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SOME RESULTS ON HARMONIC SECTIONS OF COTANGENT BUNDLES*

BY

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Abstract. Let (M,g) be a Riemannian manifold and consider a symmetric M–tensor field defined by the components $h_{ij}(q,p)$, on the cotangent bundle T^*M of M. Then T^*M may be endowed with a semi–Riemannian metric G of Riemann extension type, defined by the Levi–Civita connection of g and the M–tensor field h_{ij}. A 1–form η on M may be thought of as a map $\phi_\eta:(M,g)\longrightarrow(T^*M,G)$. We study the conditions under which the map ϕ_η is either an isometric immersion, or a totally geodesic map, or a harmonic map. Then we study the property of ϕ_η^*G to be harmonic with respect to g.

Introduction. The notion of a harmonic mapping ϕ between the Riemannian manifolds $(M,g),(N,h)$ where M is compact, arised from the problem of minimizing the energy integral ([7]):

$$E(\phi)=\frac{1}{2}\int_M \|\nabla d\phi\|^2 v_g.$$

This notion can be generalized in many ways ([6]). First of all, the minimization problem for the energy integral remains still valid when the Riemannian metric h is replaced by a semi–Riemannian one, although it is not expected to obtain zero as the minimal value of $E(\phi)$. Moreover, the formal expression of the tension of ϕ (obtained from the variation of $E(\phi)$) can be considered even if on N it is taken a linear connection $\overset{N}{\nabla}$ which is not necessarily the Levi–Civita connection of a (semi–) Riemannian metric h on

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N. Remark that the second fundamental form $\nabla d\phi$ of a differentiable mapping $\phi : M\longrightarrow N$ can be defined even in the case where M, N are endowed with two torsion free linear connections $\overset{M}{\nabla}, \overset{N}{\nabla}$ respectively. In [10] the first author has studied some problems concerning the harmonicity for a 1-form η on a Riemannian manifold (M, g) where η is thought of as a differentiable mapping from M to its cotangent bundle T^*M, endowed with a semi–Riemannian metric of Riemann extension type (see [14], [4], [12]). Remark that similar problems have been considered for vector fields and, more general, for tensor fields, when the tangent bundle or the corresponding tensor bundle are endowed with Riemannian metrics of Sasaki type ([1], [8], [16], [17], [15]).

In the present paper we study some problems related to a 1-form η defined on a Riemannian space (M, g), thought of as a section in the cotangent bundle T^*M, where T^*M is endowed with a semi–Riemannian metric G of Riemann extensiton type defined by the Levi–Civita connection ∇ of g and an arbitrary symmetric M-tensor field h of type $(0, 2)$ on T^*M. The 1-form η on M defines a map $\phi_\eta : (M, g)\longrightarrow(T^*M, G)$ and there are some interesting relations between η and h in the case where ϕ_η is totally geodesic or harmonic. We determine the Levi–Civita connection of G and its curvature tensor field. Then we study the condition under which ϕ_η is an isometric immersion. From the expression of the second quadratic form of ϕ_η we get the notion of an H-geodesic immersion. If ϕ_η is an H-geodesic isometric immersion then it is a totally geodesic immersion. We show that there always exists a pseudo–Riemannian metric G on T^*M such that the components h_{ij} which define the symmetric M–tensor h are dependent of the cotangential coordinates p_i and ϕ_η is a totally geodesic immersion. Next, there are obtained conditions for harmonicity of ϕ_η in the case where ϕ_η is an H-geodesic map (see Theorem 8). Finally we show that, in the case where ϕ_η is an H-geodesic map, the tensor field $\phi_\eta^* G$ is harmonic with respect to g if and only if ϕ_η is harmonic.

The manifolds, tensor fiels and geometric objects we consider in this paper, are assumed to be differentiable of class C^∞. We use the well known summation convention, the range for the indices i, j, k, h, ℓ, s, t being always $1, ..., n$. We shall denote by $\Gamma(TM)$ the module of smooth vector fields on TM.

1. The pseudo–Riemannian metric on T^*M. Let M be an n-dimensional smooth manifold and denote by $\pi : T^*M\longrightarrow M$ its cotangent bundle with fibres the cotangent spaces to M. Then T^*M is a $2n$–dimensional manifold and some local charts, induced naturally from local charts on M, may be used. Namely, if (U, x^i); $i = 1, ..., n$ is a local chart on M, then the local chart $(\pi^{-1}(U), q^i, p_i)$; $i = 1, ..., n$ is defined on T^*M, whee $q^i = x^i \circ \pi$; $i = 1, ..., n$ (i.e. the first n local coordinates of a cotangent vector are the local coordinates of its base point) and p_i; $i = 1, ..., n$ are the vector space coordinates of the considered cotangent vector with respect to the natural local frame $(dx^1, ..., dx^n)$ in T^*M, determined by the local chart (U, x^i).

The M-tensor fields and linear M-connections can be considered on T^*M and the usual tensor fields and linear connections on M may be considered as M-tensor fields and linear M-connections on T^*M depending only on the base point of the cotangent vector. The M-tensor fields on T^*M corresponding to the covariant tensor fields on M do coincide with the pullback of these tensor fields, by the differentiable mapping $\pi : T^*M \longrightarrow M$ (see [9] for the definition of the M-tensor field and the linear M-connection).

