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FINSLERIAN SUBSPACES SUBJECTED TO β-CHANGES

BY

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Dedicated to Prof. Dr. Radu MIRON on the occasion of his 75th birthday

1. Preliminaries. Let M^n be an n-dimensional differentiable manifold and $F^n = (M^n, L)$ be a Finsler space equipped with a fundamental function $L(x, y)(y^i = \dot{x}^i)$ on M^n. For a differential one-form $\beta(x, dx) = b_i(x)dx^i$ on M^n, we shall deal with a change of Finsler metric which is defined by

$$L(x, y) \longrightarrow \bar{L}(x, y) = f\big(L(x, y), \beta(x, y)\big),$$

where $f(L, \beta)$ is a positively homogeneous function of L and β of degree one. This is called a β-change of the metric. We have specially interesting examples of β-change of the metric, for instance,

$$\bar{L}(x, y) = L(x, y) + \beta(x, y), \tag{1}$$

$$\bar{L}(x, y) = L^2(x, y)/\beta(x, y). \tag{2}$$

The change (1) is called a Randers change and (2) a Kropina change. For a β-change $L \longrightarrow \bar{L} = f(L, \beta)$, if L is a Riemannian metric $\alpha(x, dx) = \big(a_{ij}(x)dx^i dx^j\big)^{1/2}$, then $\bar{L} = f(L, \beta)$ becomes (α, β)-metric. In particular $\bar{L} = \alpha + \beta$ is a Randers metric and $\bar{L} = \alpha^2/\beta$ a Kropina metric.

In this section we shall use the following notations :

$$f_1(= f_L) := \partial f/\partial L, \qquad f_2(= f_\beta) := \partial f/\partial \beta,$$

$$f_{11}(= f_{LL}) := \partial^2 f/\partial L \partial L, \qquad f_{12}(= f_{L\beta}) := \partial^2 f/\partial L \partial \beta,$$

etcetera.

Since $\bar{L} = f$ is a positively homogeneous function of L and β of degree one, we have

$$f = f_1 L + f_2 \beta, \quad L f_{12} + \beta f_{22} = 0, \quad L f_{11} + \beta f_{12} = 0. \tag{1.1}$$

For the later use we put

$$p = f f_1 / L, \qquad q = f f_2, \qquad q_0 = f f_{22}. \tag{1.2}$$

Paying attention to $l_i = \dot{\partial}_i L$, from (1.1) we have

$$\bar{l}_i = f_1 l_i + f_2 b_i. \tag{1.3}$$

Differentiating this by y^j, we have the angular metric tensor $\bar{h}_{ij} = \bar{L}\dot{\partial}_i \dot{\partial}_j \bar{L}$ on $\bar{F}^n$:

$$\bar{h}_{ij} = p h_{ij} + q_0 m_i m_j, \tag{1.4}$$

where the covariant vector m_i is defined by

$$m_i = b_i - \beta y_i / L^2. \tag{1.4) 1}$$

It is noted that m_i is a non-zero vector orthogonal to y^i. In fact $m_i = 0$ gives $L^2 b_i - \beta y_i = 0$. We differentiate this by y^j and get $\beta g_{ij} - 2L l_j b_i + b_j y_i = 0$, which leads to a contradiction $g_{ij} - l_i l_j = 0$.

Now, from (1.1), (1.3), (1.4) and (1.4) 1) the fundamental tensor $\bar{g}_{ij} = \dot{\partial}_i \dot{\partial}_j (\bar{L}^2/2)$ of $\bar{F}^n$ is given by

$$\bar{g}_{ij} = p g_{ij} + p_0 b_i b_j + p_{-1}(b_i y_j + b_j y_i) + p_{-2} y_i y_j, \tag{1.5}$$

