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REMARK ON GRONWALL–TYPE INEQUALITIES

BY

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Abstract. We established in this paper some new Gronwall–type inequalities which can be used in the study of differential and integral equations.

1. Introduction. The importance of Gronwall's type inequalities and their applicability to the study of differential and integral equations are well known (cf. [1], [2]). Due to various motivations, linear and nonlinear generalizations of these inequalities have been obtained and used extensively (cf. [1], [2]). In this remark, we shall consider the generalizations of Gronwall's type inequalities obtained in [1], [2] and [4]. Special cases of our results are also given.

2. Main Results. The main results of this present paper are given in Theorems 2.1 and 2.2. Although there appear to be no results of this kind for Gronwall–type inequalities. We shall need the following lemma in the prove of our main results.

Lemma 2.1. [1]. *Let $f(t)$ and $g(t)$ be nonnegative continuous functions for $t \geq a$, let $v(t)$ be a differentiable function for $t \geq a$ and suppose*

$$v'(t) \leq g(t)v(t) + f(t), \quad t \geq a, \qquad v(a) = v_0. \tag{1}$$

Then, for $t \geq a$, we have

$$v(t) \leq v_0 \exp\left(\int_a^t g(s)ds\right) + \int_a^t f(s) \exp\left(\int_s^t g(\tau)d\tau\right) ds. \tag{2}$$

Proof. (see [1, p.2])

Theorem 2.1. *Let $u(t), g(t)$ be nonnegative continuous functions in a real interval $I = [a, b]$. Suppose that $k(t, s)$ and its partial derivatives $k_t(t, s)$ are nonnegative continuous functions for almost every $t, s \in I$ and assume further that $k(t, s)$ is nondecreasing in its first variable. Suppose that*

$$u(t) \leq c + \int_a^t k(t,s)u(s)ds + \int_a^b g(s)u(s)ds \tag{3}$$

where c is a constant. Also, suppose that

$$q = \int_a^b g(s) \exp\left(\int_a^s \left[k(\tau,\tau) + \int_a^\tau k_\tau(\tau,\sigma)d\sigma\right] d\tau\right) ds < 1. \tag{4}$$

Then

$$u(t) \leq \frac{c}{1-q} \exp\left(\int_a^t \left[k(s,s) + \int_a^s k_s(s,\tau)d\tau\right] ds\right). \tag{5}$$

Proof. Let $v(t) = c + \int_a^t k(t,s)u(s)ds + \int_a^b g(s)u(s)ds$. Then

$$u(t) \leq v(t), \tag{6}$$

$$v(a) = c + \int_a^b g(s)u(s)ds. \tag{7}$$

Hence $v'(t) = k(t,t)u(t) + \int_a^t k_t(t,s)u(s)ds \leq \left[k(t,t) + \int_a^t k_t(t,s)ds\right] v(t)$ by (4).

By Lemma 2.1, we have

$$v(t) \leq v(a) \exp\left(\int_a^t \left[k(s,s) + \int_a^s k_t(s,\tau)d\tau\right] ds\right). \tag{8}$$

Substituting (8) into (7), we obtain

$$\begin{aligned} v(a) &= c + \int_a^b g(s)u(s)ds \\ &\leq c + v(a) \int_a^b g(s) \exp\left(\int_a^s \left[k(\tau,\tau) + \int_a^\tau k_\tau(\tau,\sigma)d\sigma\right] d\tau\right) ds. \end{aligned}$$

Hence $v(a)\left\{1-\int_a^b g(s)\exp\left(\int_a^s\left[k(\tau,\tau)+\int_a^\tau k_\tau(\tau,\sigma)d\sigma\right]d\tau\right)ds\right\}\leq c.$

Thus

$$v(a)\leq \frac{c}{1-\int_a^b g(s)\exp\left(\int_a^s\left[k(\tau,\tau)+\int_a^\tau k_\tau(\tau,\sigma)d\sigma\right]d\tau\right)ds} \tag{9}$$

$$=\frac{c}{1-q} \tag{10}$$

where $q=\int_a^b g(s)\exp\left(\int_a^s\left[k(\tau,\tau)+\int_a^\tau k_\tau(\tau,\sigma)d\sigma\right]d\tau\right)ds.$

Substituting (8) into (6) yields

$$u(t)\leq \frac{c}{1-q}\exp\left(\int_a^t\left[k(s,s)+\int_a^s k_s(s,\tau)d\tau\right]ds\right).$$

This completes the proof of the theorem.

We now give the following specific cases of our result.

Corollary 2.1. *If we set $k(t,s)=h(t)f(s)$ in Theorem* 2.1 *and we assume that $f(s),h(t)$ and its derivative are nonnegative and continuous in I. Suppose that*

$$u(t)\leq c+h(t)\int_a^t f(s)u(s)ds+\int_a^b g(s)u(s)ds$$

where c is a constant. If

$$q=\int_a^b g(s)\exp\left(\int_a^s\left[h(\tau)f(\tau)+h'(\tau)\int_a^\tau f(\sigma)d\sigma\right]d\tau\right)ds<1,$$

then $u(t)\leq \frac{c}{1-q}\exp\left(\int_a^t\left[h(s)f(s)+h'(s)\int_a^s f(\tau)d\tau\right]ds\right)$ *where $h'(t)$ denotes the derivative of $h(t)$ with respect to t.*

Remark 2.1. If in Theorem 2.1 we set $g(t)=0$, our estimate gives

$$u(t)\leq c\exp\left(\int_a^t\left[k(s,s)+\int_a^s k_s(s,\tau)d\tau\right]ds\right)$$

which is sharper than the estimates obtained in Theorem 2.1 in [2].