Let $VT^*M = \text{Ker } \pi_* \subset TT^*M$ be the vertical distribution over T^*M. Then VT^*M is involutive with fibre dimension n and the local vector fields $\partial^i = \dfrac{\partial}{\partial p_i}$; $i = 1, ..., n$ define a local frame in VT^*M, over the domain $\pi^{-1}(U)$ of the induced local chart $(\pi^{-1}(U), q^i, p_i)$ on T^*M. A nonlinear connection on T^*M is defined by a complementary distribution HT^*M (called a horizontal distribution) to VT^*M in TT^*M. A local frame in HT^*M, related to the induced local chart $(\pi^{-1}(U), q^i, p_i)$ is defined by the HT^*M–valued local vector fields $\delta_i = \dfrac{\delta}{\delta q^i}$; $i = 1, ..., n$ where

$$\delta_i = \frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - N_{ij}\frac{\partial}{\partial p_j}\cdot$$

The functions $N_{ij} = N_{ij}(q, p)$; $i, j = 1, ..., n$, are the connection coefficients of the considered nonlinear connection in the induced local chart $(\pi^{-1}(U), q^i, p_i)$; $i = 1, ..., n$. If the connection coefficients of the considered nonlinear connection are symmetric, i.e. $N_{ij} = N_{ji}$ then the nonlinear connection is called *symmetric*. The property of the connection coefficients N_{ij} to be symmetric is independent of the chosen induced local chart on T^*M and there always exists a symmetric nonlinear connection on T^*M. If $\overset{\circ}{N}_{ij}$ are the connection coefficients of an arbitrary nonlinear connection on T^*M then the functions $N_{ij} = \dfrac{1}{2}(\overset{\circ}{N}_{ij} + \overset{\circ}{N}_{ji})$ define a symmetric nonlinear connection on T^*M, while the functions $b_{ij} = \dfrac{1}{2}(\overset{\circ}{N}_{ij} - \overset{\circ}{N}_{ji})$ define an M-tensor field of type $(0, 2)$ on T^*M. In this paper we shall work with symmetric nonlinear connections only. If we have a symmetric nonlinear connection on T^*M, defined by the horizontal distribution HT^*M then:

$$TT^*M = VT^*M \oplus HT^*M \tag{1}$$

and $(\partial^i, \delta_i);\ i = 1, ..., n$ is a local frame in TT^*M, adapted to this direct sum decomposition. The system of local 1-forms $(dp_i + N_{ji}dq^j, dq^i);\ i = 1, ..., n$ is the dual local frame of the local frame $(\partial^i, \delta_i);\ i = 1, ..., n$. We have the following formulas giving the brackets of the vector fields from the considered adapted local frame:

$$[\partial^i, \partial^j] = 0,\ [\partial^i, \delta_j] = \Phi^i_{jk}\partial^k,\ [\delta_i, \delta_j] = -R_{kij}\partial^k \tag{2}$$

where

$$\Phi^i_{jk} = -\partial^i N_{jk} = -\frac{\partial N_{jk}}{\partial p_i},\ R_{kij} = \delta_i N_{jk} - \delta_j N_{ik} = \frac{\delta N_{jk}}{\delta q^i} - \frac{\delta N_{ik}}{\delta q^j}. \tag{3}$$

Remark that the components Φ^i_{jk} define a linear M-connection on T^*M while the components R_{kij} define an M–tensor field of type $(0,3)$ on T^*M.

It is obvious that the complete integrability of the differential system on T^*M, defined by HT^*M is equivalent to the vanishing of the components R_{kij} on $\pi^{-1}(U)$.

Assume that the base manifold M is endowed with a (pseudo–) Riemannian metric g. Then it is determined the Levi–Civita connection ∇ of this metric, with the connection coefficients the Christoffel symbols Γ^k_{ij} and the connection ∇ defines a symmetric nonlinear connection on T^*M with the connection coefficients:

$$N_{ij} = -p_k\Gamma^k_{ij} \tag{4}$$

(i.e. N_{ij} are linear in (co–)tangential coordinates p_k).

Then the formulas (3) become:

$$[\partial^i, \partial^j = 0,\ [\partial^i, \delta_j] = \Gamma^i_{jk}\partial^k,\ [\delta_i, \delta_j] = R^0_{kij}\partial^k, \tag{5}$$

where the components R^0_{kij} are defined by

$$R^0_{kij} = p_h R^h_{kij}$$

and R^h_{kij} are the components of the curvature tensor field of ∇ in the local chart (U, x^i) on M. It is obvious that the components R^0_{kij} define an M-tensor field of type $(0,3)$ on T^*M. Consider an arbitrary symmetric M-tensor field of type $(0,2)$ on T^*M, defined by the components $h_{ij} = h_{ij}(q,p)$. Then a pseudo–Riemannian metric G is defined on T^*M by:

$$G(\partial^i, \partial^j) = 0,\ G(\partial^i, \delta_j) = G(\delta_j, \partial^i) = \delta^i_j,\ G(\delta_i, \delta_j) = 2h_{ij}. \tag{6}$$

Remark that the pseudo–Riemannian metric G is very much similar to the Riemann extension defined and studied in ([14], [11], [12], [13]). In the case of Riemann extensions the M-tensor field defined by the components h_{ij} is the pullback of symmetric covariant tensor field on the base manifold, i.e. the components h_{ij} do not depend on p_k. An alternative expression for the metric G is:

$$G = 2dq^i[dp_i + (h_{ij} - \Gamma^k_{ij} p_k) dq^j].$$

The Levi–Civita connection and the Riemann Christoffel tensor field of the metric G may be obtained by a straightforward computation by using their definitions. Then we obtain the following results.

