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MULTIFUNCTIONS AND UNIFORMITY OF METRIC SPACES

BY

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Introduction. Let (X,d) be a metric space. Let $S(x,r)$ and $S[x,r]$ denote the open and closed spheres in X with center x and radius r. For every $r>0$, we define the multifunctions F_r and Φ_r by

$$x \xrightarrow{F_r} S(x,r),\ x\in X, \quad \text{and} \quad x \xrightarrow{\Phi_r} S[x,r],\ x\in X.$$

The graphs of the multifunctions $F_r,\ r>0$ ($\Phi_r,\ r>0$) are the sets $\{(x,y);\ d(x,y)<r\}$, $r>0$ ($\{(x,y);\ d(x,y)\leq r\}$, $r>0$). Since the collection of these sets is a base for the metric uniformity, consisting of the open (closed) entourages, it will exist a connection between the properties of the multifunctions F_r and Φ_r and those of the metric space.

The aim of this paper is to point out some aspects of this interdependence. More precisely, we will establish the conditions under which F_r and Φ_r are lower or upper semicontinuous multifunctions. In other words, we will see that, if d satisfies certain conditions, the sections of the above entourages (i.e. the spheres $S(x,r)$ or $S[x,r]$) vary continuously as a function of x.

A multifunction F from a topological space X to a topological space Y is called *lower* (*upper*) *semicontinuous at* $x_0\in X$ if for any open set $U\subset Y$ with $F(x_0)\cap U\neq\emptyset$ ($F(x_0)\subset U$) there is a neighbourhood V of x_0 such that $F(x)\cap U\neq\emptyset$ ($F(x)\subset U$) for any $x\in V$. Further, F is *continuous at* x_0 if it is both lower semicontinuous (l.s.c.) and upper semicontinuous (u.s.c.) at x_0 [5]. A metric space (X,d) is *convex in the sense of Menger* if for every two distinct points x and y of X there exists a point z such that $x\neq z\neq y$ and

$$d(x,y)=d(x,z)+d(z,y) \quad [11]. \tag{1}$$

The space (X,d) is *totally convex* if for $x,y \in X$ with $x \neq y$ and $\alpha \in (0, d(x,y))$ there exists $z \in X$ such that

$$d(x,z) = \alpha \quad \text{and} \quad d(x,z) + d(z,y) = d(x,y) \text{ [11].} \tag{2}$$

(X,d) is called *quasiconvex* if for each $\varepsilon > 0$ and any two distinct points $a, b \in X$ there is $\delta > 0$ and a point $x \in X$ such that

$$d(a,x) < d(a,b) - \delta \quad \text{and} \quad d(b,x) < \varepsilon \text{ [6].} \tag{3}$$

In [13], a metric space (X,d) is said to be *round* if the equality

$$\overline{S(x,r)} = S[x,r] \tag{4}$$

holds for all $x \in X$ and $r > 0$. Obviously, the normed spaces are round. E. BLANC proved in [6] that (X,d) verifies the condition (3) (i.e. it is round) if and only if it is quasiconvex. If (X,d) is totally convex, then it is also quasi convex. If a metric space is complete and convex in the sense of Menger, then it is total convex [11]. Concerning the subject of round spaces see also [1] and [3].

The metric space (X,d) is said to be *L-space* if for any two nonempty closed sets A and B such that $A \cap B = \emptyset$ we have $d(A,B) > 0$, where $d(A,B) = \inf\{d(a,b);\ a \in A \text{ and } b \in B\}$. Each L-space is complete. These spaces are characterized by the property that every real-valued continuous function on X is uniformly continuous. The L-spaces have many remarkable properties (see [2], [4], [10], [12] etc.).

2. Lower semicontinuity of F_r. This case is the simplest, because we have:

Proposition 1. *The multifunction F_r is* l.s.c. *for each $r > 0$.*

Proof. Let $x_0 \in X$ and U be an open subset of X such that $S(x_0,r) \cap U \neq \emptyset$. Hence, there is $y \in S(x_0,r) \cap U$.

We put $V = S(x_0, r - d(x_0,y))$. Consequently, for any $x \in V$ we have:

$$d(x,y) \leq d(x,x_0) + d(x_0,y) < [r - d(x_0,y)] + d(x_0,y) = r,$$

that is $d(x,y) < r$ or $y \in S(x,r)$. Hence, it follows that $y \in S(x,r) \cap U$ or $S(x,r) \cap U \neq \emptyset$. Therefore, F_r is l.s.c.

