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ON NEARLY SASAKIAN MANIFOLD

BY

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Introduction. In 1926, LEVY [3] proved that a second order symmetric parallel nonsingular tensor on a space of constant curvature is a constant multiple of the metric tensor. Recently, SHARMA [4] generalized Levy's result and also studied a second order parallel tensor on a Kähler space of constant holomorphic sectional curvature. In this paper, we have shown that in a nearly Sasakian manifold, a second order symmetric parallel tensor is a constant multiple of the associated metric tensor. In the last section we consider nearly Sasakian manifolds which are hypersurfaces of a Riemann manifold of constant curvature 1 and a necessary and sufficient condition for a nearly Sasakian manifold to be minimal has been obtained.

Let (M, g) be an n-dimensional differentiable manifold ($n = 2m + 1$, $m > 1$) with almost contact metric structure (ϕ, ξ, η, g). If in such a manifold the following relation holds

$$(\nabla_x \phi)y + (\nabla_y \phi)x = 2g(x, y)\xi - \eta(x)y - \eta(y)x \tag{1}$$

then the manifold if said to be nearly Sasakian [1].

1. Preliminaries. It is known [1] that in a nearly Sasakian manifold the following relations hold:

(1.1)	$\phi\xi = 0$	(1.2)	$\eta(\xi) = 1$
(1.3)	$\phi^2 x = -x + \eta(x)\xi$	(1.4)	$\nabla_x \xi = -\phi x$
(1.5)	$g(\phi x, \phi y) = g(x, y) - \eta(x)\eta(y)$	(1.6)	$g(x, \phi y) + g(\phi x, y) = 0$
(1.7)	$s(x, \xi) = (n - 1)\eta(x)$	(1.8)	$(\nabla_x \eta)\xi = 0$
(1.9)	$(\nabla_x \eta)x = 0$	(1.10)	$\eta(\phi x) = 0$
(1.11)	$R(x, \xi)\xi = x - \eta(x)\xi$	(1.12)	$R(\xi, x)y = g(x, y)\xi - \eta(y)x$
(1.13)	$S(\phi x, \phi y) = s(x, y) - (n-1)\eta(x)\eta(y)$		

The above results will be used in the next section.

2. Nearly Sasakian manifolds with second order parallel tensor. Let α denote a $(0,2)$–tensor field on a nearly Sasakian manifold M such that $\nabla\alpha = 0$. Then it follows that

$$\alpha(R(w,x)y,z) + \alpha(y,R(w,x)z) = 0 \tag{2.1}$$

for arbitrary vector fields x, y, z, w on M.

Taking $y = z = w = \xi$ in (2.1), we get

$$\alpha(R(\xi,x)\xi,\xi) + \alpha(\xi,R(\xi,x)\xi) = 0 \tag{2.2}$$

Using (1.11) in (2.2), we have

$$2g(x,\xi)\alpha(\xi,\xi) - \alpha(x,\xi) - \alpha(\xi,x) = 0 \tag{2.3}$$

Differentiating (2.3) along y we get

$$\begin{aligned}\{g(\nabla_y x,\xi) + g(x,\nabla_y\xi)\}\alpha(\xi,\xi) + 2g(x,\xi)\alpha(\nabla_y\xi,\xi) - \\ -\alpha(\nabla_y x,\xi) - \alpha(\nabla_y\xi,x) = 0\end{aligned} \tag{2.4}$$

Putting $x = \nabla_y x$ in (2.3), we find

$$g(\nabla_y x,\xi)\alpha(\xi,\xi) - \alpha(\nabla_y x,\xi) = 0 \tag{2.5}$$

From (2.4) and (2.5) we get, by using (1.4)

$$g(x,\phi y)\alpha(\xi,\xi) + 2g(x,\xi)\alpha(\phi y,\xi) - \alpha(x,\phi y) = 0 \tag{2.6}$$

Replacing x by ϕy in (2.3), and using (1.10), one gets

$$\alpha(\phi y,\xi) = 0 \tag{2.7}$$

From (2.6) and (2.7), we obtain

$$g(x,\phi y)\alpha(\xi,\xi) - \alpha(x,\phi y) = 0 \tag{2.8}$$

Replacing y by ϕy and using (1.3) and (2.3), we obtain

$$\alpha(x,y) = \alpha(\xi,\xi)g(x,y) \tag{2.9}$$

The fact that $\alpha(\xi,\xi)$ is a constant can be checked by differentiating it along any vector field on M. Thus we state

Theorem 2.1. *On a nearly Sasakian manifold a second order symmetric parallel tensor is a constant multiple of the associated metric tensor.*

3. Nearly Sasakian manifolds which are hypersurfaces of a Riemannian manifold of constant curvature. Let M be a nearly Sasakian manifold of dimension n which is isometrically immersed in a Riemannian manifold N of dimension $(n+1)$ of constant curvature 1. Then we have the Gauss and Codazzi equations [2]

$$R(x,y) = x\wedge y + Ax\wedge Ay \tag{3.1}$$

$$R(x,y)z = g(y,z)x - g(x,z)y + g(Ay,z)Ax - g(Ax,z)Ay \tag{3.2}$$

$$(\nabla_x A)y = (\nabla_y A)x \tag{3.3}$$

where A is a $(1-1)$ tensor field associated with the second fundamental form B by $B(x,y) = g(x,Ay)$. Here A is symmetric with respect to g and when the trace of A vanishes, M or the immersion is called minimal. The rank of A is called the type number of immersion.

Since the Ricci Curvature tensor s is given by

$$S(x,y) \longrightarrow \text{trace}\,[u \longrightarrow R(x,u)y]$$

by (3.2) we have

$$s(x,y) = (n-1)g(x,y) + (\text{trace}\,A)g(Ax,y) - g(AAx,y) \tag{3.4}$$

Replacing x and y by ϕx and ϕy in (3.4), we have

$$s(\phi x,\phi y) = (n-1)g(\phi x,\phi y) + \theta g(A\phi x,\phi y) - g(AA\phi x,\phi y) \tag{3.5}$$

where θ is the trace of A.

Again

$$g(A\phi x,\phi y) = -g(\phi A\phi x,y)$$

$$g(AA\phi x,\phi y) = -g(\phi AA\phi x,y)$$

Using (1.5) and (1.13), it follows from (3.5)

$$s(x,y) = (n-1)g(x,y) - \theta g(\phi A\phi x, y) + g(\phi AA\phi x, y) \tag{3.6}$$

Then (1.5), (3.4) and (3.6) imply

$$\theta A - AA + \theta\phi A\phi - \phi AA\phi = 0 \tag{3.7}$$

From (3.4), the scalar curvature $\gamma =$ trace s is given by

$$\gamma = n(n-1) + \theta^2 - \text{trace } AA$$

Now if trace of A vanishes, then from (3.7) we get

$$\phi AA\phi + AA = 0 \tag{3.8}$$

Again if (3.8) holds then we have from (3.7), $\theta = 0$. Hence, we can state the following theorems:

Theorem 3.1. *A necessary and sufficient condition for a nearly Sasakian manifold to be minimal is that* (3.8) *holds.*

Theorem 3.2. *If a nearly Sasakian manifold is isometrically immersed in a Riemann manifold of constant curvature* 1 *then the scalar curvature is given by*

$$\gamma = n(n-1) + \theta^2 - \text{trace } AA$$

where θ *is the* trace *of* A.

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