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ASYMPTOTIC CHARACTERISTIC SOLUTIONS OF AUTONOMOUS HAMILTON–JACOBI EQUATIONS

BY

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Abstract. In view of possible applications to infinite–horizon optimal problems, we extend Cauchy's Method of Characteristics to obtain classical and generalized solutions with certain asymptotic properties along the characteristics of a Hamilton–Jacobi equation.

1. Introduction. The aim of this paper is extend the Cauchy's Method of Characteristics in order to construct classical and generalized solutions of a Hamilton–Jacobi equation of the form:

$$H(x, DW(x)) = 0,\ x \in Y \subseteq \mathbf{R}^n, \tag{1.1}$$

that have certain asymptotic properties along characteristics.

This problem is motivated by the recently obtained aymptotic properties in [11] of the value function of an infinite–horizon optimal control problem which consists in miniminzing a cost functional of he form:

$$C(x(\cdot)) := \int_0^\infty g_0(x(t), x'(t))dt, \tag{1.2}$$

over a prescribed class, $\Omega_a(y),\ y \in Y \subseteq \mathbf{R}^n$, of admissible trajectories, that are locally–absolutely continuous mappings, $x(\cdot) : [0, \infty) \to \mathbf{R}^n$ satisfying the following constraints:

$$x'(t) \in G(x(t))\ a.e.\ ([0, \infty),\ x(0) = y, \tag{1.3}$$

$$x(t) \in Y, \forall t \in [0, \infty), \tag{1.4}$$

$$g_0(x(\cdot), x'(\cdot)) \in L_1([0, \infty); \mathbf{R}). \tag{1.5}$$

As it is well known, the *value function* of the problem (1.2)–(1.5), defined by:

$$W(y) := \inf_{x(\cdot)\in\Omega_a(y)} C(x(\cdot)),\ y \in Y, \tag{1.6}$$

is a solution, in some generalized sense, of the equation (1.1) defined by the following Hamiltonian:

$$\begin{aligned} &H(x,p) := \inf_{v\in G(x)} [< p, v > +g_0(x,v)], x \in Y, p \in \mathbf{R}^n \\ &\widehat{G}(x,p) := \{v \in G(x); H(x,p) =< p, v > +g_0(x,v)\} \end{aligned} \tag{1.7}$$

Moreover, according to the results in [11], the value function in (1.6) has the following *asymptotic property* along all the admissible trajectories:

$$(\exists)\ \lim_{t\to\infty} W(x(t)) \leq 0, (\forall)x(\cdot) \in \Omega_a(y), y \in Y, \tag{1.8}$$

and, in addition, an admissible trajectory, $\tilde{x}(\cdot) \in \Omega_a(y)$, is *optimal* with respect to the initial point $y \in Y$ if and only if the function $\omega_{\tilde{x}}(\cdot)$ defined by:

$$\omega_{\tilde{x}}(t) = W(\tilde{x}(t)) + \int_0^t g_0(\tilde{x}(s), \tilde{x}'(s))ds, (\forall)t \in [0,\infty), \tag{1.9}$$

is constant and the following asymptotic property is satisfies:

$$\lim_{t\to\infty} W(\tilde{x}(t)) = 0. \tag{1.10}$$

On the other hand, certain results obtained in the optimal control theory on finite horizon show that the optimal trajectories may be obtained in many cases as the first components of the characteristics of the equation in (1.1) (e.g., MIRICĂ [15], [16]).

In the case of inifinite–horizon problems, the use of Cauchy's Method of Characteristics to obtain suitable solutions of the Hamilton–Jacobi equation in (1.1) (that would coincide with the value function of the problem (1.2)–(1.5)) is obstructed by the absence of boudary conditions associated to the equation (1.1).

In this paper we suggest a natural way to extend Cauchy's Method of Characteristics to obtain classical and generalized solutions of the equations (1.1) that have an asymptotic property of the type in (1.10) for some characteristics and are susceptible to have also the properties in (1.8) if the Hamiltonian $H(\cdot,\cdot)$ is of the form in (1.7).

The paper is organized as follows: in Section 2 we present the notations, definitions and the auxilliary results used in the paper.

The main results of the paper are Theorems 3.7 and 3.8 describing classical and generalized solutions of the equation (1.1) that have asymptotic properties of the type in (1.10) along certain characteristics.

The theoretical resuts in Section 3 are illustrated by two simple examples of Hamilton–Jacobi equations generated by problems of the form (1.2)–(1.5), presented in the last section.

We note that the results in Section 3 may be significantly extended to more general Hamiltonians introducing generalized Hamiltonian flows in the same way as in the case of finite–horizon problems considered in MIRICĂ ([15],[16]).

2. Notations, definitions and preliminary results. We recall first the definitions and some properties of the concepts in Nonsmooth Analysis that will be used in the next section.

From the multitude of tangent cones and the corresponding differentiability concepts (e.g. MIRICĂ [17]–[19], etc.) we shall use the Bouligand–Severi contingent cones and the related differentiability concepts that are natural generalizations of the classical ones.

If $X \in \mathbf{R}^n$ is a nonempty subset then the *contingent* cones to X at a point $x \in X$ are defined by:

$$\begin{aligned} &K_x^{\pm}X := \{u\in\mathbf{R}^n | (\exists)(\theta_m,u_m)\to(0\pm,u): x+\theta_m\cdot u_m\in X,\ (\forall)m\in\mathbf{N}\},\\ &\breve{K}_xX := K_x^{+}X \cup K_x^{-}X,\ K_xX = K_x^{+}X\cap K_x^{-}X,\ K_x^{-}X = -K_x^{+}X, \end{aligned} \tag{2.1}$$

and if $g(\cdot): X \to \mathbf{R}$ is a real–valued function then its *extreme contingent derivatives* at $x \in X$ are defined by:

$$\begin{aligned} \overline{D}_K^{\pm}g(x;u) &= \limsup_{\substack{(\theta,v)\to(0\pm,u)\\ x+\theta\cdot v\in X}} \frac{g(x+\theta\cdot v)-g(x)}{\theta}, \quad (\forall)u\in K_x^{\pm}X,\\ \underline{D}_K^{\pm}g(x;u) &= \limsup_{\substack{(\theta,v)\to(0\pm,u)\\ x+\theta\cdot v\in X}} \frac{g(x+\theta\cdot v)-g(x)}{\theta}, \quad (\forall)u\in K_x^{\pm}X, \end{aligned} \tag{2.2}$$

and its *contingent semidifferentials* are defined by:

$$\begin{aligned}\overline{\partial}_K g(x) &:= \{p \in \mathbf{R}^n | < p, u > \geq \overline{D}^+_K g(x;u),\ (\forall) u \in K^+_x X\} = \\ &= \{p \in \mathbf{R}^n | < p, u > \leq \underline{D}^-_K g(x;u),\ (\forall) u \in K^-_x X\}, \\ \underline{\partial}_K g(x) &:= \{p \in \mathbf{R}^n | < p, u > \leq \underline{D}^+_K g(x;u),\ (\forall) u \in K^+_x X\} = \\ &= \{p \in \mathbf{R}^n | < p, u > \geq \overline{D}^-_K g(x;u),\ (\forall) u \in K^-_x X\}.\end{aligned} \tag{2.3}$$