Proposition 1. *The Levi–Civita connection $\widetilde{\nabla}$ of G is expressed in the local adapted frame (∂^i, δ_i) by:*

$$\begin{aligned} &\widetilde{\nabla}_{\partial^i}\partial^j = 0,\ \widetilde{\nabla}_{\partial^i}\delta_j = \partial^i h_{jk}\partial^k,\ \widetilde{\nabla}_{\delta_i}\partial^j = (\partial^j h_{ik} - \Gamma^j_{ik})\partial^k, \\ &\widetilde{\nabla}_{\delta_i}\delta_j = (\Gamma^k_{ij} - \partial^k h_{ij})\delta_k + (-R^0_{ijk} + 2h_{k\ell}\partial^\ell h_{ij} + \nabla_i h_{jk} + \\ &\qquad + \nabla_j h_{ik} - \nabla_k h_{ij})\partial^k \end{aligned} \tag{7}$$

where the components $\nabla_i h_{jk}$ are defined by

$$\nabla_i h_{jk} = \delta_i h_{jk} - \Gamma^\ell_{ij} h_{\ell k} - \Gamma^\ell_{ik} h_{j\ell}.$$

Remark that the components $\nabla_i h_{jk}$ define an M-tensor field of type $(0,3)$ on T^*M while the components $\partial^i h_{jk}$, $\partial^i\partial^j h_{k\ell}$ define M-tensor fields on T^*M of type $(1,2)$ and $(2,2)$, respectively.

Before we give the expression of the curvature tensor field of the connection $\widetilde{\nabla}$ we shall present some Ricci type commutation formulas for the operators ∂ and ∇ acting on the M-tensor field defined by the components h_{jk} on T^*M. Then we have by a straightforward computation:

$$\partial^i \nabla_j h_{k\ell} = \nabla_j \partial^i h_{k\ell} \tag{8}$$

and

$$(9)\qquad \nabla_i\nabla_j h_{kh} - \nabla_j\nabla_i h_{kh} = R^0_{\ell ij}\partial^\ell h_{kh} - R^\ell_{kij}h_{\ell h} - R^\ell_{hij}h_{k\ell}.$$

Remark that the last two terms in the right hand side of the relation (9) are usual in the Ricci type commutation formulas while the first term in a new one, arising from the dependence of the components h_{jk} on the coordinates p_i.

Proposition 2. *The curvature tensor field K of the connection $\widetilde{\nabla}$ on T^*M is expressed in the local adapted frame (∂^i, δ_i) by:*

$$
\begin{aligned}
&K(\partial^i,\partial^j)\partial^k = K(\partial^i,\partial^j)\delta_k = 0,\ \ K(\partial^i,\delta_j)\partial^k = \partial^i\partial^k h_{j\ell}\partial^\ell,\\
&K(\delta_i,\delta_j)\partial^k = (-R^k_{hij} + \partial^k\nabla_i h_{jh} - \partial^k\nabla_j h_{ih} + \partial^k h_{j\ell}\partial^\ell h_{ih} - \partial^k h_{i\ell}\partial^\ell h_{jh})\partial^h,\\
&K(\partial^i,\delta_j)\delta_k = -\partial^h\partial^i h_{jk}\delta_h + (-R^i_{jkh} + \partial^i\nabla_k h_{jh} - \partial^i\nabla_h h_{jk} + \partial^i h_{h\ell}\partial^\ell h_{jk} -\\
&\qquad\qquad -\partial^i h_{k\ell}\partial^\ell h_{jh} + 2h_{h\ell}\partial^i\partial^\ell h_{jk})\partial^h,\\
&K(\delta_i,\delta_j)\delta_k = (R^h_{kij} - \partial^h\nabla_i h_{jk} + \partial^h\nabla_j h_{ik} + \partial^h h_{i\ell}\partial^\ell h_{jk} - \partial^h h_{j\ell}\partial^\ell h_{ik})\delta_h +\\
&\quad + [\nabla_i(\nabla_k h_{jh} - \nabla_h h_{jk}) - \nabla_j(\nabla_k h_{ih} - \nabla_h h_{ik}) - \nabla_0 R_{ijkh} -\\
&\quad - h_{k\ell}R^\ell_{hij} - h_{h\ell}R^\ell_{kij} + R^0_{ik\ell}\partial^\ell h_{jh} + R^0_{jh\ell}\partial^\ell h_{ik} - R^0_{ih\ell}\partial^\ell h_{jk} - R^0_{jk\ell}\partial^\ell h_{ih} +\\
&\quad + \partial^\ell h_{jk}(\nabla_h h_{i\ell} - \nabla_\ell h_{ih} + \nabla_i h_{h\ell}) - \partial^\ell h_{ik}(\nabla_h h_{j\ell} - \nabla_\ell h_{jh} + \nabla_j h_{h\ell}) +\\
&\quad + \partial^\ell h_{ih}(\nabla_j h_{k\ell} + \nabla_k h_{j\ell} - \nabla_\ell h_{jk}) - \partial^\ell h_{jh}(\nabla_i h_{k\ell} + \nabla_k h_{i\ell} - \nabla_\ell h_{ik}) +\\
&\quad + 2h_{hm}(\partial^m h_{j\ell}\partial^\ell h_{ik} - \partial^m h_{i\ell}\partial^\ell h_{jk}) + 2h_{\ell m}(\partial^m h_{jk}\partial^\ell h_{ih} - \partial^m h_{ik}\partial^\ell h_{jh}) +\\
&\quad + 2h_{h\ell}(\partial^\ell\nabla_i h_{jk} - \partial^\ell\nabla_j h_{ik}]\partial^h
\end{aligned}
\qquad (10)
$$

where R_{ijkh} are the components of the Riemann Christoffel tensor field of ∇ i.e. $R_{ijkh} = g_{i\ell}R^\ell_{jkh}$ and $\nabla_0 R_{ijkh} = g^{ab}p_a\nabla_b R_{ijkh}$.