where we put

$$p_0 = q_0 + {f_2}^2,$$

$$q_{-1} = f f_{12}/L, \qquad p_{-1} = q_{-1} + p f_2 / f, \tag{1.5) 1}$$

$$q_{-2} = f(f_{11} - f_1/L)/L^2, \qquad p_{-2} = q_{-2} + p^2/f^2.$$

The reciprocal tensor $\bar{g}^{ij}$ of $\bar{g}_{ij}$ can be written as

$$\bar{g}^{ij} = (1/p) g^{ij} - s_0 b^i b^j - s_{-1}(b^i y^j + b^j y^i) - s_{-2} y^i y^j, \tag{1.6}$$

where we put

$$b^i = g^{ij}b_j, \quad b^2 = g^{ij}b_ib_j, \quad \nu = b^2 - \beta^2/L,$$

$$s_0 = \bar{L}^2 q_0/\tau p L^2, \qquad s_{-1} = p_{-1}\bar{L}^2/p\tau L^2, \tag{1.6 1)}$$

$$s_{-2} = p_{-1}(\nu p L^2 - b^2\bar{L}^2)/\tau p\beta L^2, \quad \tau = \bar{L}^2(p + \nu q_0)/L^2.$$

From the homogeneity it follows that these quantities satisfy

$$\begin{aligned} &q_0\beta + q_{-1}L^2 = 0, && q_{-1}\beta + q_{-2}L^2 = -p, \\ &p_0\beta + p_{-1}L^2 = q, && q\beta + pL^2 = f^2, \\ &p_{-1}\beta + p_{-2}L^2 = 0, && s_0\beta + s_{-1}L^2 = q/\tau, \end{aligned} \tag{1.7}$$

$$s_{-1}b^2 + s_{-2}\beta = p_{-1}\nu/\tau.$$

As to the torsion tensor $\bar{C}_{ijk} = \dot{\partial}_k(\bar{g}_{ij}/2)$ of $\bar{F}^n$, from (1.5) and (1.7) we get

$$\bar{C}_{ijk} = pC_{ijk} + p_{-1}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j)/2 + p_{02}m_im_jm_k/2, \tag{1.8}$$

where we put $p_{02} = \partial p_0/\partial\beta$. Contracting this by $\bar{g}^{hk}$, we have

$$\bar{C}_i{}^h{}_j = C_i{}^h{}_j - V_i{}^h{}_j, \tag{1.9}$$

where we put

$$\begin{aligned} V_i{}^h{}_j = {} & Q^h(pC_{imj}b^m - p_{-1}m_im_j) - p_{-1}(h^h{}_im_j + h^h{}_jm_i)/2p \\ & -(m^h/p - \nu Q^h)(p_{02}m_im_j + p_{-1}h_{ij})/2, \end{aligned} \tag{1.9 1)}$$

$$Q^h = s_0b^h + s_{-1}y^h, \quad h^h{}_i = g^{hr}h_{ir}, \quad m^h = g^{hr}m_r.$$

We denote by the symbol $(\,|\,)$ the h-covariant differention with respect to the Cartan connection $C\Gamma$ and put

$$2E_{jk} = b_{j|k} + b_{k|j}, \qquad 2F_{jk} = b_{j|k} - b_{k|j}. \tag{1.10}$$

Now we deal with well-known function $G^i(x,y)$ which are $(2)p$- homogeneous in y^i and are written as $G^i = \gamma_j{}^i{}_k y^j y^k/2$, by putting

$$\gamma_j{}^i{}_k = g^{ir}(\partial_k g_{jr} + \partial_j g_{kr} - \partial_r g_{jk})/2.$$

Owing to (1.5) and (1.6), a straightforward calculation leads to

$$\bar{G}^i(x,y) := (\bar{\gamma}_j{}^i{}_k y^j y^k)/2 = G^i + D^i \tag{1.11}$$

where the vector D^i is given by

$$\begin{aligned} D^i &= (q/p)F^i{}_0 + (pE_{00} - 2qF_{r0}b^r)(s_{-1}y^i + s_0 b^i)/2, \\ F^i{}_j &= g^{ir}F_{rj}, \end{aligned} \tag{1.11 1)}$$

and the subscript 0 (excluding s_0) means the contraction by y^i.

