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ON THE BRYANT–SCHNEIDER GROUP OF A CONJUGACY CLOSED LOOP

BY

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Abstract. In this work, a necessary and sufficient condition for an element of the Bryant–Schneider group of a conjugacy closed loop to be an autotopism is obtained.

Introduction. Conjugacy closed loops have attracted a lot of consideration. Recent studies on this subject include those of GOODAIRE and ROBINSON [5] where it was proved that conjugacy closed loops are G-loops, CHIBOKA and SOLARIN [4] where it was proved that extra loops are conjugacy closed and that of ADENIRAN and SOLARIN [1] where it was shown that conjugacy closed loops are flexible and are M-loops.

ROBINSON [6] has characterized the Bryant–Schneider group of a loop as a generalized holomorph of the loop. He also proved that the Bryant–Schneider of a loop is an isotopic invariant. CHIBOKA [3] considered the Bryant–Schneider group of an extra loop. Extra loops are examples of conjugacy closed loops. The object of this work is to obtain the necessary and sufficient condition for an element of the Bryant–Schneider group of a conjugacy closed loop to be an automorphism.

For the definition of concepts in the theory of loops readers may consult BRUCK [2]. If x is an element of a loop $(Q,\cdot)$ the translation maps R_x and L_x of $(Q,\cdot)$ are defined by $aR_x = ax$ and $aL_x = xa \forall a, x \in Q$. If $(Q,\cdot)$ is a quasigroup, the translation mappings R_x and L_x are permutations and so the inverse mappings $R_{x^{-1}}$ and $L_{x^{-1}}$ are also permutations, although they may not be translations.

The left nucleus N_λ, middle nucleus N_μ, right nucleus N_ρ, nucleus N, centrum C and centre Z of $(Q, \cdot)$ are respectively defined as follows

$$\begin{aligned}
N_\lambda &= \{x \in Q : xy \cdot z = x \cdot yz,\ \forall y, z \in Q\} \\
N_\mu &= \{y \in Q : xy \cdot z = x \cdot yz,\ \forall x, z \in Q\} \\
N_\rho &= \{x \in Q : xy \cdot z = x \cdot yz,\ \forall y, z \in Q\} \\
N &= N_\lambda \cap N_\mu \cap N_\rho \\
C &= \{x \in Q : xy = yx,\ \forall y \in Q\} \\
Z &= N \cap C
\end{aligned}$$

An ordered triple $< \alpha, \beta, \gamma >$ of permutations of a loop $(Q, \cdot)$ is called an autotopism of the loop $(Q, \cdot)$ if

$$x\alpha \cdot y\beta = (x \cdot y)\gamma \qquad \forall x, y \in Q$$

The set of all autotopism of a loop $(Q, \cdot)$ form a group under componentwise multiplication.

A loop $(Q, \cdot)$ is conjugacy closed if $L_x L_y L_x^{-1} \in \Pi_L$ and $R_x R_y R_x^{-1} \in \Pi_R$ where Π_L and Π_R are the set of all right and left translations respectively of the loop $(Q, \cdot)$.

Let $(Q, \cdot)$ be a loop, a permutation α of Q is called a pseudo–automorphism of $(Q, \cdot)$ if and only if there exist an element $c \in Q$ such that $< \alpha, \alpha R_c, \alpha R_c >$ is an autotopism of Q. The set of all pseudo-automorphism of Q, denoted by $P(Q, \cdot)$ is a subgroup of the set of all permutations of Q.

Let $(Q, \cdot)$ be a loop and $\mathrm{BS}(Q, \cdot)$ the set of all permutations α of Q such that $< \alpha R_{g^{-1}}, \alpha L_{f^{-1}}, \alpha >$ is an autotopism of Q for some $f, g \in Q$. Then, $\mathrm{BS}(Q, \cdot)$ is called Bryant–Schneider group of the loop.

2. Preliminary Results.

Theorem 2.1. [2]

1. *If $(Q, \cdot)$ is an inverse property loop, then the nuclie N_λ, N_μ and N_ρ coincide with the nucleus N of Q.*
2. *If $< U, V, W >$ is an autotopism of an inverse property loop then $< W, JVJ, U >$ and $< JUJ, W, V >$ are autotopisms of $(Q, \cdot)$.*

3. *Every loop isotopic to a Moufang loop is isomorphic to a principal isotope* $(Q, \cdot)$ *with the operation "$\circ$" given by* $x \circ y = (xf) \cdot (f^{-1}y)$ *for some fixed element* $f \in Q$.

Theorem 2.2. [5] *If* $(Q, \cdot)$ *is a conjugacy closed loop, then*

$$< R_{x^{-1}}, L_x R_{x^{-1}}, R_{x^{-1}} > \text{ and } < R_x L_{x^{-1}}, L_{x^{-1}}, L_{x^{-1}} >$$

are autotopisms for all $x \in Q$.

3. Main Results.

3.1. Statement of Results.

Theorem 3.1.1 *Let* $(Q, \cdot)$ *be a conjugacy loop,* $\alpha \in BS(Q, \cdot)$ *such that* $V =< \alpha R_f, \alpha L_{x^{-1}}, \alpha >$ *is an autotopism of* Q *for some* $f \in Q$. *Then* α *is an automorphism of* Q *if and only if* Q *is of exponent* 2 *and* $f \in C$ *(centrum of* Q*).*

Remark. Theorem 3.1.1 agrees with that of CHIBOKA [3] on extra loops which is an example of a conjugacy closed.