Remark. The same formulas concerning the connection $\widetilde{\nabla}$ and its curvature tensor field may be obtained by using another nonlinear connection on T^*M and the results from [12], [13]. Namely, consider the nonlinear connection on T^*M defined by the connection coefficients

$$\overline{N}_{ij} = N_{ij} + h_{ij} = -p_k\Gamma^k_{ij} + h_{ij}. \tag{11}$$

Then the corresponding local adapted frame is $(\partial^i, \bar{\delta}_k)$, where

$$\bar{\delta}_k = \delta_k - h_{k\ell}\partial^\ell \tag{12}$$

and the pseudo–Riemannian metric G es expressed by the formulas:

$$G(\partial^i, \partial^j) = G(\bar{\delta}_i, \bar{\delta}_j) = 0,\ G(\partial^i, \bar{\delta}_j) = G(\bar{\delta}_j, \partial^i) = \delta^i_j.$$

The Levi–Civita connection $\widetilde{\nabla}$ of the pseudo–Riemannian metric G on T^*M with respect to the new adapted local frame is obtained by using the formulas from [12] and [13].

$$\widetilde{\nabla}_{\partial^i}\partial^j = 0,\ \widetilde{\nabla}_{\partial^i}\bar{\delta}_j = 0,\ \widetilde{\nabla}_{\bar{\delta}_i}\partial^j = -\Phi^j_{ik}\partial^k,\ \widetilde{\nabla}_{\bar{\delta}_i}\bar{\delta}_j = \Phi^k_{ij}\bar{\delta}_k + \overline{R}_{ijk}\partial^k \tag{13}$$

where

$$\begin{aligned} \Phi^k_{ij} &= \Gamma^k_{ij} - \partial^k h_{ij}, \\ \overline{R}_{kij} &= \bar{\delta}_i\overline{N}_{jk} - \bar{\delta}_j\overline{N}_{ik} = -R^0_{kij} + \nabla_i h_{jk} - \\ &\quad -\nabla_j h_{ik} - h_{i\ell}\partial^\ell h_{jk} + h_{j\ell}\partial^\ell h_{ik}. \end{aligned} \tag{14}$$

Remark that the components Φ^k_{ij} define a linear M-conncetion on T^*M and the components $\overline{R}_{kij}$ define an M-tensor field of type $(0,3)$ on T^*M. It can be seen easily that we obtain, in fact, the same connection if we use the relations (12) between the operators δ_i and $\bar{\delta}_i$. If we introduce the components:

$$\overline{\nabla}_i\overline{R}_{jk\ell} = \bar{\delta}_i\overline{R}_{jk\ell} - \Phi^h_{ij}\overline{R}_{hk\ell} - \Phi^h_{ik}\overline{R}_{jh\ell} - \Phi^h_{i\ell}\overline{R}_{jkh}$$

which define an M-tensor field of type $(0,4)$ on T^*M, then the curvature tensor field K of the Levi–Civita connection $\widetilde{\nabla}$ is given by:

$$\begin{aligned} &K(\partial^i, \partial^j)\partial^k = K(\partial^i, \partial^j)\bar{\delta}_k = 0,\ K(\partial^i, \bar{\delta}_j)\partial^k = \partial^i\partial^k h_{jh}\partial^h, \\ &K(\bar{\delta}_i, \bar{\delta}_j)\partial^k = \partial^k\overline{R}_{hij}\partial^h,\ K(\partial^i, \bar{\delta}_j)\bar{\delta}_k = -\partial^i\partial^h h_{jk}\bar{\delta}_h + \partial^i\overline{R}_{jkh}\partial^h, \\ &K(\bar{\delta}_i, \bar{\delta}_j)\bar{\delta}_k = -\partial^h\overline{R}_{kij}\bar{\delta}_h + (\overline{\nabla}_i\overline{R}_{jkh} - \overline{\nabla}_j\overline{R}_{ikh})\partial^h \end{aligned} \tag{15}$$

By using the relation (12) between the operators δ_i and $\bar{\delta}_i$ it can be seen easily that we have the same tensor field as that given by the formulas (10).

Now, we shall study the conditions under which the curvature tensor field K of the connection $\widetilde{\nabla}$ vanishes, i.e. the conditions under which

$$K = 0.$$

By using the relations (15) we see that the M-tensor field deffined by the components h_{ij} and the Riemannian metric g on M must satisfy the relations

$$\partial^k \partial^\ell h_{ij} = 0,\ \partial^h \overline{R}_{kij} = 0,\ \overline{\nabla}_i \overline{R}_{jkh} - \overline{\nabla}_j \overline{R}_{ikh} = 0. \tag{16}$$

From the first condition (16) it follows that the components h_{ij} are given as:

$$h_{ij} = c_{ij} + b^k_{ij} p_k \tag{17}$$

where the components c_{ij}, b^k_{ij} define tensor fields on the base manifold M, thought of as M-tensor fields on T^*M. Remark that, due to the symmetry property $h_{ij} = h_{ji}$, we have $c_{ij} = c_{ji}$, $b^k_{ij} = b^k_{ji}$.