3. Upper semicontinuity of F_r. Generally, the multifunctions F_r may not be u.s.c. The Euclidean space $\mathbb{R}^n$ is an example in this sense.

Indeed, let $x_0 \in \mathbb{R}^n$ and $U = S(x_0, r)$. Obviously, $S(x_0, r) \subset U$, but the inclusion $S(x, r) \subset S(x_0, r)$ is not true if $x \neq x_0$.

The following assertion follows immediately from the definition of the upper semicontinuity, and is stated without proof:

Proposition 2. *Let (X, d) be a metric space. The following properties are equivalent:*

1° *F_r is* u.s.c. *for any $r > 0$;*
2° *there exists $\varphi : X \times \mathbb{R}_+^* \to (0, 1]$ such that the implication*

$$y \in S(x, r\varphi(x, r)) \Longrightarrow S(y, r) \subset S(x, r)$$

holds for each $x \in X$ and $r > 0$.

The condition 2° appears in [7, p.202] as the condition I_2 concerning the uniformization of the topological structures satisfying the first axiom of countability.

Naturally, there is an interdependence between the structure of the metric space and the properties of φ.

To make evident a class of metric spaces for which all F_r are u.s.c., we need the following

Lemma 1. *Let (X, d) be a metric space and $\psi : X \to (0, 1]$ be an arbitrary function. Then, the following conditions are equivalent:*

1° *for each $x \in X$ and $r > 0$, we have:*

$$y \in S(x, r\psi(x)) \Longrightarrow S(y, r) \subset S(x, r); \tag{5}$$

2° *the metric d verifies the inequality*

$$d(x, z) \leq \max\left\{\frac{1}{\psi(x)} d(x, y), d(y, z)\right\}, \ x, y, z \in X. \tag{6}$$

Proof. 1°$\Longrightarrow$2° First we observe that 1° can be written in the form

$$\begin{gathered} d(x, y) < r\psi(x) \ \text{ and } \ d(y, z) < r \Longrightarrow d(x, z) < r, \\ \text{for each } x \in X \text{ and } r > 0. \end{gathered} \tag{7}$$

Suppose that there exist three points $x, y, z \in X$ for which the contrary inequality of (6) holds. Then, there is $\alpha > 0$ such that we have

$$\max\left\{\frac{1}{\psi(x)} d(x, y), d(y, z)\right\} < \alpha < d(x, z).$$

Consequently, we obtain

$$d(x,y) < \alpha\psi(x),\ d(y,z) < \alpha \ \text{ and } \ \alpha < d(x,z).$$

These inequalities together with (7) lead us to

$$d(x,z) < \alpha \ \text{ and } \ \alpha < d(x,z)$$

which is absurd.

$2^\circ \Longrightarrow 1^\circ$ Indeed, by (6) and the relations $d(x,y) < r\psi(x)$ and $d(y,z) < r$ it follows that $d(x,r) < r$. Therefore (7) holds and, hence, the condition 1° is established.

Proposition 3. *Let (X,d) be a metric space and $\psi : X \to (0,1]$ be an arbitrary function. If the relation* (6) *holds, then the multifunctions F_r, $r > 0$, are* u.s.c.*, and, hence, they are continuous.*

Proof. The fact that F_r is u.s.c. follows from Lemma 1 and Proposition 2. According to Proposition 1, F_r is also l.s.c. Hence F_r is continuous.

Corollary 1. *If (X,d) is a ultrametric space, i.e., the metric d verifies the strong triangle inequality*

$$d(x,z) \le \max\{d(x,y), d(y,z)\},\ x,y,z \in X, \tag{8}$$

then the multifunctions F_r, $r > 0$, are continuous.

Proof. We put $\psi(x) = 1$, $x \in X$, in the last proposition.