The function $g(\cdot)$ is said to be *contingent* (homogeneously) *differentiable* at $x \in X$ if the upper and lower derivatives in (2.2) coincide, i.e. if the following limits exist:

$$\begin{aligned} &(\exists) g^+_K(x;u) = \lim_{\substack{(\theta,v)\to(0+,u)\\ x+\theta\cdot v\in X}} \frac{g(x+\theta\cdot v)-g(x)}{\theta},\ (\forall) u \in K^+_x X \\ &g^-_K(x;u) = \lim_{\substack{(\theta,v)\to(0-,u)\\ x+\theta\cdot v\in X}} \frac{g(x+\theta\cdot v)-g(x)}{\theta} = -g^+_K(x;-u), \\ &\qquad\qquad\qquad\qquad\qquad\qquad (\forall) u \in K^-_x X.\end{aligned} \tag{2.4}$$

As it is well known (e.g. MIRICĂ [15], $g(\cdot)$ is (Fréchet) differentiable at $x \in Int(X)$ iff it is contingent differentiable and the mapping $K^+_x X = \mathbf{R}^n \ni u \mapsto g^+_K(x;u)$ is linear; in this case, the classical derivative, $Dg(x)$ coincides with the contingent one, $g^+_K(x;\cdot)$. This concepts allows, in particular, to consider restrictions of differentiable functions to arbitrary subsets of their domains.

The generalized derivatives in (2.2) are particularly useful in the study of *marginal functions* of the form:

$$\begin{cases} f(x) & := \inf\limits_{h(y)=x, y\in Y} g(y), x \in X \subseteq \mathbf{R}^n \\ \widehat{h}(x) & := \{y \in Y : h(y) = x, g(y) = f(x)\}, \\ \widetilde{X} & := \{x \in X; \widehat{h}(x) \neq \Phi\}, \end{cases} \tag{2.5}$$

defined by the given mapping $g(\cdot) : Y \subseteq \mathbf{R}^m \to \mathbf{R}, h(\cdot) : Y \to X$.

As very simple examples show (e.g. MIRICĂ [15],[18]) the marginal functions in (2.5) are, generally, not differentiable even if the data $g(\cdot), h(\cdot)$ are smooth mappings.

We shall use the following result estimating the upper–left contingent derivatives of the marginal function $f(\cdot)$ in (2.5):

Proposition 2.1. ([16],[18]). *If $f(\cdot), \widehat{h}(\cdot), \widetilde{X}$ are objects defined in (2.5) and $h(\cdot) : Y \to X$ is contingent differentiable then one has:*

$$\overline{D}_K^- f(x;u) \geq \sup_{y \in \hat{h}(x)} \sup_{\substack{v \in K_y^- Y \\ u = h_K^-(y;v)}} \underline{D}_K^- g(y;v), \ (\forall) u \in K_x^- \widetilde{X}, \ x \in \widetilde{X}. \tag{2.6}$$

We shall use also the following refinement in [12] of the classical theorem on differentiability with respect to parameters of the Lebesgue integral of MC SHANE ([10]):

Theorem 2.2. ([12]). *Let $I \subseteq \mathbf{R}$ be an interval, let $Z \subseteq \mathbf{R}^k$, $z_0 \in Z$ and let $g(\cdot,\cdot) : I \times Z \to \mathbf{R}$ be a measurable function with the following properties:*

1. $g(\cdot, z) \in L_1(I; \mathbf{R}), (\forall) z \in Z$;
2. *There exists a null subset $I_0 \subset I$ (of zero Lebsgue measure) such that $g(t,\cdot)$ is contingent differentiable at z_0 for any $t \in I \setminus I_0$;*
3. *There exist $r > 0$ and an integrable function $k(\cdot) \in L_1(I; \mathbf{R}_+)$ such that:*

$$|g(t,z) - g(t,z_0)| \leq k(t) \cdot \|z - z_0\|, \ (\forall) z \in Z \cap \mathbf{B}_r(z_0), \ t \in I \setminus I_0. \tag{2.7}$$

Then for any $\overline{z} \in K_{z_0}^+ Z$, the function $I \ni t \mapsto (g(t,\cdot))_K^+(z_0;\overline{z})$ is integrable, the function $f(\cdot) : Z \to \mathbf{R}$ defined by:

$$f(z) = \int_I g(t,z)dt, \ (\forall) z \in Z, \tag{2.8}$$

is contingent differentiable at z_0 and its contingent derivative is given by:

$$f_K^+(z_0;\overline{z}) := \int_I (g(t,\cdot))_K^+(z_0;\overline{z})dt, \ (\forall) \overline{z} \in K_{z_0}^+ Z. \tag{2.9}$$

3. Asymptotic characteristic solutions. In this section we extend the classical Cauchy's Method of Characteristics (e.g. HARTMAN [8], MIRICĂ [13], [15], [16], SUBBOTIN [22], etc.) to obtain classical and generalized solutions of the autonomous Hamilton–Jacobi equation in (1.1) that have asymptotic properties of the form in (1.10) along certain characteristics.