Next, from the second condition (16) we get, by using the second expression (14) of $\overline{R}_{kij}$ the following condition:

$$\nabla_i b^h_{jk} - \nabla_j b^h_{ik} = R^h_{kij} + b^h_{i\ell} b^\ell_{jk} - b^h_{j\ell} b^\ell_{ik}. \tag{18}$$

Finally, from the third condition (16) or, equivalently, from the vanishing of the coefficient of ∂^h in the expression (10) of $K(\delta_i, \delta_j)\delta_k$, we obtain

$$\begin{aligned}
&\nabla_i(\nabla_k c_{jh} - \nabla_h c_{jk} + c_{h\ell} b^\ell_{jk} - c_{k\ell} b^\ell_{jh}) - \\
&-\nabla_j(\nabla_k c_{ih} - \nabla_h c_{ik} + c_{h\ell} b^\ell_{ik} - c_{k\ell} b^\ell_{ih}) + \\
&+b^\ell_{ih}(\nabla_k c_{j\ell} - \nabla_\ell c_{jk} + c_{\ell a} b^a_{jk} - c_{ka} b^a_{j\ell}) - \\
&-b^\ell_{jh}(\nabla_k c_{i\ell} - \nabla_\ell c_{ik} + c_{\ell a} b^a_{ik} - c_{ka} b^a_{i\ell}) - \\
&-b^\ell_{ik}(\nabla_h c_{j\ell} - \nabla_\ell c_{jh} + c_{\ell a} b^a_{jh} - c_{ha} b^a_{j\ell}) + \\
&+b^\ell_{jk}(\nabla_h c_{i\ell} - \nabla_\ell c_{ih} + c_{\ell a} b^a_{ih} - c_{ha} b^a_{i\ell}) = 0.
\end{aligned} \tag{19}$$

To obtain this relation we have used the conditions (18) fulfilled by the components b_{ij}^k and we should remark that, due to same conditions, the coefficients of the (co–)tangential coordinates vanished identically.

Hence, we may state:

Theorem 3. *The connections* $\widetilde{\nabla}$ *is flat if and only if the* M*–tensor field defined by the components* h_{ij} *is given by* (17) *and the tensor fields defined by* b_{ij}^k, c_{ij} *satisfy the conditions* (18), (19).

Now we shall study the integrability conditions for the relations (18) fulfilled by the tensor field defined by the components b_{ij}^k on M, under the condition $K = 0$, thought of as a partial differential system, as well as of the relations:

$$\nabla_j c_{k\ell} - \nabla_k c_{j\ell} = c_{jh} b_{k\ell}^h - c_{kh} b_{j\ell}^h \tag{20}$$

which assures the fulfillement of the condition (19).

The integrability conditions are expressed by using some Ricci commutation formulas for the tensor fields on M defined by the components b_{ij}^k and c_{ij} and the relations (18), (20). Due to the specific aspect of the relations (18), (20), similar to the expression of the exterior differential of a 1-form, the integrability conditions are expressed by covariant differentiation of the relations (18), (20) with respect to the connection ∇, then summing up after appropriate cyclic permutations and using the relations (18), (20) themselves. After a quite straightforward but long computation we find that the integrability conditions for the relations (18), (20) are identically fulfilled.

2. Harmonic sections of T^*M defined by 1-forms. Let η be a 1-form on M. Then η may be thought of as a map $\phi_\eta : M \longrightarrow T^*M$ such that $\pi \circ \phi_\eta = 1_M$. Using the local coordinates on M and T^*M, we have that ϕ_η is defined by

$$q^i = x^i, \ p_i = \eta_i(x); \ i = 1, ..., n.$$

Consider the pseudo–Riemannian metric G on T^*M defined in section 1 by the Levi–Civita connection of the Riemannian metric g on M and the arbitrary symmetric M-tensor field h of type $(0, 2)$ on T^*M. We shall study the properties of ϕ_η related to the Riemannian metric g on M and the pseudo–Riemannian metric G on T^*M.

Proposition 4. *The* 1*-form* η *defines an isometric immersion* ϕ_η *if and only if*

$$\nabla_i \eta_j + \nabla_j \eta_i = g_{ij} - 2h_{ij}(x, \eta). \tag{21}$$

Proof. Use the local coordinate expression (6) of G. Then

$$\phi_\eta^*(dq^i) = dx^i, \;\; \phi_\eta^*(dp_i - \Gamma_{ij}^k p_k dq^j) = \nabla_j \eta_i dx^j$$

and

$$\phi_\eta^* G = (\nabla_i \eta_j + \nabla_j \eta_i + 2h_{ij}(x, \eta)) dx^i dx^j.$$

Thus the condition $\phi_\eta^* G = g$ is equivalent to (21).

From (21) we get

$$\nabla_k \nabla_i \eta_j + \nabla_k \nabla_j \eta_i = -2\nabla_k h_{ij}(x, \eta) - 2\nabla_k \eta_\ell \partial^\ell h_{ij}. \tag{22}$$

Next, from (22) we get by a straightforward computation, by using cyclic permutations of i, j, k, the following relation

$$\begin{aligned} \nabla_i \nabla_j \eta_k \;&= \eta_\ell R_{ijk}^\ell + \nabla_k h_{ij} - \nabla_i h_{jk} - \nabla_j h_{ki} + \\ &+ \nabla_k \eta_\ell \partial^\ell h_{ij} - \nabla_i \eta_\ell \partial^\ell h_{jk} - \nabla_j \eta_\ell \partial^\ell h_{ki} \end{aligned} \tag{23}$$

satisfied by η in the case of an isometric immersion $\phi_\eta : M \longrightarrow T^*M$.