Theorem 3.1.2 *Let* $(Q, \cdot)$ *be a conjugacy closed loop. If* $\psi \in BS(Q, \cdot)$ *then* $\psi R_x L_{x^{-1}}$ *is a pseudo–automorphism with companion* f $\forall x \in Q$.

Corollary to Theorem 3.1.2. *If* $(Q, \cdot)$ *is a conjugacy closed loop,* $\psi \in BS(Q, \cdot)$ *and* Q *is of exponent* 2. *Then* ψ *is a pseudo–automorphism with companion* f.

3.2. Proof of Results.

3.2.1. Proof of Theorem 3.1.1. Since Q is conjugacy closed then by Theorem 2.2

$$T =< R_{x^{-1}}, L_x R_{x^{-1}}, R_{x^{-1}} >$$

is an autotopism for all $x \in Q$ and

$$U =< R_x L_{x^{-1}}, L_{x^{-1}},_{x^{-1}} >$$

is also an autotopism for all $x \in Q$

$$W = TU =< L_{x^{-1}}, R_{x^{-1}} R_x, R_x >$$

is also an autotopism $\forall x \in Q$.

Now

$$T^{-1} =< R_x, L_{x^{-1}}R_x, R_x >$$

$$(3.2.1a) \qquad VW =< \alpha R_f L_{x^{-1}}, \alpha L_{f^{-1}}R_{x^{-1}}, \alpha L_{x^{-1}}R_{x^{-1}}$$

Now if $x = f$ in (3.2.1a) we have

$$(3.2.1b) \qquad VW =< \alpha R_x L_{x^{-1}}, \alpha L_{x^{-1}}R_{x^{-1}}, \alpha L_{x^{-1}}R_{x^{-1}} >$$

If $x \in C$ and $x^2 = 1$ (i.e of exponent 2)

$$ax = xa$$

hence $L_x = R_x$, using this in (3.2.1b) we obtain

$$VW =< \alpha, \alpha(L_{x^{-1}})^2, \alpha(L_{x^{-1}})^2 >$$

Again since all $x \in Q$ is of exponent 2, so is x^{-1}, therefore

$$(L_{x^{-1}})^2 = (L_{x^2})^{-1} = I \quad \text{(the identity mapping)}$$

hence

$$=< \alpha, \alpha, \alpha >$$

is an autotopism of Q. This implies that α is an automorphism.

Conversely,

If α is an automorphism of Q the $< \alpha, \alpha, \alpha >$ is an autotopism, also

$$VW =< \alpha R_x L_{x^{-1}}, \alpha R_x L_{x^{-1}}, \alpha R_x L_{x^{-1}} >$$

is also an autotopism of Q which implies that $R_x L_{x^{-1}} = I$ (the identity mapping) implying that $R_x = L_x \forall x \in Q$, hence $x \in C$ and $x^2 = 1$.

3.2.2. Proof of Theorem 3.1.2.

$$X =< R_x L_{x^{-1}}, L_{x^{-1}}, L_{x^{-1}} >$$

is an autotopism $\forall x \in Q$. Since $\psi \in \mathrm{BS}(Q, \cdot)$ therefore there exist $f, g \in Q$ such that

$$Y =< \psi R_{g-1}, \psi L_{f^{-1}}, \psi >$$

is an autotopism for some $f, g \in Q$.

Now by Lemma 2.2 of [1], Q is an inverse property loop, hence by Theorem 2.1

$$T =< \psi, J\psi L_{f^{-1}}J, \psi R_{g^{-1}} >$$

is an autotopism of Q

$$TX =< \psi R_x L_{x^{-1}}, J\psi L_{f^{-1}}JL_{x^{-1}}, \psi R_{g^{-1}}L_{x^{-1}} >$$

is an autotopism for all x and some $g, f \in Q$. Therefore for $y, z \in Q$ we have

$$y\psi R_x L_{x^{-1}} \cdot zJ\psi L_{f^{-1}}JL_{x^{-1}} = yz\psi R_{g^{-1}}L_{x^{-1}}$$

if $z = e$

$$y\psi R_x L_{x^{-1}} \cdot (x^{-1} \cdot xf) = y\psi R_{g^{-1}}L_{x^{-1}}$$

$$y\psi R_x L_{x^{-1}} R(x^{-1} \cdot xf) = y\psi R_{g^{-1}}L_{x^{-1}}$$

since Q is an inverse property loop [1],

$$y\psi R_x L_{x^{-1}} R_f = y\psi R_{g^{-1}}L_{x^{-1}}$$

Therefore,

$$TX =< \psi R_x L_{x^{-1}}, \psi R_x L_{x^{-1}} R_f, \psi R_x L_{x^{-1}} R_f >$$

is an autotopism for all x and some $x \in Q$. Hence $\psi R_x L_{x^{-1}}$ is a pseudo–automorphism with companion f.

3.3.3. Proof of corollary to Theorem 3.1.2. By Theorem 3.1.2

$$< \psi R_x L_{x^{-1}}, \psi R_x L_{x^{-1}} R_f, \psi R_x L_{x^{-1}} >$$

is an autotopism for all x and some $f \in Q$.

Now $x^2 = 1$ hence $R_x = L_x$ i.e $R_x L_{x^{-1}} = I$ which implies that

$$< \psi, \psi R_f, \psi R_f >$$

is an autotopism of Q which implies that ψ is a pseudo–automorphism with companion f.

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