We shall examine how the Cartan connection $C\Gamma$ changes by a β-change of the metric. Let $C\bar{\Gamma} = (\bar{F}_j{}^i{}_k, \bar{N}^i{}_j, \bar{C}_j{}^i{}_k)$ be the Cartan connection on the space $\bar{F}^n = (M^n, \bar{L})$. For coefficients $N^i{}_j = \dot{\partial}_j G^i$ of the non-linear connection, we differentiate (1.11) by y^i and get

$$\bar{N}^i{}_j = N^i{}_j + D^i{}_j, \tag{1.12}$$

where the tensor $D^i{}_j = \dot{\partial}_j D^i$ is given by

$$\begin{aligned} D^i{}_j &:= (1/p)A^i{}_j - Q^i A_{rj}b^r - qb_{0|j}(s_{-1}b^i + s_{-2}y^i), \\ A_{ij} &:= (1/2)E_{00}B_{ij} + qF_{ij} + F_{i0}Q_j - (pC_{ijm} + V_{ijm})D^m, \\ A^i{}_j &= g^{ir}A_{rj}, \quad V_{ijm} = g_{sj}V_i{}^s{}_m, \quad Q_i = p_{-1}y_i + p_0 b_i \\ B_{jk} &= (p_{-1}h_{jk} + p_{02}m_j m_k)/2. \end{aligned} \tag{1.12 1)}$$

Here, for the covariant vector Q_i it is noted that

$$Q_0 = q, \qquad \dot{\partial}_k Q_j = B_{jk}/2. \tag{1.13}$$

Let $B\bar{\Gamma} = (\bar{G}_j{}^i{}_k, \bar{N}^i{}_j, 0)$ be the Berwald connection on $\bar{F}^n$. Differentiating (1.12) by y^k, we have connection coefficients $\bar{G}_j{}^i{}_k = \dot{\partial}_k \bar{N}^i{}_j$ of $B\bar{\Gamma}$ which are given by

$$\bar{G}_j{}^i{}_k := G_j{}^i{}_k + B_j{}^i{}_k, \qquad B_j{}^i{}_k = \dot{\partial}_k D^i{}_j, \tag{1.14}$$

where $G_j{}^i{}_k$ are connection coefficients of $B\Gamma$ on F^n. Therefore from (1.6), (1.8), (1.9) and (1.12) we obtain connection coefficients $\bar{F}_j{}^i{}_k$ of the Cartan connection $C\bar{\Gamma}$ on $\bar{F}^n$ as follows :

$$\begin{aligned}\bar{F}_j{}^i{}_k &:= \bar{\gamma}_j{}^i{}_k + \bar{C}_{jkr}\bar{N}^r{}_m\bar{g}^{im} - \bar{C}_k{}^i{}_r\bar{N}^r{}_j - \bar{C}_j{}^i{}_r\bar{N}^r{}_k = \\ &= F_j{}^i{}_k + D_j{}^i{}_k,\end{aligned} \tag{1.15}$$

where we put

$$\begin{aligned}D_j{}^i{}_k = \quad & \{g^{is}/p - Q^i b^s - y^s(s_{-1}b^i + s_{-2}y^i)\} \\ & \cdot(B_{sj}b_{0|k} + B_{sk}b_{0|j} - B_{kj}b_{0|s} + F_{sj}Q_k + F_{sk}Q_j \\ & + E_{kj}Q_s + pC_{jkr}D^r{}_s + V_{jkr}D^r{}_s - pC_{skm}D^m{}_j \\ & - V_{sjm}D^m{}_k - pC_{sjm}D^m{}_k - V_{skm}D^m{}_j).\end{aligned} \tag{1.15) 1}$$

