ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLV, s.I a, Matematică, 1999, f.2.

A CONTRIBUTION TO THE STUDY OF BOL–MOUFANG TYPE ALGEBRAS

BY

J.0. ADENIRAN and J.A. OGUNTUASE

Abstract. It is proved that loops satisfying the RC(LC,C) identities generate algebras satisfying the same identities. Relationships between RC(LC,C) algebras with certain Bol–Moufang type identities are also established.

1. Introduction. The class of loops and algebras satisfying the Bol–Moufang type identities have been studied and extended in various directions. See for example [1], [2], [3], [4], [5] and the references therein.

In this paper we shall study algebras satisftying the following identities

1. (a) $(xy \cdot z)z = x(yz \cdot z)$
   (b) $(xy \cdot z)z = x(y \cdot zz)$
   (c) $xy \cdot zz = x(yz \cdot z)$
2. (a) $x(x \cdot yz) = (x \cdot xy)z$
   (b) $x(x \cdot yz) = (xx \cdot y)z$
   (c) $(x \cdot xy)z = xx \cdot yz$
3. $(zx \cdot x)y = z(x \cdot xy)$

Identities (1) are called the RC–identities and any loop satisfying any one of them satisfies the others. Identities (2) are called the LC–identites and any loop satisfying any one of them satifies the others. Identity (3) is called the C–identity. It is the object of this study to contribute to the study of Bol–Moufang type algebras by examining the properties of algebras satisfying the aforementioned identities.

Throughout this paper, we shall assume that A is an algebra (non–associative) over a scalar field F with an identity element 1.

The mapping $R_a : A \longrightarrow A$ is defined as $R_a : x \longrightarrow xa$. Similarly the mapping $L_a : A \longrightarrow A$ is defined as $L_a : x \longrightarrow ax$.

2. Main results.

Theorem 2.1. *Let A be an algebra over a field F. Then:*

(i) *A is an* RC*–algebra if and only if A satifies one of the following equivalent identities:*

$$(xy \cdot z)w + (xy \cdot w)z - x(yz \cdot w) - x(yw \cdot z) = 0$$

or

$$(xy \cdot z)w + (xy \cdot w)z - x(y \cdot zw) - x(y \cdot wz) = 0$$

or

$$(xy \cdot zw) + (xy \cdot wz) - x(yz \cdot w) - x(yw \cdot z) = 0$$

(ii) *A is an* LC*–algebra over F (where the characteristic of $F \neq 2$) if and only if A satifies one of the following equivalent identities*

$$x(w \cdot yz) + w(x \cdot yz) - (x \cdot wy)z - (w \cdot xy)z = 0$$

or

$$x(w \cdot yz) + w(x \cdot yz) - (xw \cdot y)z - (wx \cdot y)z = 0$$

or

$$(x \cdot wy)z + (w \cdot xy)z - xw \cdot yz - wx \cdot yz = 0$$

(iii) *A is a* C*–algebra if and only if A satifies one of the following identity:*

$$(zx \cdot w)y + (zw \cdot x)y - z(x \cdot wy) - z(w \cdot xy) = 0$$

for all $x, y, z, w \in A$.

Proof. (i) Let A be an RC–algebra, then it follows that A satisfies $(1a, b, c)$ for all $x, y, z \in A$. Thus A satisfies (1a) if and only if

$$(xy \cdot z)z - x(yz \cdot z) = 0.$$

We linearize (4) by substituting $z = z + w$ and we obtain

$$(xy \cdot (z + w))(z + w) - x(y(z + w) \cdot (z + w)) = 0.$$

That is

$$(xy \cdot z)w + (xy \cdot w)z - x(yw \cdot z) - x(yz \cdot w) = 0.$$

Hence (4).

Conversely let A be an algebra which satisfies (4), putting $w = z$ in (4) we obtain

$$(xy \cdot z)z + (xy \cdot z)z - x(yz \cdot z) - x(yz \cdot z) = 0$$

which, because $\mathrm{Car}F \neq 2$ implies that

$$(xy \cdot z)z = x(yz \cdot z) \text{ for all } x, y, z \in A.$$

The proofs of (1b,c), (2a,b.c) and 3 are similar, hence the details are omitted.

Theorem 2.2. *Let A be the loop algebra generated by an* RC*–loop over F. Then A is an* RC*–algebra* (*respectively* LC, C *algebra*).

Proof. Let A be the loop algebra generated by an RC–loop $(L, \cdot)$. Then $x, y, z \in A$ can be expressed in the form

$$x = \sum_{i=1}^{n_1} \alpha_i x_i, \quad y = \sum_{j=1}^{n_2} \beta_j y_j, \quad z = \sum_{k=1}^{n_3} \gamma_k z_k$$

where $x_i, y_j, z_k \in \mathrm{L}$ and $\alpha_i, \beta_j, \gamma_k \in \mathrm{F}$. Then

$$\begin{aligned} (xy \cdot z)z &= \sum \alpha_i\beta_j\gamma_k\gamma_l(x_iy_j \cdot z_k)z_l \\ &= \sum_{k=l} \alpha_i\beta_j\gamma_k^2(x_iy_j \cdot z_k)z_k + \sum_{k\neq l} \alpha_i\beta_j\gamma_k\gamma_l[(x_iy_j \cdot z_k)z_l + (x_iy_j \cdot z_l)z_k] \end{aligned} \tag{6}$$

for all $x_i, y_j, z_k, z_l \in L$ and $\alpha_i, \beta_j, \gamma_k, \gamma_l \in F$.