Concerning the condition of complete integrability of the partial differential system defined by (21) with the consequence (23) we get the following results. First of all, from the Ricci identity

$$\nabla_i \nabla_j \nabla_k \eta_\ell - \nabla_j \nabla_i \nabla_k \eta_\ell = -\nabla_h \eta_\ell R_{kij}^h - \nabla_k \eta_h R_{\ell ij}^h. \tag{24}$$

by taking into account that the sum of terms which contain $\nabla \eta \nabla \eta$ must be zero and assuming that $n = \dim M > 2$, we get

$$h_{ij}(x, \eta) = c_{ij} + b_{ij}^k \eta_k, \tag{25}$$

where the components c_{ij}, b_{ij}^k define tensor fields on the base manifold M, thought of as M-tensor fields on T^*M. Remark that, due to the symmetry property $h_{ij} = h_{ji}$, we have $c_{ij} = c_{ji}$, $b_{ij}^k = b_{ji}^k$. Next, by using (25) in (23) we get

$$\begin{aligned} \nabla_i \nabla_j \eta_k \;&= \eta_\ell R_{ijk}^\ell + \nabla_k c_{ij} - \nabla_i c_{jk} - \nabla_j c_{ik} + \nabla_k b_{ij}^h \eta_h - \\ &- \nabla_i b_{jk}^h \eta_h - \nabla_j b_{ik}^h \eta_h + \nabla_k \eta_h b_{ij}^h - \nabla_i \eta_h b_{jk}^h - \nabla_j \eta_h b_{ik}^h. \end{aligned} \tag{26}$$

Then, replacing (26) in (24), we get by a straightforward computation that the condition of complete integrability of the partial differential system defined by (21) with the consequence (23) is given by

$$\begin{aligned}
&(\nabla_h\eta_i-\nabla_i\eta_h)R^h_{jk\ell}-(\nabla_h\eta_j-\nabla_j\eta_h)R^h_{ik\ell}+(\nabla_h\eta_k-\nabla_k\eta_h)R^h_{\ell ij}-\\
&-(\nabla_h\eta_\ell-\nabla_\ell\eta_h)R^h_{kij}=2\eta^h\nabla_hR_{ijk\ell}+2R_{ijk\ell}-2c_{ih}R^h_{jk\ell}+2c_{jh}R^h_{ik\ell}+\\
&+2\nabla_i(\nabla_\ell c_{jk}-\nabla_k c_{j\ell})-2\nabla_j(\nabla_\ell c_{ik}-\nabla_k c_{i\ell})-2\eta_s b^s_{ih}R^h_{jk\ell}+\\
&+2\eta_s b^s_{jh}R^h_{ik\ell}+2\nabla_i(\nabla_\ell b^h_{jk}-\nabla_k b^h_{j\ell})\eta_h-2\nabla_j(\nabla_\ell b^h_{ik}-\nabla_k b^h_{i\ell})\eta_h+\\
&+2(\nabla_\ell b^h_{jk}-\nabla_k b^h_{j\ell})\nabla_i\eta_h-2(\nabla_\ell b^h_{ik}-\nabla_k b^h_{i\ell})\nabla_j\eta_h+\\
&+2(\nabla_i b^h_{jk}-\nabla_j b^h_{ik})\nabla_\ell\eta_h-2(\nabla_i b^h_{\ell j}-\nabla_j b^h_{\ell i})\nabla_k\eta_h+\\
&+2b^h_{jk}\nabla_i\nabla_\ell\eta_h-2b^h_{ik}\nabla_j\nabla_\ell\eta_h-2b^h_{j\ell}\nabla_i\nabla_k\eta_h+2b^h_{i\ell}\nabla_j\nabla_k\eta_h.
\end{aligned}\tag{27}$$

In order to obtain some classes of manifolds for which the condition of complete integrability of the partial differential system defined by (21) with the consequence (23) are satisfied, we assume that h_{ij} is of the form (25) and the tensor fields defined by the components c_{ij} and b^k_{ij} satisfy the conditions (18) and (20) from section 1. Then, from Theorem 3 it follows that the connection $\widetilde{\nabla}$ on T^*M is flat.

Using (27) we get by a straightforward computation.

Proposition 5. *Under the above assumptions, the condition of complete integrability of the partial differential system* (21) *with the consequence* (23) *is equivalent to*

$$R_{ijk\ell}=g_{hs}(b^h_{jk}b^s_{i\ell}-b^h_{ik}b^s_{j\ell}).\tag{28}$$

Now we shall determine the second quadratic form $\nabla d\phi_\eta$ of the mapping $\phi_\eta:(M,g)\longrightarrow(T^*M,G)$. Since ϕ_η is an immersion we may use the formula (see [5], [16], [17])

$$(\nabla d\phi_\eta)(X,Y)=\widetilde{\nabla}_{\phi_{\eta^*}X}\phi_{\eta^*}T-\phi_{\eta^*}(\nabla_XY);\ X,Y\in\Gamma(M).$$

We have

$$\phi_{\eta^*}\left(\frac{\partial}{\partial x^i}\right)=\frac{\delta}{\delta q^i}+\nabla_i\eta_j\frac{\partial}{\partial p_j},$$

so we get the following local coordinate expression of $\nabla d\phi_\eta$

$$(29)\qquad \begin{aligned}(\nabla d\phi_\eta)\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) &= -\partial^k h_{ij}\delta_k + [\nabla_i\nabla_j\eta_k - R^h_{ijk}\eta_h + \\ &+\nabla_i h_{jk} + \nabla_j h_{ik} - \nabla_k h_{ij} + \nabla_j\eta_\ell\partial^\ell h_{ik} + \nabla_i\eta_\ell\partial^\ell h_{jk} + 2h_{k\ell}\partial^\ell h_{ij}]\partial^k.\end{aligned}$$

We say that the map $\phi_\eta : M\longrightarrow T^*M$ is *H-geodesic* if we have

$$\nabla d\phi_\eta \in L_2(TM;VT^*M).$$

From (29) and the above definition it follows that ϕ_η is *H-geodesic* if and only if we have

$$(30)\qquad \partial^k h_{ij}(x,\eta(x)) = 0.$$

Remarks. (i) If the components h_{ij} define a symmetric tensor field on the base manifold M thought of as an M-tensor field on T^*M then the relations (30) are identically satisfied. Therefore, in the case where h_{ij} are independent of the cotangential coordinates p_k then the map ϕ_η is H-geodesic for any 1-form η on M. This case has been completely studied by the first author in [10].