The tensor $D_j{}^i{}_k$, called the *difference tensor*, has the following properties :

$$(1)\ \ D_j{}^i{}_0 = B_j{}^i{}_0 = D^i{}_j, \qquad (2)\ \ D_0{}^i{}_0 = 2D^i. \tag{1.16}$$

Lemma 1.1 [4]. *If the covariant vector, the components $b_i(x)$ of which are coefficients of the one-form β, is parallel with respect to the Cartan connection $C\Gamma$ on F^n, then the difference tensor $D_j{}^i{}_k(= \bar{F}_j{}^i{}_k - F_j{}^i{}_k)$ vanishes.*

2. Subspaces given by a β- change. Let M^n be an n-dimensional smooth manifold and $F^n = (M^n, L)$ be an n-dimensional Finsler space equipped with a fundamental function $L(x, y)$ on M^n. Then the metric tensor $g_{ij}(x, y)$ and Cartan's C-tenson $C_{ijk}(x, y)$ are given by

$$g_{ij} = (\partial^2 L^2/\partial y^i \partial y^j)/2, \quad C_{ijk} = (\partial g_{ij}/\partial y^k)/2,$$

and we can introduce in F^n the Cartan connection $C\Gamma = (F_j{}^i{}_k, G^i{}_j, C_j{}^i{}_k)$.

An m-dimensional subspace M^m of the underlying smooth manifold M^n may be parametrically represented by the equation $x^i = x^i(u^\alpha)(i = 1, 2, \cdots, n)$, where u^α are Gaussian coordinates on M^m and Greek indices

run from 1 to m. Here, we shall assume that the matrix consisting of the pojection factors $B_\alpha{}^i = \partial x^i/\partial u^\alpha$ is of rank m. The following notations are also employed : $B_\alpha{}^i{}_\beta := \partial^2 x^i/\partial u^\alpha \partial u^\beta$, $B_{\bar{0}}{}^i{}_\beta := v^\alpha B_\alpha{}^i{}_\beta$, $B^{ij\cdots}_{\alpha\beta\ldots} := B^i_\alpha B_\beta{}^j \cdots$. If the supporting element y^i at a point (u^α) of M^m is assumed to be tangential to M^m, we may then write $y^i = B_\alpha{}^i(u)v^\alpha$, so that v^α is thought of as the supporting element of M^m at the point (u^α). Since the function $\underline{L}(u,v) := L(x(u), y(u,v))$ gives rise to a Finsler metric of M^m, we get an m-dimensional Finsler space $F^m = (M^m, \underline{L}(u,v))$.

At each point (u^α) of F^m, the unit normal vectors $N^i_a(u,v)$ are defined by

$$g_{ij}B_\alpha{}^i N^j_a = 0, \quad g_{ij}N^i_a N^j_b = \delta_{ab} \quad (a,b,\cdots = m+1,\cdots,n). \tag{2.1}$$

If $(B^\alpha{}_i, N^a_i)$ is the inverse matrix of $(B_\alpha{}^i, N^i_a)$, we have

$$B_\alpha{}^i B^\beta{}_i = \delta_\alpha{}^\beta, \quad B_\alpha{}^i N^a_i = 0, \quad N^i_a B^\alpha{}_i = 0, \quad N^i_a N^b_i = \delta^b_a, \tag{2.2}$$

and further

$$B_\alpha{}^i B^\alpha{}_j + N^i_a N^a_j = \delta^i{}_j. \tag{2.3}$$

Making use of the inverse matrix $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get $B^\alpha{}_i = g^{\alpha\beta}g_{ij}B_\beta{}^j$. By (2.1) and (2.3), we also have $\delta_{ab}N^b_i = g_{ij}N^j_a$.