Similarly, we have

$$\begin{aligned} x(yz \cdot z) &= \sum \alpha_i\beta_j\gamma_k\gamma_l x_i(y_jz_k \cdot z_l) \\ &= \sum_{k=l} \alpha_i\beta_j\gamma_k^2 x_i(y_jz_k \cdot z_k) + \sum_{k\neq l} \alpha_i\beta_j\gamma_k\gamma_l[x_i(y_jz_k \cdot z_l) + x_i(y_jz_l \cdot z_k)] \end{aligned} \tag{7}$$

for all $x_i, y_j, z_k, z_l \in L$ and $\alpha_i, \beta_j, \gamma_k, \gamma_l \in F$. Hence

$$(8)\quad \begin{aligned}(xy\cdot z)z - x(yz\cdot z) &= \sum_{k=l} \alpha_i\beta_j\gamma_k^2[(x_iy_j\cdot z_k)z_k - x_i(y_jz_k\cdot z_k)] \\ +\sum_{k\neq l} \alpha_i\beta_j\gamma_k\gamma_l[(x_iy_j\cdot z_k)z_l &+ (x_iy_j\cdot z_l)z_k - x_i(y_jz_k\cdot z_l) - x_i(y_jz_l\cdot z_k)]\end{aligned}$$

By 1(a)

$$(x_iy_j \cdot z_k)z_k = x_i(y_jz_k \cdot z_k)$$

and by (4), equation (15) becomes

$$(x_iy_j \cdot z_k)\cdot z_l + (x_iy_j \cdot z_l)\cdot z_k = x_i(y_jz_k)\cdot z_l) + x_i(y_jz_l)\cdot z_k).$$

Thus using (8) we obtain

$$(xy \cdot z)z = x(yz \cdot z) \quad \text{for all } x, y, z \in A.$$

Hence A is an RC–algebra.

The proofs for LC and C are similar and omitted.

In the following theorem we shall assume that A^* is the loop of units in A.

Theorem 2.3. *In an* RC*–algebra A the following are true*

$$R_yR_zR_w + R_yR_wR_z - R_{yz\cdot w} - R_{yw\cdot z} = 0 \quad \forall y, z, w \in A$$

$$R_yR_zR_y^{-1} = R_{yz\cdot y^{-1}} \qquad \forall y \in A^*, z \in A$$

$$A \text{ is a Bol algebra and } \quad y^2 \in N_\mu \qquad \forall y \in A.$$

Proof.

(16) Let A be an RC–algebra. From (4) we obtain

$$xR_yR_zR_w + xR_yR_wR_z - xR_{yz\cdot w} - xR_{yw\cdot z} = 0$$

which implies that (16) holds for all $x, y, z \in A$.

(17) Let $w = y^{-1}$ in (16) we obtain

$$R_yR_zR_{y^{-1}} + R_yR_{y^{-1}}R_z - R_{yz\cdot y^{-1}} - R_z = 0.$$

Now Lemma 5[4] implies that

$$R_{y^{-1}} = R_y^{-1} \quad \forall y \in A.$$

Hence

$$R_y R_z R_y^{-1} = R_{yz \cdot y^{-1}} \quad \forall y, z \in A^* \quad (A^* \text{ is the loop of units in } A)$$

(18) Let $w = y$ in (4), we obtain

$$(xy \cdot z)y + (xy \cdot y)z - x(yz \cdot y) - x(yy \cdot z) = 0.$$

This implies by Lemma 5[4] that

$$(xy \cdot z)y + xy^2 \cdot z - x(yz \cdot y) - x^2 \cdot y^2 z = 0$$

which is only true if only (18) holds.

Theorem 2.4. *In an* LC*–algebra A the following are true*

$$L_y L_w L_x + L_y L_x - L_{x \cdot wy} - L_{w \cdot xy} = 0 \quad \forall x, y, w \in A$$

$$L_y L_x L_y^{-1} = L_{y^{-1} \cdot xy} \quad \forall x \in A, y \in A^*$$

$$A \text{ is a left} - \text{Bol algebra and } y^2 \in N_\mu \quad \forall y \in A.$$

Proof. Similar to that of Theorem 2.3.

Corollary 2.1. *In a* C*–algebra A, the following are true*

1. *A is conjugacy closed*
2. *A is a Moufang algebra.*

Proof. Since C–algebra are essentially those algebra that are both RC and LC, then the proof follows from the proofs of Theorems 2.3 and 2.4.

Acknowledgements. The authors would like to thank the referee for the careful review of the manuscript and his useful suggestions which have improved the final version of this paper.

REFERENCES

1. BRUCK, R.H. – *A Survey of Binary Systems,* Springer–Verlag, Berlin, 1958.
2. FENYVES, F. – *Extra loops* I, Publ. Math. Debrecen 15(1968), 235–238, MR 38, No 5976.
3. FENYVES, F. – *Extra loops* II, Publ. Math. Debrecen 16(1969), 187–192, MR 41, No 7017.
4. RAMAMURTHI, V.S. & SOLARIN, A.R.T. – *On finite Right–Central loops,* Publ. Math. Debrecen 35 (1988), 261–264.
5. SOLARIN, A.R.T. – *A short note on Bol algebra,* Riv. Mat. Pura Appl., 8 (1991), 7–11.
6. SOLARIN, A.R.T. – *On algebra of Bol–Moufang type,* An. Ştiinţ. Univ. "Al.I. Cuza" Iaşi Mat.(N.S.), XLI (1995), 45–56.

Received: 10.VIII.1998
Revised: 18.X.1999

Department of Mathematical Sciences
University of Agriculture
Abeokuta
NIGERIA
e-mail: adedayo@unaab.edu.ng