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ON METRICAL FINSLER CONNECTIONS WITH TORSION $T_j{}^i{}_k$ OF GENERALIZED FINSLER SPACES

BY

M. KITAYAMA* **and S. KIKUCHI****

1. Introduction. Let us consider an n–dimensional manifold M^n on which a symmetric and nondegenerate Finsler tensor field $g_{ij}(x,y)$ of type (0,2) is defined, we shall call (M^n, g_{ij}) a generalized Finsler space, and the tensor field g_{ij} is called a metrical Finsler structure. The generalized Finsler spaces are studied by many authors.

The purpose of the present paper is to extend directly KIKUCHI's results [2] and the fruit is a natural extension of HASHIGUCHI [1]. In §1, we obtain a sufficient condition that the metrical Finsler connection with a given deflection and torsion be determined uniquely. In §2, we shall consider generalized Finsler spaces corresponding to generalized Berwald spaces. The terminologies and notations in the present paper are referred to MATSUMOTO's book [4].

1. Metrical Finsler connections with deflection and torsion. A metrical Finsler connection $F\Gamma T = (N^i{}_k, F_j{}^i{}_k, \overset{\circ}{C}{}_j{}^i{}_k)$ with deflection $D^i{}_k$ and torsion $T_j{}^i{}_k$ on (M^n, g_{ij}) is a Finsler connection satisfying the following axioms:

$$g_{ij|k} = \partial g_{ij}/\partial x^k - N^r{}_k\,\partial g_{ij}/\partial y^r - g_{rj}F_i{}^r{}_k - g_{ir}F_j{}^r{}_k = 0, \tag{1.1}$$

$$g_{ij}\big|_k = \partial g_{ij}/\partial y^k - g_{rj}\,\overset{\circ}{C}{}_i{}^r{}_k - g_{ir}\,\overset{\circ}{C}{}_j{}^r{}_k = 0, \tag{1.2}$$

$$T_j{}^i{}_k = F_j{}^i{}_k - F_k{}^i{}_j, \tag{1.3}$$

$$D^i{}_k = F_0{}^i{}_k - N^i{}_k, \tag{1.4}$$

$$\overset{\circ}{C}{}_j{}^i{}_k = \overset{\circ}{C}{}_k{}^i{}_j, \tag{1.5}$$

where the subscript 0 means the contraction for y^j. Then we can see

$$\overset{\circ}{C}{}_j{}^i{}_k = \frac{1}{2}g^{ir}(\partial g_{kr}/\partial y^j + \partial g_{jr}/\partial y^k - \partial g_{jk}/\partial y^r). \tag{1.6}$$

From (1.1), (1.3) and (1.4), the metrical Finsler connection $F\Gamma T$ with deflection and torsion of g_{ij} satisfies the equations

$$\begin{aligned} F_j{}^i{}_k = \gamma_j{}^i{}_k - C_k{}^i{}_\ell F_0{}^\ell{}_j - C_j{}^i{}_\ell F_0{}^\ell{}_k + C_{jk\ell}F_0{}^\ell{}_r g^{ri} + \\ C_k{}^i{}_\ell D^\ell{}_j + C_j{}^i{}_\ell D^\ell{}_k - C_{jk\ell}D^\ell{}_r g^{ri} + A_j{}^i{}_k, \end{aligned} \tag{1.7}$$

where $C_{ij\ell} = \frac{1}{2}\partial g_{ij}/\partial y^\ell$, $g^{jr}C_{ir\ell} = C_i{}^j{}_\ell$, $\gamma_j{}^i{}_k = \frac{1}{2}g^{ir}(\partial g_{kr}/\partial x^j + \partial g_{jr}/\partial x^k - -\partial g_{jk}/\partial x^r)$, and $A_j{}^i{}_k = \frac{1}{2}(T_j{}^i{}_k - T^i{}_{jk} + T_{jk}{}^i)$.

Contracting (1.7) with y^j, we obtain

$$W^{im}_{\ell k}F_0{}^\ell{}_m = \gamma_j{}^i{}_k y^j + C_k{}^i{}_\ell D^\ell{}_0 + C_0{}^i{}_\ell D^\ell{}_k - C_{0k\ell}D^\ell{}_r g^{ri} + A_0{}^i{}_k, \tag{1.8}$$

where

$$W^{im}_{\ell k} = \delta^i_\ell \delta^m_k + C_k{}^i{}_\ell y^m + C_0{}^i{}_\ell \delta^m_k - C_{0k\ell}g^{mi}. \tag{1.9}$$

Therefore, if we assume that $n^2 \times n^2$ matrix $(W^{im}_{\ell k})$ with respect to the system of indices $(1, m)$ and (i, k) is regular, there exists the inverse matrix $(\overset{*}{W}{}^{pk}_{iq})$ such that $\overset{*}{W}{}^{pk}_{iq}W^{im}_{\ell k} = \delta^p_\ell \delta^m_q$. On contracting (1.8) with $\overset{*}{W}{}^{pk}_{iq}$, we obtain

$$F_0{}^p{}_q = \overset{*}{W}{}^{pk}_{iq}(\gamma_j{}^i{}_k y^j + C_k{}^i{}_\ell D^\ell{}_0 + C_0{}^i{}_\ell D^\ell{}_k - C_{0k\ell}D^\ell{}_r g^{ri} + A_0{}^i{}_k). \tag{1.10}$$

Given a deflection tensor and a torsion tensor, by substituting these equations in (1.7), we get $F_j{}^i{}_k$ completely. Then we have

Theorem 1.1. *Let g_{ij} be a generalized Finsler metric structure with the matrix $(W^{im}_{\ell k})$ being regular. Then given a deflection tensor $D^i{}_k$ and a torsion tensor $T_j{}^i{}_k$, there exists uniquely a metrical Finsler connection $F\Gamma T = (N^i{}_k, F_j{}^i{}_k, \overset{\circ}{C}{}_j{}^i{}_k$ with deflection $D^i{}_k$ and torsion $T_j{}^i{}_k$ of g_{ij}, and $\overset{\circ}{C}{}_j{}^i{}_k$ is given by* (1.6) *and $F_j{}^i{}_k$ is given by* (1.7)*, where $F_0{}^i{}_k$ is defined by* (1.10) *and so $N^i{}_k$ is given by* (1.4).