For the induced Cartan connection $IC\Gamma = (F_\beta{}_\gamma{}^\alpha, G^\alpha_\beta, C_\beta{}_\gamma{}^\alpha)$ on F^m, the second fundamental h-tensor $H^a_{\alpha\beta}$ and the normal curvature vector H^a_α in a normal direction N^i_a are given by

$$\begin{aligned} H^a_{\alpha\beta} &:= N^a_i(B_{\alpha\beta}{}^i + F_j{}_k{}^i\ B_\alpha{}^j B_\beta{}^k) + M_\alpha{}^a{}_b H^b_\beta, \\ H^a_\alpha &:= N^a_i(B_{\bar{0}\alpha}{}^i + G^i{}_j B_\alpha{}^j), \end{aligned} \tag{2.4}$$

where $M_\alpha{}^a{}_b := C_i{}^j{}_k B_\alpha{}^i N^a_j N^k_b$ and $B_{\bar{0}\alpha}{}^i = B_{\beta\alpha}{}^i\ v^\beta$.

Contracting $H^a_{\beta\alpha}$ by v^β, we immediately get

$$H^a_{\bar{0}\alpha} := H^a_{\beta\alpha}v^\beta = H^a_\alpha. \tag{2.5}$$

Let's introduce in $\bar{F}^n = (M^n, \bar{L})$ the Cartan connection $C\bar{\Gamma} = (\bar{F}_j{}^i{}_k, \bar{G}^i_j, \bar{C}_j{}^i{}_k)$ from a β-change of the metric.

We now consider a Finslerian subspace $F^m = (M^m, \underline{L}(u,v))$ of F^n and another Finslerian subspace $\bar{F}^m = (M^m, \underline{\bar{L}}(u,v))$ of the $\bar{F}^n$ given by the β-change. Let N^i_a be unit normal vectors at each point of F^m, and $(B^\alpha{}_i, N^a_i)$ be the inverse matrix of $(B_\alpha{}^i, N^i_a)$. The functions $B_\alpha{}^i(u)$ may be considered as components of m linearly independent vectors tangent to F^m and they are invariant under the β-change. The unit normal vectors $\bar{N}^i_a(u,v)$ of $\bar{F}^m$ are uniquely determined by

$$\bar{g}_{ij} B_\alpha{}^i \bar{N}^j_a = 0, \qquad \bar{g}_{ij} \bar{N}^i_a \bar{N}^j_b = \delta_{ab}. \tag{2.6}$$

The fundamental tensor $\bar{g}_{ij} = (\partial^2 \bar{L}^2 / \partial y^i \partial y^j)/2$ of the Finsler space $\bar{F}^n$ given by (1.5), (1.5) 1.

Now contracting (2.1) by v^α, we immediately get

$$y_i N^i_a = 0. \tag{2.7}$$

Further contracting (1.5) by $N^i_a N^j_b$ and paying attention to (2.1), (2.6) and (2.7), we have

$$\bar{g}_{ij} N^i_a N^j_b = p\delta_{ab} + p_0 (b_i N^i_a)(b_j N^j_b). \tag{2.8}$$

Putting $a = b$, then we obtain

$$\bar{g}_{ij} (\pm N^i_a / \sqrt{p + p_0 (b_i N^i_a)^2})(\pm N^j_a / \sqrt{p + p_0 (b_i N^i_a)^2}) = 1, \tag{2.9}$$

provided $p + p_0 (b_i N^i_a)^2 > 0$.

Therefore we can put

$$\bar{N}^i_a = N^i_a / \sqrt{p + p_0 (b_i N^i_a)^2}, \tag{2.10}$$

where we have chosen the sign "+" in order to fix an orientation. On using (2.1) and (2.7), the first condition of (2.6) gives us

$$(b_i N^i_a)(p_0 b_j B_\alpha{}^j + p_{-1} y_j B_\alpha{}^j) = 0. \tag{2.11}$$

Now, assuming that $p_0 b_j B_\alpha{}^j + p_{-1} y_j B_\alpha{}^j = 0$ and contracting this by v^α, we find $p_0 \beta + p_{-1} L^2 = 0$. By (1.5) 1) this equation lead us to $f f_\beta = 0$, where we have used $L f_{L\beta} + \beta f_{\beta\beta} = 0$ and $L f_L + \beta f_\beta = f$ owing to the homogeneity