(ii) If the components h_{ij} depend on the cotangential coordinates p_k, we have that ϕ_η is H-geodesic if and only if the components η_ℓ of the 1-form η verify the relations (30).

By using (30) and comparing (29) with (23) we obtain

Theorem 6. *If $\phi_\eta : M\longrightarrow T^*M$ is an H-geodesic isometric immersion then it is totally geodesic immersion.*

Remark that in the case where h_{ij} are independent of the cotangential coordinates p_k, Theorem 6 becomes Theorem 4 from [10].

In order to obtain some examples of H-geodesic isometric immersions ϕ_η in the essential case where the components h_{ij} depend on the cotangential coordinates p_k, we take

$$h_{ij}(x,p) = c_{ij}(x) + b^k_{ij}(x)p_k - \frac{1}{2}K^{kh}_{ij}(x)p_kp_h$$

where the components $c_{ij}, b^k_{ij}, K^{kh}_{ij}$ define tensor fields on M thought of as M-tensor fields on T^*M. Due to the symmetry property $h_{ij} = h_{ji}$, we have

$c_{ij} = c_{ji}$, $b^k_{ij} = b^k_{ji}$, and $K^{kh}_{ij} = K^{kh}_{ji} = K^{hk}_{ij}$. From the conditions (30) we get

$$b^k_{ij} = K^{kh}_{ij}\eta_h.$$

Therefore, if the components $h_{ij}(x,p)$ are expressed by

$$h_{ij}(x,p) = c_{ij}(x) + K^{kh}_{ij}(x)\eta_h(x)p_k - \frac{1}{2}K^{kh}_{ij}(x)p_k p_h \tag{31}$$

then the map ϕ_η is H-geodesic. Assuming that $h_{ij}(x,p)$ are defined by (31), the condition (21) which must be fulfilled by ϕ_η in order to be an isometric immersion becomes

$$2c_{ij} + K^{kh}_{ij}\eta_k\eta_h = g_{ij} - \nabla_i\eta_j - \nabla_j\eta_i. \tag{32}$$

Hence, if we take the components c_{ij} defined by

$$c_{ij} = \frac{1}{2}(g_{ij} - \nabla_i\eta_j - \nabla_j\eta_i - K^{kh}_{ij}\eta_k\eta_h), \tag{33}$$

where K^{kh}_{ij} satisfy the conditions $K^{kh}_{ij} = K^{kh}_{ji} = K^{hk}_{ji}$ we have that (32) are identically satisfied.

Hence we have

Proposition 7. *If the components $h_{ij}(x,p)$ are given by*

$$h_{ij} = \frac{1}{2}(g_{ij} - \nabla_i\eta_j - \nabla_j\eta_i - K^{kh}_{ij}\eta_k\eta_h) + K^{kh}_{ij}\eta_h p_k - \frac{1}{2}K^{kh}_{ij}p_k p_h,$$

where K^{kh}_{ij} define a tensor field of type (2.2) on M thought of as an M-tensor field on T^*M satisfying the conditions $K^{kh}_{ij} = K^{kh}_{ji} = K^{hk}_{ij}$, then $\phi_\eta : (M,g)\longrightarrow(T^*M,G)$ is an H-geodesic isometric immersion, therefore ϕ_η is a totally geodesic immersion.

Remark. If we take in Proposition 7 either $K^{kh}_{ij} = \delta^k_i\delta^h_j + \delta^h_i\delta^k_j$ or $K^{kh}_{ij} = \delta^k_i\delta^h_j + \delta^h_i\delta^k_j - g_{ij}g^{kh}$ we get two concrete examples of $h_{ij}(x,p)$ such that ϕ_η is a totally geodesic immersion.

Corollary. *There always exists a pseudo–Riemannian metric G of type (6) on T^*M such that ϕ_η is a totally geodesic immesion.*

The tension $\tau(\phi_\eta)$ of ϕ_η is defined as being the trace of $\nabla d\phi_\eta$ and the map ϕ_η is harmonic if $\tau(\phi_\eta) = 0$ (see [5]). Assume that ϕ_η is an H-geodesic map. Then by using (29) and (30) we get that the local coordinate expression of $\tau(\phi_\eta)$ is

$$(34) \qquad \tau(\phi_\eta) = [g^{ij}\nabla_i\nabla_j\eta_k + R^i_k\eta_i + 2\nabla_i h^i_k - \nabla_k(tr\ h)]\frac{\partial}{\partial p_k},$$

where R_{jk} is the local coordinate expression of the Ricci tensor field $\rho(R)$ of ∇, $R^i_k = g^{ij}R_{jk}$ and $tr\ h = h_{ij}g^{ij}$ is the trace of $h(x,\eta)$.