In the case of the metrical Finsler connection without deflection, we have

Theorem 1.2. *Let g_{ij} be a generalized Finsler metric structure with the matrix $(W^{im}_{\ell k})$ being regular. Then given a torsion tensor $T_j{}^i{}_k$, there exists uniquely a metrical Finsler connection $F\Gamma T = (N^i{}_k, F_j{}^i{}_k, \overset{\circ}{C}{}_j{}^i{}_k)$ with torsion $T_j{}^i{}_k$ of g_{ij}, and $\overset{\circ}{C}{}_j{}^i{}_k$ is given by* (1.6) *and $F_j{}^i{}_k$ is given by*

$$F_j{}^i{}_k = \gamma_j{}^i{}_k - C_k{}^i{}_\ell F_0{}^\ell{}_j - C_j{}^i{}_\ell F_0{}^\ell{}_k + C_{jk\ell} F_0{}^\ell{}_r g^{ri} + A_j{}^i{}_k, \tag{1.11}$$

where $F_0{}^p{}_q$ is defined by

$$F_0{}^p{}_q = \overset{*}{W}{}^{pk}_{iq}(\gamma_j{}^i{}_k y^j + A_0{}^i{}_k). \tag{1.12}$$

Remark. In the case of that g_{ij} is derived from a Finsler fundamental function, the matrix $(W^{im}_{\ell k})$ is regular. Indeed the inverse matrix $\overset{*}{W}{}^{pk}_{iq}$ of $(W^{im}_{\ell k})$ is expressed as $\overset{*}{W}{}^{pk}_{iq} = \delta^p_i \delta^k_q - C_q{}^p{}_i y^k$. Further if we deal with a special metrical Finsler structure [6] $g_{ij} = e^{2\sigma(x,y)}\gamma_{ij}(x)$, where $\gamma_{ij}(x)$ is a Riemannian metric tensor and $\sigma(x,y)$ is a function of x and y, the matrix $(W^{im}_{\ell k})$ is naturally regular under the assumption $1 + \sigma_0 \neq 0$ [2].

2. A condition for $F_j{}^i{}_k$ to be functions of position only. In this section, we shall find the condition for $F_j{}^i{}_k$ to be functions of position only. Differentiating (1.7) by y^ℓ, we have

$$\begin{aligned} \dot\partial_\ell F_j{}^i{}_k &= C_k{}^i{}_{\ell|j} + C_j{}^i{}_{\ell|k} - C_{jk\ell|}{}^i - \\ &-(C_j{}^i{}_s \dot\partial_\ell F_m{}^s{}_k + C_k{}^i{}_s \dot\partial_\ell F_m{}^s{}_j - C_{jks} g^{ri} \dot\partial_\ell F_m{}^s{}_r) y^m + \\ &+ C_k{}^i{}_m \dot\partial_\ell D^m{}_j + C_j{}^i{}_m \dot\partial_\ell D^m{}_k - C_{jkm} g^{ri} \dot\partial_\ell D^m{}_r + \Lambda_j{}^i{}_{k\ell}, \end{aligned} \tag{2.1}$$

where we put $\dot\partial_\ell = \partial/\partial y^\ell$, $C_{jk\ell|}{}^i = g^{ih} C_{jk\ell|h}$ and

$$\Lambda_j{}^i{}_{k\ell} = \frac{1}{2} g^{ih}(g_{hm}\dot\partial_\ell T_j{}^m{}_k - g_{jm}\dot\partial_\ell T_h{}^m{}_k + g_{km}\dot\partial_\ell T_j{}^m{}_h). \tag{2.2}$$

In view of (2.1), we have

Theorem 2.1. *We are concerned with a metrical Finsler connection $F\Gamma T$ with deflection $D^i{}_k$ and torsion $T_j{}^i{}_k$. Then a necessary and sufficient condition for the connection coefficients $F_j{}^i{}_k$ to be functions of position only is the following equations are satisfied*

$$T_j{}^i{}_k = T_j{}^i{}_k(x), \tag{2.3}$$

$$C_{ijk|\ell} = -C_{ijm}\dot\partial_k D^m{}_\ell. \tag{2.4}$$

In the case of the metrical Finsler connection without deflection, we have

Theorem 2.2. *We are concerned with a metrical Finsler connection $F\Gamma T$ with torsion $T_j{}^i{}_k$. Then a necessary and sufficient condition for the connection coefficients $F_j{}^i{}_k$ to be functions of position only is* (2.3) *and*

$$C_{ijk|\ell} = 0 \tag{2.5}$$

are satisfied.

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* *Department of Mathematics,*
Kushiro Campus, Hokkaido
University of Education
Kushiro, Hokkaido 085, JAPAN

** *Department of Mathematics*
College of Science and Technology
Nihon University
Kanda–Surugadai
Tokyo 101, JAPAN