Troughout this section we are assuming the Hamiltonian $H(\cdot,\cdot)$ in (1.1) to have the following properties:

Hypothesis 3.1. The subset $A \subseteq \mathbf{R}^n \times \mathbf{R}^n$ is a class C^2 differentiable submanifold, $H(\cdot,\cdot) : A \to \mathbf{R}$ is of class C^1 and the *generalized Hamiltonian orientor field* $d^{\#}H(\cdot,\cdot)$ defined by:

$$\begin{aligned} d^{\#}H(x,p) := \{(x',p') \in T_{(x,p)}A; < x', \overline{q} > - < p', \overline{y} > = \\ = DH(x,p)\cdot(\overline{y},\overline{q}), (\forall)(\overline{y},\overline{q}) \in T_{(x,p)}A\} \in A \end{aligned} \tag{3.1}$$

has a smooth selection, $h(\cdot,\cdot) : A \to T(A)$ of class C^1, i.e. such that:

$$h(x,p) \in d^{\#}H(x,p), (\forall)(x,p) \in A. \tag{3.2}$$

Remark 3.2. As it is easy to see (e.g. LANG [9], MIRICĂ [15], etc.) in the case $A = Int(A) \subseteq \mathbf{R}^n \times \mathbf{R}^n$ is an open subset and $H(\cdot,\cdot) : A \to \mathbf{R}$ is of class C^2, the Hamiltonian orientor field in (3.1) concides with the classical Hamiltonian vector field:

$$d^{\#}H(x,p) = \left\{ \left(\frac{\partial H}{\partial p}(x,p), -\frac{\partial H}{\partial x}(x,p) \right) \right\}, \ (\forall)(x,p) \in A, \tag{3.3}$$

and therefore Hypothesis 3.1 is satisfied; a mor general case is that in which $A \subseteq \mathbf{R}^n \times \mathbf{R}^n$ is a symplectic submanifold (e.g. GODBILLON [7]) and $H(\cdot,\cdot) : A \to \mathbf{R}$ is of class C^2.

However, as Example 4.2 shows, there are many cases in which Hypothesis 3.1 is satisfied.

In the case $H(\cdot,\cdot)$ is the Hamiltonian in (1.7) associated to an optimal control problem of the form (1.2)–(1.5) the orientor field in (3.1) should be modified as follows:

$$\begin{aligned} d_F^{\#}H(x,p) := \{(x',p') \in T_{(x,p)}A;\ x' \in \widehat{G}(x,p), < x', \overline{q} > - < p', \overline{y} > = \\ = DH(x,p)\cdot(\overline{y},\overline{q}),\ (\forall)(\overline{y},\overline{q}) \in T_{(x,p)}A\},\ (x,p) \in A. \end{aligned} \tag{3.4}$$

Remark 3.3. If Hypothesis 3.1 is satisfied then the selection $h(\cdot,\cdot)$ in (3.2) is a smooth vector field on the differentiable manifold $A \subseteq \mathbf{R}^n \times \mathbf{R}^n$ and therefore from the general theory of Ordinary Differential Equation (e.g. HARTMAN [8], LANG [9], MIRICĂ [14], etc.) it follows that there exists a unique maximal flow,

$$X_h^*(\cdot,\cdot) = (X(\cdot,\cdot), P(\cdot,\cdot)) : D_h \subseteq \mathbf{R} \times A \to A,$$

of class C^1, defined by the following property: for any $a = (y, q) \in A$ the partial mapping, $X_h^*(\cdot, a) = (X(\cdot, a), P(\cdot, a)) : I(a) \subseteq \mathbf{R} \to A$ is the unique, maximal (i.e. noncontinuable) solution of the Cauchy problem:

$$(x', p') := h(x, p),\ (x(0), p(0)) = a = (y, q) \in A, \tag{3.5}$$

hance satisfies the identities:

$$\begin{array}{l} (X'(t, a), P'(t, a)) \equiv h(X(t, a), P(t, a)), (\forall)t \in I(a),\ a \in A \\ (X(0, a), P(0, a)) = a = (y, q) \in A, \end{array} \tag{3.6}$$

and $I(a) \subseteq \mathbf{R}$ is an open interval.

Moreover, since $h(\cdot, \cdot)$ and $X_h^*(\cdot, \cdot)$ are of class C^1, using Schwartz's Theorem, from (3.6) one obtains the identity:

$$\frac{d}{dt} D_2 X_h^*(t, a) \cdot \bar{a} \equiv Dh(X_h^*(t, a)) \cdot (D_2 X_h^*(t, a) \cdot \bar{a}),\ (\forall) \bar{a} \in T_a A,\ t \in I(a), \tag{3.7}$$

which, usually is interpreted as follows: for any $a = (y, q) \in A$ and for any tangent vector $\bar{a} = (\bar{y}, \bar{q}) \in T_a A$, the mapping $t \mapsto D_2 X_h^*(t, a) \cdot \bar{a}$ is a solution of the associated variational equation $z' = Dh(X_h^*(t, a)) \cdot z$.

The basic properties of the Hamiltonian flow $X_h^*(\cdot, \cdot)$ in (3.6) are contained in the following result that generalizes the classical one in the case $A = Int(A)$ is open and $d^{\#}H(\cdot, \cdot)$ is given by (3.3) (e.g. GODBILLON [7], HARTMAN [8], MIRICĂ [14], etc.):

Theorem 3.4. (MIRICĂ [16]). *If Hypothesis* 3.1 *is satisfied then the Hamiltonian flow* $X_h^*(\cdot, \cdot) := (X(\cdot, \cdot), P(\cdot, \cdot))$ *in* (3.6) *has the following properties:*

$$H(X(t, a), P(t, a)) \equiv H(a), (\forall)t \in I(a),\ a = (y, q) \in A; \tag{3.8}$$

$$\begin{array}{c} \dfrac{\partial}{\partial a} [<P(t, a), X'(t, a)>] \cdot \bar{a} = \dfrac{d}{dt} [<P(t, a), D_2 X(t, a) \cdot \bar{a}>] + DH(a) \cdot \bar{a}, \\ (\forall)t \in I(a),\ a = (y, q) \in A,\ \bar{a} = (\bar{y}, \bar{q}) \in T_a A. \end{array} \tag{3.9}$$

In particular, the restriction to the subset $A_0 \subseteq A$ *defined by:*

$$A_0 := \{a = (y, q) \in A;\ H(y, q) = 0\}, \tag{3.10}$$

has the following properties:

$$H(X(t, a), P(t, a)) \equiv 0,\ (\forall)t \in I(a),\ a = (y, q) \in A_0; \tag{3.11}$$

$$\frac{\partial}{\partial a}[<P(t,a),X'(t,a)>]\cdot\bar{a}=\frac{d}{dt}[<P(t,a),D_2X(t,a)\cdot\bar{a}>],\tag{3.12}$$
$$(\forall)t\in I(a),\ a=(y,q)\in A_0,\ \bar{a}=(\bar{y},\bar{q})\in\breve{K}_aA_0:=K_a^+A_0\cup K_a^-A_0.$$

We note that even in the classical case in which $A\subseteq\mathbf{R}^n\times\mathbf{R}^n$ is open and $H(\cdot,\cdot)$ is of class C^2, the subset $A_0\subseteq A$ may not be a differentiable manifold and therefore the properties in (3.12) should be expressed only in terms of the contingent cones in (2.1).