Remark 2.2. If we set $k(t,s)=h(t)$ in Theorem 2.1, our result reduces to Theorem 1.7 in [1].

Theorem 2.2. *Let $u(t), g(t)$ be nonnegative continuous functions in a real interval $I = [a, b]$. Suppose that $k(t, s)$ and its partial derivatives $k_t(t, s)$ are nonnegative continuous functions for almost every $t, s \in I$ and let $p \geq 0$, $p \neq 1$ and c a positive constant. Assume that*

$$u(t) \leq c + \int_a^t k(t,s)u(s)ds + \int_a^t g(s)u^p(s)ds \tag{11}$$

Then

$$u(t) \leq \exp\left(\int_a^t w(s)ds\right)\left[c^q + q\int_a^t g(s)\exp\left(-q\int_a^s w(\tau)d\tau\right)ds\right]^{1/q} \tag{12}$$

for $t \in [a, \beta)$, where $q = 1 - p$ and β is chosen so that

$$c^q + q\int_a^t g(s)\exp\left(-q\int_a^s w(\tau)d\tau\right)ds$$

is positive in the subinterval $[a, \beta)$, $(\beta = b$ if $q > 0)$ and

$$w(t) = k(t,t) + \int_a^t k_t(t,s)ds. \tag{13}$$

Proof. Let $v(t) = c + \int_a^t k(t,s)u(s)ds + \int_a^t g(s)u^p(s)ds$. Then

$$u(t) \leq v(t), \quad v(a) = c. \tag{14}$$

Hence

$$\begin{aligned} v'(t) &= k(t,t)u(t) + g(t)u^p(t) + \int_a^t k_t(t,s)u(s)ds \\ &\leq \left[k(t,t) + \int_a^t k_t(t,s)ds\right]v(t) + g(t)v^p(t) = w(t)v(t) + g(t)v^p(t). \end{aligned}$$

That is

$$v'(t) \leq w(t)v(t) + g(t)v^p(t). \tag{15}$$

where $w(t) = k(t,t) + \int_a^t k_t(t,s)ds$.

We simplify (13) by the following substitution. Let

$$z(t) = \frac{v^q(t)}{q}. \tag{16}$$

Then it follows that $z(a) = \dfrac{v^q(a)}{q}$. Hence

$$\begin{aligned} z'(t) &= v^{q-1}(t) \times v'(t) \leq v^{q-1}(t)(w(t)v(t) + g(t)v^p(t)) \qquad (\text{by } (13)) \\ &= qw(t)z(t) + g(t) \qquad (\text{since } q = 1-p) \end{aligned}$$

That is

$$z'(t) \leq qw(t)z(t) + g(t), \quad t \geq a \;\; z(a) = \frac{v^q(a)}{q} \tag{17}$$

Comparing (17) with (1) and noting that $z(a) = \dfrac{v^q(a)}{q}$, it follows from Lemma 2.1 that

$$z(t) \leq \frac{v^q(a)}{q} \exp\left(\int_a^t qw(s)ds\right) + \int_a^t g(s) \exp\left(\int_a^s qw(\tau)d\tau\right) ds.$$

That is

$$v^q(t) \overset{\leq}{\geq} \exp\left(\int_a^t qw(s)ds\right) \left[v^q(a) + \int_a^t qg(s) \exp\left(-\int_a^s qw(\tau)d\tau\right) ds\right] \tag{18}$$

where $\leq$ (respectively, $\geq$) holds for $q > 0$ (respectively, $q < 0$). In both cases if we substitute (14) into (18), we obtain

$$u(t) \leq \exp\left(\int_a^t w(s)ds\right) \left[c^q + q\int_a^t g(s) \exp\left(-q\int_a^s w(\tau)d\tau\right) ds\right]^{1/q}.$$

This ends the proof of the theorem.

The following special cases of our result are immediate.

Corollary 2.2. *If we set $k(t,s) = h(t)f(s)$ in Theorem* 3.1 *and we assume that $f(s), h(t)$ and its derivative are nonnegative and continuous in I. If we assume that*

$$u(t) \leq c + h(t)\int_a^t f(s)u(s)ds + \int_a^t g(s)u^p(s)ds.$$

Then

$$u(t) \leq \exp\left(\int_a^t \left[h(s)f(s) + h'(s)\int_a^s f(\tau)d\tau\right] ds\right)$$
$$\times \left\{c^q + q\int_a^t g(s)\exp\left(-q\int_a^s \left[h(\tau)f(\tau) + h'(\tau)\int_a^\tau f(\sigma)d\sigma\right] d\tau\right) ds\right\}^{1/q}$$

where $h'(t)$ denotes the derivative of $h(t)$ with respect to t.

Remark 2.3. In Theorem 3.1, set $k(t,s) = h(t)$, then our result yields

$$u(t) \leq \exp\left(\int_a^t h(s)ds\right)\left[c^q + q\int_a^t g(s)\exp\left(-q\int_a^s h(\tau)d\tau\right) ds\right]^{1/q}$$

which is a result obtained by WILLETT and WONG [4].

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