$ and $(A\setminus\Phi_{\mathcal{I}_1\otimes\mathcal{I}_2}(A))_{x_1}\subset A_{x_1}\setminus\Phi_{\mathcal{I}_2}(A_{x_1})$. Since $\mathcal{I}_2$ is a τ_2-complete σ-ideal, $A_{x_1}\setminus\Phi_{\mathcal{I}_2}(A_{x_1})\in\mathcal{I}_2$ and hence $(A\setminus\Phi_{\mathcal{I}_1\otimes\mathcal{I}_2}(A))_{x_1}\in\mathcal{I}_2$. As a conclusion

$$\{x_1\in X_1;\ (A\setminus\Phi_{\mathcal{I}_1\otimes\mathcal{I}_2}(A))_{x_1}\notin\mathcal{I}_2\}\subset E.$$

It results that

$$\{x_1\in X_1;\ (A\setminus\Phi_{\mathcal{I}_1\otimes\mathcal{I}_2}(A))_{x_1}\notin\mathcal{I}_2\}\in\mathcal{I}_1\text{ and }A\setminus\Phi_{\mathcal{I}_1\otimes\mathcal{I}_2}(A)\in\mathcal{I}_1\otimes\mathcal{I}_2.$$

Now we consider that the set A is F_σ. Let (A_n) be a sequence of closed subsets of $X_1\times X_2$ so that $A=\bigcup\limits_n A_n$. We have that $A\setminus\Phi_{\mathcal{I}_1\otimes\mathcal{I}_2}(A)=$ $=(\bigcup\limits_n A_n)\setminus\Phi_{\mathcal{I}_1\otimes\mathcal{I}_2}(\bigcup\limits_n A_n)\subset(\bigcup\limits_n An)\setminus(\bigcup\limits_m\Phi_{\mathcal{I}_1\otimes\mathcal{I}_2}(A_m))\subset\bigcup\limits_n(A_n\setminus\Phi_{\mathcal{I}_1\otimes\mathcal{I}_2}(A_n))$. Since $A_n\setminus\Phi_{\mathcal{I}_1\otimes\mathcal{I}_2}(A_n)\in\mathcal{I}_1\otimes\mathcal{I}_2,\forall n$, it results that $\bigcup\limits_n(A_n\setminus\Phi_{\mathcal{I}_1\otimes\mathcal{I}_2}(A_n))\in$ $\in\mathcal{I}_1\otimes\mathcal{I}_2$ and hence $A\setminus\Phi_{\mathcal{I}_1\otimes\mathcal{I}_2}(A)\in\mathcal{I}_1\otimes\mathcal{I}_2$. ∎

Theorem 5. *Let X_2 be a separable space, $\mathcal{I}_2$ be a τ_2-complete σ-ideal and $\mathcal{I}_1$ be a $\mathcal{D}_1$-complete σ-ideal which satisfies the conditions C_1 and $\tau_1 \subset \mathcal{I}_1 \dot{\Delta} \mathcal{D}_1$. Let $\mathcal{B}$ the family of Borel sets in $X_1 \times X_2$. If there exists $(B_n) \subset \mathcal{B} \backslash (\mathcal{I}_1 \otimes \mathcal{I}_2)$ a sequence of F_σ sets so that $\forall B \in \mathcal{B} \backslash (\mathcal{I}_1 \otimes \mathcal{I}_2), \exists n$ so that $B_n \subset B$ and $B \backslash B_n \in \mathcal{I}_1 \otimes \mathcal{I}_2$, then $\mathcal{I}_1 \otimes \mathcal{I}_2$ is a $\mathcal{B}$-complete σ-ideal.*

Proof. By Proposition 1, $\mathcal{I}_1 \otimes \mathcal{I}_2$ is a σ-ideal and since $\mathcal{B}$ is a σ-algebra it follows that $(\mathcal{I}_1 \otimes \mathcal{I}_2) \dot{\Delta} \mathcal{B}$ is a σ-algebra and hence $\mathcal{I}_1 \otimes \mathcal{I}_2$ satisfies the condition $(\mathcal{B})_1$. By Proposition 2 and Theorem 3 it results that $\mathcal{I}_1 \otimes \mathcal{I}_2$ satisfies the conditions $(i_2), (i_3)$ and $(\mathcal{B})_2$. Since the conditions by Theorem 4 are verified it results that $A \backslash \Phi_{\mathcal{I}_1 \otimes \mathcal{I}_2}(A) \in \mathcal{I}_1 \otimes \mathcal{I}_2, \forall A$ a F_σ subset of $X_1 \times X_2$. If $B \in \mathcal{B} \backslash (\mathcal{I}_1 \otimes \mathcal{I}_2)$ there exists $n \in \mathbf{N}$ so that $B_n \subset B$ and $B \backslash B_n \in \mathcal{I}_1 \otimes \mathcal{I}_2$. Since $B_n \backslash \Phi_{\mathcal{I}_1 \otimes \mathcal{I}_2}(B_n) \in \mathcal{I}_1 \otimes \mathcal{I}_2$, $B = B_n \cup (B \backslash B_n)$ and $\Phi_{\mathcal{I}_1 \otimes \mathcal{I}_2}(B_n) = \Phi_{\mathcal{I}_1 \otimes \mathcal{I}_2}(B)$ it results that $B \backslash \Phi_{\mathcal{I}_1 \otimes \mathcal{I}_2}(B) \in \mathcal{I}_1 \otimes \mathcal{I}_2$. Therefore $\mathcal{I}_1 \otimes \mathcal{I}_2$ satisfies the condition $B \backslash \Phi_{\mathcal{I}_1 \otimes \mathcal{I}_2}(B) \in \mathcal{I}_1 \otimes \mathcal{I}_2, \forall B \in \mathcal{B}$ and hence $\mathcal{I}_1 \otimes \mathcal{I}_2$ is a $\mathcal{B}$-complete σ-ideal. ∎

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