In the absence of boundary conditions accompanying equation (1.1) and in view of necessity of asymptotic properties of the type in (1.10), we suggest the following addaptation of the classical Cauchy's Method of Characteristics:

– one considers the subset $\widetilde{A}_0\subseteq A_0$ defined by:

$$\begin{aligned}\widetilde{A}_0:=\{a=(y,q)\in A_0;\ I_a:=[0,\infty)\subseteq I(a),\\ g(\cdot,a)=<P(\cdot,a),X'(\cdot,a)>\in L_1(I_a;\mathbf{R})\},\end{aligned}\tag{3.13}$$

on which one defines the real–valued function $V(\cdot):\widetilde{A}_0\to\mathbf{R}$ as follows:

$$V(a)=-\int_0^\infty<P(t,a),X'(t,a)>dt,\ (\forall)a=(y,q)\in\widetilde{A}_0;\tag{3.14}$$

– one chooses either a selection, $\tilde{q}(\cdot)):\widetilde{Y}\subseteq pr_1(A_0)\to\mathbf{R}^n$ of the multifunction $Q_0(\cdot)$ defined by:

$$Q_0(y):=\{q\in\mathbf{R}^n;(y,q)\in\widetilde{A}_0\},\ y\in\widetilde{Y}\subseteq X,\tag{3.15}$$

i.e. such that:

$$\tilde{q}(y)\in Q_0(y),\tilde{a}(y):=(y,\tilde{q}(y))\in\widetilde{A}_0,\ (\forall)y\in\widetilde{Y},\tag{3.16}$$

or a set–valued selection, $\widetilde{Q}(y)\subseteq Q_0(y),(\forall)y\in\widetilde{Y}$ such that

$$V(y,q_1)=V(y,q_2),\ (\forall)y\in\widetilde{Y},\ q_1,q_2\in\widetilde{Q}(y),\tag{3.17}$$

and one seeks sufficient conditions under which the associated function:

$$W(y):=V(y,\tilde{q}(y)),\ (\forall)y\in\widetilde{Y},\tag{3.18}$$

is either a classical or a generalized solution in some sense of the equation (1.1).

We note that in the case the Hamiltonian $H(\cdot,\cdot)$ is of the form in (1.7) and one tries to solve an optimal control problem of the form (1.2)–(1.5), the natural candidate for the value function in (1.6) (provided that Hypothesis 3.1 is satisfied) is the marginal function $\widetilde{W}(\cdot):\widetilde{Y}\subseteq X\to\mathbf{R}$ defined by:

$$\begin{cases} \widetilde{W}(y)=\inf\limits_{q\in Q_0(y)} V(y,q) \\ \widetilde{Q}(y)=\{q\in Q_0(y); V(y,q)=\widetilde{W}(y)\}, (\forall)y\in\widetilde{Y}\subseteq pr_1(\widetilde{A}_0). \end{cases} \tag{3.19}$$

In view of the differentiability result in Theorem 2.2 to be applied to the function $V(\cdot)$ in (3.14) we shall assume the following additional property of the Hamiltonian flow in (3.6):

Hypothesis 3.5. The Hamiltonian $H(\cdot,\cdot)$ satisfies Hypothesis 3.1 and if $\widetilde{A}_0$, $g(\cdot,\cdot)$ are defined in (3.13) then for any $a=(y,q)\in\widetilde{A}_0$ there exist $r>0$, a null subset $I_0\subset[0,\infty)$ and integrable function $k_a(\cdot)\in L_1([0,\infty);\mathbf{R}^+)$ such that

$$|g(t,b)-g(t,a)|\le k_a(t)\cdot\|b-a\|,\ (\forall)b\in B_r(a)\cap\widetilde{A}_0,\ (\forall)t\in[0,\infty)\setminus I_0. \tag{3.20}$$

In what follows we shall use the following subsets of the contingent cones $K_a^-\widetilde{A}_0,\ a=(y,q)\in\widetilde{A}_0$:

$$K_a^\infty\widetilde{A}_0:=\{\bar a=(\bar y,\bar q)\in K_a^-\widetilde{A}_0;\ \lim_{t\to\infty}<P(t,a),D_2X(t,a)\cdot\bar a>=0\}. \tag{3.21}$$

We prove now the following preliminary result providing additional properties of the Hamiltonian flow in (3.6):

Proposition 3.6. *If Hypothesys* 3.5 *is satisfied then the function* $V(\cdot)$ *defined in* (3.14) *is contingent differentiable and its contingent derivative in* (2.4) *is given by:*

$$\begin{gathered} V_K^-(a;\bar a)=<q,\bar y>-\lim_{t\to\infty}<P(t,a),D_2X(t,a)\cdot\bar a>, \\ (\forall)a=(y,q)\in\widetilde{A}_0,\ \bar a=(\bar y,\bar q)\in K_a^-\widetilde{A}_0. \end{gathered} \tag{3.22}$$

In particular, if $\bar a=(\bar y,\bar q)\in K_a^\infty\widetilde{A}_0$ *then one has:*

$$V_K^-(a;\bar a)=<q,\bar y>,\ (\forall)a=(y,q)\in\widetilde{A}_0,\ \bar a=(\bar y,\bar q)\in K_a^\infty\widetilde{A}_0. \tag{3.23}$$

Proof. If Hypothesis 3.5 is satisfied the Theorem 2.2 is applicable to the functions $g(\cdot,\cdot)$ and $V(\cdot)$ in (3.13)–(3.14) and therefore $V(\cdot)$ is contingent differentiable; moreover since according to Theorem 3.4, the functions $g(t,\cdot) := < P(t,\cdot), X'(t,\cdot) >$, $t \in [0,\infty)$ are contingent differentiable on $\widetilde{A}_0$, from (2.9) and (3.12) it follows:

$$V_K^-(a;\bar{a}) = -\int_0^\infty \frac{d}{dt}[< P(t,a), D_2X(t,a)\cdot\bar{a} >]dt, \tag{3.24}$$
$$(\forall) a = (y,q) \in \widetilde{A}_0,\ \bar{a} = (\bar{y},\bar{q}) \in K_a^- A.$$