Let now $d : X \times X \longrightarrow \mathbb{R}_+$ be a function satisfying the conditions: a) $d(x,y) = 0$ if and only if $x = y$; b) $d(x,y) = d(y,x),\ x,y \in X$; c) the triangle inequality (6), where $\psi : X \longrightarrow (0,1]$ is such that $\psi(x) \ge \alpha > 0$, $x \in X$. So, we do not suppose that d is a metric on X. The next result point out the importance of the inequality (6). We first remind an useful result:

Lemma 2. [8, Th.1] *If the function $d : X \times X \longrightarrow \mathbb{R}_+$ verifies the above conditions* a), b) *and the following triangle inequality*

$$d(x,z) \le \max\{\gamma d(x,y), d(y,z)\},\ x,y,z \in X, \tag{9}$$

where $\gamma \ge 1$, then there is an ultrametric d^* which induces the same topology on X as d.

Proposition 4. *Let X be an arbitrary nonempty set and*

$$d : X \times X \longrightarrow \mathbb{R}_+$$

be a function verifying the conditions a), b), *and* c). *Then*

1° *d induces a topology on X in an obvious way ($S(x,r)$ will denote the open sphere centred at x and of radius r with respect to d);*
2° *the multifunction F_r defined by $x \longrightarrow S(x,r)$, $x \in X$, is continuous, for any $r > 0$;*
3° *there is an ultrametric d^* on X which is equivalent to d.*

Proof. 1° It is trivial to check the axioms $(V_I) - (V_{III})$ of the neighbourhood systems with respect to d [9, p.15]. For the axiom (V_{IV}), let $x \in X$ and V be a neighbourhood of x. There exists $r > 0$ such that $V \supset S(x,r)$. Choose $W = S(x, \alpha r)$. Using the triangle inequality (6), it is not hard to prove that $S(y,r) \subset S(x,r)$ for all $y \in W$. It results that $V \supset S(y,r)$, i.e. V is a neighbourhood for any $y \in W$. Thus, (V_{IV}) is verified.

2° Let $x_0 \in X$. Therefore $F_r(x_0) = S(x_0, r)$. We mention that the sphere $S(x_0, r)$ is not necessarily an open set in the topology induced by d. We can prove the lower semicontinuity of F_r at the point x_0 as in Proposition 1, if we pick $V = S(x_0, \alpha r)$. The upper semicontinuity of F_r follows directly from the inclusion $S(x,r) \subset S(x_0, r)$ which is true for all $x \in S(x_0, \alpha r)$.

3° Under the assumption relative to ψ, it results that

$$d(x,z) \leq \max\left\{\frac{1}{\alpha} d(x,y), d(y,z)\right\}, \ x, y, z \in X.$$

This inequality is of the form (9) with $\gamma = \dfrac{1}{\alpha} > 1$. Now, the desired affirmation is a consequence of Lemma 2.

4. Lower semicontinuity of Φ_r. In view of the study of the lower or upper semicontinuity of Φ_r it is necessary to assume that (X, d) satisfies some conditions of convexity.

Proposition 5. *If (X, d) is a round metric space, then the multifunction Φ_r is* l.s.c., *for any $r > 0$.*

Proof. Let $x_0 \in X$ and U be an open subset such that $U \cap S[x_0, r] \neq \emptyset$. Hence, there exists $y \in U \cap S[x_0, r]$. Further, let $\varepsilon > 0$ such that $S(y, \varepsilon) \subset U$. Because a round space is exactly quasiconvex one, there exist $\delta > 0$ and $z \in X$ satisfying the next conditions:

$$d(y,z) < \varepsilon \ \text{ and } \ d(x_0, z) < d(x_0, y) - \delta.$$

We put $V = S(x_0, \delta)$. So, for any point $x \in V$, we have:

$$d(x,z) \leq d(x, x_0) + d(x_0, z) < \delta + [d(x_0, y) - \delta] = d(x_0, y) \leq r,$$

hence $z \in S(x,r)$. Therefore, $z \in U \cap S[x,r]$. Thus, for any $x \in V$ we have $U \cap \Phi_r(x) \neq \emptyset$, that is Φ_r is l.s.c. at x_0.

Remark. The converse of the previous result is false. Every discrete metric space is an example in this sense.

5. Upper semicontinuity of Φ_r. First of all, we introduce a notion less restrictive than that of L-space.

A metric space (X,d) is called *wL-space* if for any nonempty closed set A and any closed sphere $S[x,r]$ such that $A \cap S[x,r] = \emptyset$ we have $d(A, S[x,r]) > 0$.