It is known (see [18], [10]) that the 1-form η is geodesic if the vector field $\eta^{\#}$ defined by the components $\eta^i = g^{ij}\eta_j$ is geodesic, i.e.

$$(35) \qquad g^{ij}\nabla_i\nabla_j\eta^k + R^k_i\eta^i = 0.$$

A symmetric tensor field c of type $(0,2)$ on M is harmonic with respect to the (pseudo–) Riemannian metric g if $\delta c + \frac{1}{2}d(tr\ c) = 0$ (see [18], [2]) i.e.

$$(36) \qquad -\nabla_i c^i_k + \frac{1}{2}\nabla(tr\ c) = 0.$$

Now we can state

Theorem 8. *Let $\phi_\eta : (M,G) \longrightarrow (T^*M, G)$ be an H-geodesic map. Then the following three assertions hold*

(i) *If η is a geodesic 1-form on M and the symmetric tensor field h of type $(0,2)$ on M defined by the components $h_{ij}(x,\eta)$ is harmonic then the map ϕ_η is harmonic.*

(ii) *If (M,J,g) is a Kähler manifold and η is a 1-form on M such that $\eta^{\#}$ is a holomorphic vector field on M then ϕ_η is harmonic if and only if the symmetric tensor field h of type $(0,2)$ on M defined by the components $h_{ij}(x,\eta)$ is a harmonic tensor field.*

(iii) *If the M-tensor field h is defined by the components $h_{ij}(x,p)$ given in Proposition 7 then ϕ_η is harmonic.*

Proof. (i) is obvious from the formula (34) and the definitions (35), (35) of the geodesic 1-form and the harmonic symmetric tensor field of type $(0,2)$, by taking into account that ϕ_η is assumed to be H-geodesic. The result of (ii) is obtained easily since it is known, (see [18], [10]), that a holomorphic vector field on a Kähler manifold is geodesic. The assertion (iii) is obvious from Proposition 7.

Now we study the harmonicity of the tensor field $\phi_\eta^* G$ with respect to g. If the restriction of G to $\phi_\eta(M) \subset T^*M$ is nondegenerate then the harmonicity of $\phi_\eta^* G$ with respect to g is equivalent to the harmonicity of the map $\phi_\eta : M \longrightarrow \phi_\eta(M)$ and this happens if and only if $\tau(\phi_\eta)$ is orthogonal to $\phi_\eta(M)$. We get by a straightforward computation

$$G\left(\tau(\phi_\eta), \phi_{\eta^*}\left(\frac{\partial}{\partial x^k}\right)\right) = G\left(\tau(\phi_\eta), \frac{\delta}{\delta q^k}\right) =$$
$$= \tau(\phi_\eta)_k - 2g^{ij}\partial^\ell h_{ij}(x, \eta(x)) h_{\ell k}(x, \eta(x)).$$

Thus we may state

Theorem 9. *Let $\phi_\eta : (M, g) \longrightarrow (T^*M, G)$ be an H-geodesic map. Then $\phi_\eta^* G$ is harmonic if and only if ϕ_η is harmonic.*

REFERENCES

1. CADDEO, R., SANINI, A. – *Metriche armoniche indotte da campi vettoriali*, Rend. Sem. Fac. Sci. Univ. Cagliari, 57(2), (1987), 121–130.
2. CHEN, B.Y., NAGANO, T. – *Harmonic metrics, harmonic tensors and Gauss maps*, J. Math. Soc. Japan, (36)(2), (1984), 295–313.
3. CRAMPIN, M. – *On the differential geometry of the Euler–Lagrange equations and the inverse problem of Lagrangian dynamics*, J. Physics, A. Math. Gen. 14(1981), 2567–2575.
4. CRAMPIN, M., PIRANI, F.A. – *Applicable Differential Geometry*, Cambridge University Press, 1986, ii, 394 pp.
5. EELLS, J., LEMAIRE, L. – *Selected topics in harmonic maps*, Conf. Board of the Math. Sci. A.M.S., 50(1983), 85 pp.
6. EELLS, J., RATTO, A. – *Harmonic maps and minimal immersions with symmetries. Method of ordinary differential equations applied to elliptic variational problems*, Ann. Math. Studies 130, Princeton University Press, 1993.
7. EELLS, J., SAMPSON, J.H. – *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86, (1964), 106–160.
8. HANGAN, T. – *Problemi variazionali connessi con campi tensoriali*, Rend. Sem. Fac. Sci. Univ. Cagliari, 52(2), (1982), 23–33.

9. MOK, K.R., PATTERSON, E.M., WONG, Y.C. – *Structure of symmetric tensors of type $(0, 2)$ and of tensors of type $(1, 1)$ on the tangent bundle*, Trans. Am. Math. Soc., 234(1977), 253–278.

10. OPROIU, V. – *On the harmonic sections of cotangent bundles*, Rend. Fac. Sci. Cagliari, 59(1989), 177–184.

11. OPROIU, V. – *Some aspectes from the geometry of the cotangent bundle*, Conference on the Differential Geometry, University "Politehnica", Bucharest, Romania, September 23–28, 1996, An. Univ. Timişoara, 34(1996), 117–134.

12. OPROIU, V., PAPAGHIUC, N. – *A pseudo–Riemannian structure on the cotangent bundle*, An. St. Univ. "Al.I. Cuza", Mat. 36(1990), 265–276.

13. OPROIU, V., PAPAGHIUC, N. – *Locally symmetric contangent bundles,* Matematicki Vesnik, Beograd, 42, (1990), 221–232.

14. PATTERSON, E.M., WALKER, A.G. – *Riemann extensions*, Quarterly J. Math. (2), (3), (1952), 19–28.

15. PIU, M.P. – *Campi di vettori ed applicazioni armoniche* Rend. Sem. Fac. Sci. Univ. Cagliari, 52(1), (1982), 85–94.

16. SANINI, A. – *Applicazioni armoniche tra fibratti tangenti unitari*, Rend. Sem. Mat. Univ. Polit. Torino, 43(1), (1985), 159–170.

17. SANINI, A. – *Applicazioni armoniche tra fibratti tangenti di varietà riemanniane*, Boll. U.M.I., 6(2A), (1983), 55–63.

18. YANO, K., NAGANO, T. – *On geodesic vector fiels in a compact orientable Riemannian space*, Comment. Math. Helv., 35(1), (1961), 55–64.

19. YANO, K., ISHIHARA, S. – *Tangent and Cotangent Bundles*, M. Dekker, New York, 1973.

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