Further on, using the fact that $P(0,a)=q, X(0,a)=y$ hence $D_2X(0,a)\cdot\bar{a}=\bar{y}$ if $a = (y,q) \in A$, $\bar{a} = (\bar{y},\bar{q}) \in K_a^- A$, and using also the properties of the Lebesgue integral and Leibnitz–Newton formula, from (3.24) we obtain successively:

$$V_k^-(a;\bar{a}) = -\lim_{s\to\infty}\int_0^s \frac{d}{dt}[< P(t,a), D_2X(t,a)\cdot\bar{a} >]dt =$$
$$= < P(0,a), D_2X(0,a)\cdot\bar{a} > - \lim_{s\to\infty} < P(s,a), D_2X(s,a)\cdot\bar{a} > =$$
$$= < q,\bar{y} > - \lim_{s\to\infty} < P(s,a), D_2X(s,a)\cdot\bar{a} >$$

and Proposition 3.6 is proved. ∎

Using the results in Proposition 3.6 we obtain first classical solutions of the equation (1.1) as follows:

Theorem 3.7. *Let us assume the Hypothesis* 3.5 *is satisfied and there exist an open subset* $\widetilde{Y} \subseteq pr_1(\widetilde{A}_0)$ *and a differentiable mapping* $\tilde{q}(\cdot) : \widetilde{Y} \to pr_2(A_0)$ *satisfying* (3.16) *and the following condition:*

$$(\bar{y}, D\tilde{q}(y)\cdot\bar{y}) \in K_{\tilde{a}(y)}^\infty \widetilde{A}_0,\ (\forall)\bar{y} \in \mathbf{R}^n = K_y^- \widetilde{Y}, y \in \widetilde{Y}, \tilde{a}(y) = (y,\tilde{q}(y)) \in \widetilde{A}_0. \tag{3.25}$$

Then the function $W(\cdot) : \widetilde{Y} \to \mathbf{R}$ *defined by:*

$$W(y) := V(y,\tilde{q}(y)), (\forall) y \in \widetilde{Y}, \tag{3.26}$$

is a classical solution of (1.1).

Moreover, if the mapping $\tilde{a}(\cdot) : \widetilde{Y} \to \widetilde{A}_0$ *is "flow–invariant" in the sense that:*

$$X(t,\tilde{a}(y)) \in \widetilde{Y},\ (\forall) y \in \widetilde{Y},\ t \in [0,\infty) \tag{3.27}$$

$$\tilde{a}(X(t,\tilde{a}(y))) = X_h^*(t,\tilde{a}(y)),\ (\forall) y \in \widetilde{Y},\ t \in [0,\infty), \tag{3.28}$$

then the solution $W(\cdot)$ *in* (3.26) *has the following asymptotic property along the characteristics:*

$$(\exists)\lim_{t\to\infty} W(X(t,\tilde{a}(y)))=0,\ (\forall)y\in\widetilde{Y}. \tag{3.29}$$

Proof. From Proposition 3.6 and from the hypotesis in (3.5) it follows that the function $W(\cdot)$ in (3.26) is differentiable and its derivative is given by:

$$DW(y)\cdot\bar{y}=<\tilde{q}(y),\bar{y}>,\ (\forall)\bar{y}\in K_y^{-}\widetilde{Y}=\mathbf{R}^n,$$

hence $DW(y)\equiv\tilde{q}(y)$; on the other hand, from the definition in (3.10) of the subset $A_0\subseteq domH(\cdot,\cdot)$ it follows that $H(y,\tilde{q}(y))\equiv H(y,DW(y))\equiv 0$ hence $W(\cdot)$ is a classical solution of (1.1).

If, in addition, the mapping $\tilde{a}(\cdot)$ has the properties in (3.27)–(3.28) then from (3.26), (3.14) it follows that:

$$\begin{aligned} W(X(t,\tilde{a}(y)))&=V(\tilde{a}(X(t,\tilde{a}(Y))))=\\ &=V(X_h^*(t,\tilde{a}(Y)))=-\int_0^\infty g(s,X_h^*(t,\tilde{a}(y)))ds. \end{aligned} \tag{3.30}$$

On the other hand, since the differential equation in (3.5) is an autonomous one, its maximal flow has the following Dynamical System ("semigroup") property:

$$X_h^*(s,X_h^*(t,a))=X_h^*(s+t,a),(\forall)a\in A,\ s,t\geq 0, \tag{3.31}$$

and therefore the function $g(\cdot,\cdot)$ in (3.13) has the same type of property:

$$g(s,X_h^*(t,a))=g(s+t,a),(\forall)a\in A,\ s,t\geq 0. \tag{3.32}$$

Using an obvious change of variable, from (3.30) it follows:

$$\begin{aligned} W(X(t,\tilde{a}(y)))=-\int_0^\infty g(s,X_h^*(t,\tilde{a}(y)))ds=-\int_0^\infty g(s+t,a)ds=\\ =-\int_0^\infty g(\tau,a)d\tau, \end{aligned}$$

and therefore, one of the basic properties of the Lebesgue integral (e.g. DUNFORD and SCHWARTZ [6], PRECUPANU [21], etc.) leads to the asymptotic property in (3.29). ∎

As the simple examples in next section show, equation (1.1) may have more than one classical solution of the type in Theorem 3.7; on the other hand, if $H(\cdot,\cdot)$ is the Hamiltonian in (1.7), we are looking for a solution of (1.1), even in some generalized sense, that could be the value function in (1.6) of the problem (1.2)–(1.5).