It is clear that a L-space is at the same time a wL-space. In contrast to L-spaces, wL-spaces are not complete. For example, $X = (0,1)$ endowed with the Euclidean metric is a wL-space, but is not complete. Moreover, a complete wL-space is not necessary a L-space. Thus, the Euclidean space $\mathbb{R}$ is an example in this sense.

We need a preliminary lemma.

Lemma 3. *If (X,d) is totally convex, then for each $x, y \in \mathbb{R}$ and $r > 0$ we have*

$$d(y, S[x,r]) = \begin{cases} 0, & y \in S[x,r] \\ d(x,y) - r, & y \notin S[x,r]. \end{cases} \tag{10}$$

We are now ready to prove the principal result of this section.

Proposition 6. *If (X,d) is complete and convex in the sense of Menger, then the following conditions are equivalent:*

1° *X is a wL-space;*
2° *Φ_r is* u.s.c., *for any $r > 0$.*

Proof. Observe that, under our hypotheses, (X,d) is totally convex.

1°$\Longrightarrow$2° Let $x_0 \in X$ and U be an open subset with $U \supset S[x_0, r]$. We put $A = X - U$ and observe that this is a nonempty closed subset. Since $A \cap S[x_0, r] = \emptyset$, and taking into account the condition 1°, it follows that

$$a = d(A, S[x_0, r]) > 0.$$

Let $V = S(x_0, a)$. To show that Φ_r is u.s.c. at x_0 it suffices to verify that

$$S[x,r] \subset U, \quad \text{for all } x \in V. \tag{11}$$

For this, assume the contrary. So, there exists $y \in A \cap S[x,r]$. Then, in view of Lemma 3, we have

$$a \le d(y, S[x_0,r]) = d(y,x_0) - r \le d(y,x) + d(x,x_0) - r < r + a - r = a,$$

i.e. $a < a$. This contradiction proves the relation (11).

$2^\circ \Longrightarrow 1^\circ$ To prove this implication, we assume that (X,d) satisfies 2°, but it is not a wL-space, and reach a contradiction. Hence, there exist a nonempty closed set A and a sphere $S[x_0,r]$ such that (i) $A \cap S[x_0,r] = \emptyset$ and (ii) $d(A, S[x_0,r]) = 0$. The set $U = X - A$ is open, and we have $U \supset S[x_0,r]$. Since Φ_r is u.s.c. at x_0, there is $\varepsilon > 0$ such that $S[x,r] \subset U$, if $x \in S[x_0,\varepsilon]$. Again, by virtue of (ii), there exist $a \in A$ and $s \in S[x_0,r]$ such that $d(a,s) < \varepsilon$.

Now, (X,d) being totally convex, there is $z \in X$ satisfying the conditions

$$d(x_0,z) = \varepsilon \quad \text{and} \quad d(x_0,z) + d(z,a) = d(x_0,a). \tag{12}$$

Consequently, $z \in S[x_0,\varepsilon]$ and, hence, we have

$$S[z,r] \subset U. \tag{13}$$

By (12), we can write

$$d(z,a) = d(x_0,a) - \varepsilon \le d(x_0,s) + d(s,a) - \varepsilon < r + \varepsilon - \varepsilon = r,$$

that is $d(z,a) < r$ or $a \in S(z,r)$. From this and (13), it follows that $a \in U$, i.e. $a \in X - A$. On the other hand, $a \in A$, and we have a contradiction.

Corollary 2. *If a L-space (X,d) is convex in the sense of Menger, then Φ_r is* u.s.c.*, for any $r > 0$.*

Corollary 3. *If $(X, \|\cdot\|)$ is a Banach space, then the conditions* 1° *and* 2° *of Proposition* 6 *are equivalent.*

Remark. The convexity condition which appears in the previous results is not necessary for the upper semicontinuity of Φ_r. Thus, a discrete metric space is not convex in the sense of Menger, but the multifunctions Φ_r, $r > 0$, are u.s.c.

The next assertion follows without difficulty.

Proposition 7. *In every ultrametric space (X,d) the multifunctions Φ_r, $r > 0$, are continuous.*

6. Remark on pseudometric spaces. Suppose that (X, d) is a pseudometric space having at least two points x and y with $d(x, y) \neq 0$. All the results in the previous sections remain valid in this new framework if in the convexity conditions the relation $x \neq y$ is replaced by $d(x, y) \neq 0$.

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