To this aim we prove now the following result:

Theorem 3.8. *Let us assume that Hypothesis* 3.5 *is satisfied and let* $\widetilde{W_0}(\cdot) : \widetilde{Y}_0 \subseteq pr_1(\widetilde{A}_0) \to \mathbf{R}$ *be the function defined by:*

$$\text{(3.33)} \qquad \begin{cases} \widetilde{W_0}(y) := \inf\limits_{q \in Q_0(y)} V(y,q), \\ \widetilde{Q}_0(y) := \{q \in Q_0(y); \widetilde{W_0}(y) = V(y,q)\}, \\ \widetilde{Y}_0 := \{y \in pr_1(\widetilde{A}_0); \widetilde{Q}_0(y) \neq \Phi\}. \end{cases}$$

Then the upper left contingent derivatives of $\widetilde{W_0}(\cdot)$ *satisfy the following relations:*

$$\text{(3.34)} \qquad \begin{array}{c} \overline{D}_K^{-}\widetilde{W_0}(y,\bar{y}) \geq \sup\limits_{q \in \widetilde{Q}_0(y)} < q, \bar{y} >, \\ (\forall)\bar{y} \in K_y^{\infty}\widetilde{Y}_0 = \{\bar{y} \in K_y^{-}\widetilde{Y}_0;\ (\exists) q \in \widetilde{Q}_0(y),\ \bar{q} \in \mathbf{R}^n : (\bar{y},\bar{q}) \in K_{(y,q)}^{\infty}\widetilde{A}_0\} \end{array}$$

and if $Q_0(y,\widetilde{Y}_0)$ *is defined by:*

$$\text{(3.35)} \qquad Q_0(y,\widetilde{Y}_0) = \{q \in Q_0(y); X(t,(y,q)) \in \widetilde{Y}_0, (\forall) t \geq 0\}$$

then $\widetilde{W_0}(\cdot)$ *has the following asymptotic property along characteristics:*

$$\text{(3.36)} \qquad \limsup_{t \to \infty} \widetilde{W_0}(X(t,(y,q))) \leq 0,\ \ (\forall) y \in \widetilde{Y}_0, q \in Q_0(y,\widetilde{Y}_0).$$

If, in addition, the multifunction $\widetilde{Q}_0(\cdot)$ *in* (3.33) *has a selection* $\tilde{q}(\cdot)$*:*

$$\text{(3.37)} \qquad \tilde{q}(y) \in \widetilde{Q}_0(y),\ \ (\forall) y \in \widetilde{Y}_0,$$

which has the invariance propertities in (3.27)–(3.28) *then* $\widetilde{W_0}(\cdot)$ *has also the asymptotic property in* (3.29).

In particular, if the subset $\widetilde{Y} \subseteq \widetilde{Y}_0$ *is defined by:*

$$\widetilde{Y} := \{y \in \widetilde{Y}_0; \overline{co} K_y^{\infty} \widetilde{Y}_0 = \mathbf{R}^n\}, \tag{3.38}$$

then $\widetilde{W_0}(\cdot)$ *is a strict viscosity subsolution of* (1.1) *at any point* $y \in \widetilde{Y}$ *in the sense that:*

$$H(y, p) = 0, (\forall) \in \widetilde{Y}, p \in \underline{\partial}_K \widetilde{W_0}(y). \tag{3.39}$$

Proof. We apply Proposition 2.1 to the marginal function $\widetilde{W_0}(\cdot)$ in (3.33) to obtain:

$$\overline{D}_k^- \widetilde{W_0}(y; \bar{y}) \geq \sup_{q \in \widetilde{Q}_0(y)} \sup_{(\bar{y}, \bar{q}) \in K^{\infty}_{(y,q)} \widetilde{A}_0} V_K^-((y, q); (\bar{y}, \bar{q}))$$

and therefore from (3.23) we obtain (3.34).

In order to prove (3.36) we note that from (3.33), (3.35), (3.32) it follows that if $y \in \widetilde{Y}_0,\ q \in Q_0(y, \widetilde{Y}_0)$ then one has:

$$W(X(t, (y, q))) \leq V(X_h^*(t, (y, q))) = -\int_t^{\infty} g(\tau, (y, q)) d\tau \to 0 \text{ as } t \to \infty.$$

If the selection $\tilde{q}(\cdot)$ of $\widetilde{Q}_0(\cdot)$ has the invariace property in (3.27)–(3.28) then the asymptotic property in (3.29) follows as in the proof of Theorem 3.7.

Finally, if $\widetilde{Y} \subseteq \widetilde{Y}_0$ is asubset in (3.38) then from the definition in (2.3) of the contingent subdifferential it follows that if $y \in \widetilde{Y}$, $p \in \underline{\partial}_K \widetilde{W_0}(y)$ then

$$< p, \bar{y} > \geq \overline{D}_K^- \widetilde{W_0}(y; \bar{y}),\ (\forall) \bar{y} \in K_y^- \widetilde{Y}_0$$

hence from (3.34) it follows

$$< p, \bar{y} > \geq \sup_{q \in \widetilde{Q}_0(y)} < q, \bar{y} >, (\forall) \bar{y} \in K_y^{\infty} \widetilde{Y}_0 = \mathbf{R}^n,$$

which, in turn, implies the fact that $\widetilde{Q}_0(y) = \{p\}$; on the other hand, since $\widetilde{Q}_0(y) \subseteq Q_0(y), (\forall) y \in \widetilde{Y}_0$, from the definitions in (3.10), (3.15) it follows that $H(y, q) = 0, (\forall) q \in \widetilde{Q}_0(y)$ and (3.39) is proved. ∎

Remark 3.9. In the case the Hamiltonian $H(\cdot,\cdot)$ is associated as in (1.7) to an optimal control problem of the form (1.2)–(1.5) and if one uses the modified orientor field in (3.4) then the Hamiltonian flow in (3.6) has the following additional property:

$$\begin{gathered} < P(t,a), X'(t,a) > + g_0(X(t,a), X'(t,a)) = H(X_h^*(t,a)) = 0, \\ (\forall)\, t \in I(a),\ a \in A_0, \end{gathered} \tag{3.40}$$

hence the function $V(\cdot)$ in (3.14) may equivalently be defined by:

$$V(a) = \int_0^\infty g_0(X(t,a), X'(t,a))dt = C(X(\cdot,a)),\ (\forall) a \in A_0, \tag{3.41}$$

which means that for any $a \in A_0$, the first component $X(\cdot,a)$ of the Hamiltonian flow is an admissible trajectory hence the Hamiltonian flow in (3.6) defines the multi–selection of admissible trajectories:

$$pr_1(A_0) \ni y \mapsto \mathcal{A}(y) := \{X(\cdot,(y,q)); q \in Q_0(y)\}$$

from which, the minimization in (3.33) chooses a possible optimal one:

$$\widetilde{\mathcal{A}}(y) = \{X(\cdot,(y,q)); q \in \widetilde{Q}_0(y)\}, y \in \widetilde{Y}_0,$$

whose associated value function is the marginal function $\widetilde{W}_0(\cdot)$ in (3.33).

However, for the optimality of the selection $\widetilde{\mathcal{A}}(\cdot)$ above one needs additional properties allowing the application of suitable verification theorems among which the asymptotic properties in (1.8) are essential.

4. Examples. In this section we illustrate the results in Section 3 be two simple examples of Hamilton–Jacobi equations associated to problems of the form (1.2)–(1.5).

Examples 4.1. Let us consider the optimal control problem in (1.2)–(1.5) defined by the following data:

$$Y = \mathbf{R}, g_0(x,v) = \frac{1}{2}(v-x)^2, G(x) = \mathbf{R}, \Omega = AC^{loc}([0,\infty);\mathbf{R}), \tag{4.1}$$

where $AC^{loc}([0,\infty);\mathbf{R})$ denotes the set of functions $x(\cdot) : [0,\infty) \to \mathbf{R}$ that are (locally) absolutely continuous on any compact subinterval $I \subset [0,\infty)$.

As it is easy to see, the Hamiltonian in (1.7) is given by:

$$\begin{aligned} &H(x,p) = p \cdot x - \frac{p^2}{2} \\ &\widehat{G}(x,p) = \{x - p; (x,p) \in A := \mathbf{R}^2\} \end{aligned} \tag{4.2}$$

hence Hypothesis 3.1 is trivially satisfied, the Hamiltonian system in (3.5) being the following classical ones:

$$\begin{cases} x' = x - p, \ x(0) = y \in \mathbf{R} \\ p' = -p, \ p(0) = q \in \mathbf{R}. \end{cases} \tag{4.3}$$

Standard computations show that the Hamiltonian flow in (3.6) is given by:

$$X(t,a) = \left(y - \frac{q}{2}\right) \cdot e^t + \frac{q}{2} \cdot e^{-t}, \ P(t,a) = q \cdot e^{-t}, \ (\forall) a = (y,q) \in A, \tag{4.4}$$

the subset $A_0 \subseteq A$ in (3.10) and the function $g(\cdot,\cdot)$ in (3.13) are given by:

$$A_0 = \{(y,q) \in \mathbf{R}^2; q \cdot \left(y - \frac{q}{2}\right) = 0\}, \ g(t,a) = \frac{-q^2}{2} \cdot e^{-2 \cdot t} \quad \text{if } a = (y,q) \in A_0, \tag{4.5}$$

hence the function $V(\cdot)$ in (3.14) is given by:

$$V(a) = \frac{q^2}{4}, \ (\forall) a = (y,q) \in \widetilde{A}_0 = A_0. \tag{4.6}$$

As it is easy to see, the stratified subset $A_0 \subset \mathbf{R}^2$ in (4.5) has only the following two differentiable selection, $\tilde{q}_1(\cdot), \tilde{q}_2(\cdot)$, satisfying the properties in (3.15):

$$\tilde{q}_1(y) \equiv 0, \tilde{q}_2(y) \equiv 2 \cdot y, (\forall) y \in \widetilde{Y} = \mathbf{R}, \tag{4.7}$$

which provide the following distinct classical solutions of the equation (1.1) (defined by the Hamiltonian in (4.2)):

$$W_1(y) \equiv 0, \ W_2(y) \equiv y^2, \ (\forall) y \in \widetilde{Y} = \mathbf{R}, \tag{4.8}$$

which have the asymptotic property in (3.29).

However, the solution $W_2(y) \equiv y^2, y \in \mathbf{R}$ does not have either of the asymptotic properties in (3.36), (1.8) while the "minimal" solution, $\widetilde{W_0}(\cdot)$ in (3.33) coincides with the other smooth solution, $W_1(y) \equiv 0$ which has both asymptotic properties. In fact, using an "elementary" verification theorem one may prove that $W_2(\cdot)$ is the value function of the problem (1.2)–(1.5) defined by the data in (4.1) and for any $y \in \mathbf{R}$, the function $X(t,y) = y \cdot e^t, t \in [0,\infty)$ is the unique optimal trajectory.

Example 4.2. We consider now the optimal control problem (1.2)–(1.5) defined by the following data:

$$\begin{array}{ll}(4.9) & \begin{array}{l} Y = \mathbf{R}^n,\ G(x) = \mathbf{R}^n,\ g_0(x,v) = \|v\| = \sqrt{<v,v>},\ (\forall)x, v \in \mathbf{R}^n, \\ \Omega_a = AC^{loc}([0,\infty); \mathbf{R}^n). \end{array}\end{array}$$

Standard computations show that the Hamiltonian in (1.7) is given in this case by:

$$(4.10)\quad \begin{cases} H(x,p) = \begin{cases} -\infty & \text{if } \|p\| > 1 \\ & \qquad x, p \in \mathbf{R}^n \\ 0 & \text{if } \|p\| \leq 1 \end{cases} \\ \widehat{G}(x,p) = \begin{cases} \Phi & \text{if } \|p\| > 1 \\ \{0\} & \text{if } \|p\| < 1, \quad x, p \in \mathbf{R}^n, \\ \{\mu \cdot p; \mu \leq 0\} & \text{if } \|p\| = 1 \end{cases} \\ A = domH(\cdot,\cdot) = \mathbf{R}^n \times \overline{B}_1(0),\ \overline{B}_1(0) := \{p \in \mathbf{R}^n;\ \|p\| \leq 1\}. \end{cases}$$

As one may see, the domain $A = \mathbf{R}^n \times \overline{B}_1(0)$ of the Hamiltonian as well as the Hamiltonian itself are stratified (e.g. MIRICĂ [13], [15]) being partitioned as follows:

$$(4.11)\quad \begin{array}{l} A = A_1 \cup A_2,\ A_1 := \mathbf{R}^n \times B_1(0),\ A_2 := \mathbf{R}^n \times S_1(0), \\ S_1(0) := \{p \in \mathbf{R}^n; \|p\| = 1\} \\ H_1(\cdot,\cdot) := H(\cdot,\cdot)|_{A_1}, H_2(\cdot,\cdot) := H(\cdot,\cdot)|_{A_2} \in C^1. \end{array}$$

Since the stratum $A_1 = \mathbf{R}^n \times B_1(0) \subset \mathbf{R}^n \times \mathbf{R}^n$ is an open subset, it is easy to see that the procedure in Section 3 for the restriction $H_1(\cdot,\cdot) := H(\cdot,\cdot)|_{A_1} \equiv 0$ provides the trivial solution $W_1(y) \equiv 0, y \in \mathbf{R}^n$ of the equation in (1.1).

However, the results in Section 3 applied to the restriction $H_2(\cdot,\cdot) := H(\cdot,\cdot)|_{A_2} \equiv 0$, though of the same type, provides the non trivial solution:

$$(4.12)\qquad W_2(y) = \|y\|, y \in Y := \mathbf{R}^n \setminus \{0\},$$

which solves the problem of the "minimal lenght path" joining a point $y \in \mathbf{R}^n \setminus \{0\}$ with the origin $0 \in \mathbf{R}^n$.

Since the tangent space to the $(2 \cdot n - 1)$–dimensional submanifold $A_2 = \mathbf{R}^n \times S_1(0) \subset \mathbf{R}^n \times \mathbf{R}^n$ is given by:

$$T_{(x,p)}A_2 = \{(\bar{x},\bar{p}) \in \mathbf{R}^n \times \mathbf{R}^n; <\bar{p}, p>= 0\}$$

it is easy to see that the modified orientor field in (3.4) is given by:

$$d_F^{\#} H_2(x,p) := \{(\mu \cdot p, 0); \mu \leq 0\},\ (\forall)(x,p) \in A_2 = \mathbf{R}^n \times S_1(0). \tag{4.13}$$

From the very large set of possible selections $h(\cdot,\cdot)$ in (3.2) we choose the smooth function defined as follows:

$$h(x,p) = (-<x,p> \cdot p, 0), (\forall)(x,p) \in A_2^+ := \{(x,p) \in A_2; <x,p> > 0\}. \tag{4.14}$$

Since $A_2^+ \subset A_2$ is an open submanifold and $h(\cdot,\cdot)$ is analytical, then exists a unique global Hamiltonian flow, $X_h^*(\cdot,\cdot) = (X(\cdot,\cdot), P(\cdot,\cdot))$ which has the following properties:

$$\begin{array}{l} P(t,a) \equiv q, \alpha(t,a) := <P(t,a), X(t,a)> \equiv e^{-t} \cdot <y,q> \\ g(t,a) \equiv \alpha'(t,a) \equiv -e^{-t} \cdot <y,q>,\ (\forall)t \geq 0,\ a = (y,q) \in A_2^+, \end{array} \tag{4.15}$$

and therefore the function $V(\cdot)$ in (3.14) is given by:

$$V(a) = <y,q>,\ (\forall)a = (y,q) \in A_2^+. \tag{4.16}$$

If we choose now the selection: $\tilde{q}(y) = \dfrac{y}{\|y\|}, (\forall)y \in Y_0,\ Y_0 = \mathbf{R}^n \backslash \{0\}$, which, obviously satisfies (3.16) then we obtain the non–trivial classical solution $W_2(\cdot)$ in (4.12).

REFERENCES

1. AUBIN, J.P., CELLINA, A. – *Differential Inclusions*, Springer–Verlag, New York, Berlin, 1984.
2. BARBU, V., DA PRATO, G. – *Hamilton–Jacobi Equations in a Hilbert Spaces*, Pitman, Boston, 1983.
3. CARLSON, D.A. – *A Carathéodory–Hamilton–Jacobi Theory for Infinite Horizon Optimal Control Problems*, Journal of Optimization Theory and Applications, 48, 1986, 265–287.
4. CARLSON, D.A., HAURIE, A., LEIZAROWITZ, A. – *Infinite Horizon Optimal Control. Deterministic and Stochastic Systems*, Springer–Verlag, New York, Berlin, Heidelberg, 1991.
5. CRANDALL, M.G., LIONS, P.L. – *Viscosity Solutions of Hamilton–Jacobi Equations*, Trans. A.M.S., 277 (1983), 1–42.

6. DUNFORD, N., SCHWARTZ, J.T. – *Linear Operator.* Part I: *General Theory*, Interscience, New York, 1958.

7. GODBILLON, G. – *Géométrie Différéntielle et Méchanique Analitique*, Hermann, Paris, 1969.

8. HARTMAN, PH. – *Ordinary Differential Equations*, Wiley, New York, 1964.

9. LANG, S. – *Introduction to Differential Manifolds*, Interscience, New York, 1962.

10.MC SHANE, E.J. – *Integration*, Princeton Univ. Press, 1947.

11.MIRICĂ, I. – *Basic properties of the value functions of infinite–horizon optimal control problems*, Anal. Univ. din Galaţi (to appear).

12.MIRICĂ, I. – *On differentiability with respect to parameters of the Lebesgue integral*, Anal. Univ. din Galaţi (to appear).

13.MIRICĂ, Şt. – *Generalized solutions by Cauchy's Method of Characteristics*, Rend. Sem. Mat. Univ. Padova, 77, 1987, 317–349.

14.MIRICĂ, Şt. – *Ecuaţii diferentiale şi cu derivate parţiale de ordinul întâi*, Curs, Univ. din Bucureşti, 1989.

15.MIRICĂ, Şt. – *Control optimal. Condiţii suficiente şi sinteză*, Ed.Ştiinţifică, Bucureşti, 1990.

16.MIRICĂ, Şt. – *Nonsmooth fields of extremals and constructive dynamic programming in optimal control,* Part I: *Autonomous problems*, Preprint N. 18, Dipartamento di Mat. Pura ed Appl., Padova, Italia, 1993.

17.MIRICĂ, Şt., MIRICĂ, I. – *Geometric properties of classical derivatives leading to generalized derivatives* , Stud. Cerc. Mat., Tom 40, Nr. 6, 1988, 477–505.

18.MIRICĂ, Şt., MIRICĂ, I. – *Derivate contingenţiale extreme ale funcţiilor marginale*, Lucrările Ses. Jubiliare, Vol. 5, Inst. Pol. "Gh. Asachi", Iaşi, 1988, 197–202.

19.MIRICĂ, Şt., MIRICĂ, I. – *Distinct types of intrinsic tangent cones to arbitrary subsets of Euclidian spaces*, Anal. Univ. Bucureşti, 38, Nr.1, 1989, 21–39.

20.PENOT, J.P. – *Calcul sous–différentiel et optimisation*, J.Func. Anal., 27, 1978, 248–276.

21.PRECUPANU, A. – *Analiză matematică. Funcţii reale*, Ed. Did. şi Ped. Bucureşti, 1976.

22.SUBBOTIN, A.I. – *Generalized Solutions of First Order*, P.D.E., Birkhäuser, Boston, 1995.

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