of f. Thus we have $f_\beta = 0$ because of $f \neq 0$. This fact means $\bar{L} = f(L)$ and contradicts the definition of a β-change of metric. Consequently (2.11) gives us

$$b_i N^i_a = 0. \tag{2.12}$$

Therefore (2.10) is rewritten as

$$\bar{N}^i_a = N^i_a/\sqrt{p} \qquad (p > 0), \tag{2.13}$$

and then it is clear $\bar{N}^i_a$ satistifies (2.6). Summarizing the above, we obtain

Theorem 2.1. *For a field of linear frame $(B^i_1, \cdots, B^i_m, N^i_{m+1}, \cdots, N^i_n)$ of F^n, there exists a field of linear frame $(B^i_1, \cdots, B^i_m, \bar{N}^i_{m+1}, \cdots, \bar{N}^i_n)$ of the $\bar{F}^n$ given by the β-change such that (2.6) is satisfied along $\bar{F}^m$, and then we get (2.12).*

The quantities $\bar{B}^\alpha{}_i$ are uniquely defined along $\bar{F}^m$ by

$$\bar{B}^\alpha{}_i = \bar{g}^{\alpha\beta}\bar{g}_{ij}B_\beta{}^j,$$

where $(\bar{g}^{\alpha\beta})$ is the inverse matrix of $(\bar{g}_{\alpha\beta})$.

Let $(\bar{B}^\alpha{}_i, \bar{N}^a_i)$ be the inverse matrix of $(B_\alpha{}^i, \bar{N}^i_a)$, and then we have

$$B_\alpha{}^i\bar{B}^\beta{}_i = \delta_\alpha{}^\beta,\ B_\alpha{}^i\bar{N}^a_i = 0,\ \bar{N}^i_a\bar{B}^\alpha{}_i = 0,\ \bar{N}^i_a\bar{N}^b_i = \delta^b_a, \tag{2.14}$$

and further

$$B_\alpha{}^i\bar{B}^\alpha{}_j + \bar{N}^i_a\bar{N}^a_j = \delta^i{}_j. \tag{2.15}$$

We also get $\delta_{ab}\bar{N}^b_i = \bar{g}_{ij}\bar{N}^j_a$, that is,

$$\bar{N}^a_i = \sqrt{p}N^a_i. \tag{2.16}$$

Now assuming that the covector field $b_i(x)$ is gradient, we have from (1.11)1)

$$N^a_i D^i = 0. \tag{2.17}$$

Differentiating (2.17) by y^j and contracting it by B^j_α, we get

$$N^a_i D^i_j B^j_\alpha = 0. \tag{2.18}$$

If each geodesic of F^m with respect to the induced metric is also a geodesic of F^n, then F^m is called totally geodesic. A totally geodesic subspace F^m is characterized by each $H^a_\alpha = 0$. ¿From (2.4) and (2.16) we have

$$\bar{H}^a_\alpha = \sqrt{p}(H^a_\alpha + N^a_i D^i_j B^j_\alpha). \tag{2.19}$$

Thus from (2.18) we obtain $\bar{H}^a_\alpha = \sqrt{p}H^a_\alpha$. Hence we have

Theorem 2.2 *Assume that the covector field $b_i(x)$ is gradient. Then the subspace F^m is totally geodesic, if and only if the subspace $\bar{F}^m$ is totally geodesic.*

From (2.4), (2.16) and Lemma 1.1, we have $\bar{H}^a_\alpha = \sqrt{p}H^a_\alpha$. Thus we obtain

Theorem 2.3 *Let $b_i(x)$ be parallel with respect to $C\Gamma$ on F^n. Then the subspace F^m is totally geodesic, if and only if the subspace $\bar{F}^m$ is totally geodesic.*

If each h-path of F^m with respect to the induced connection is also an h-path of F^n, then F^m is called totally h-autoparallel. A totally h-autoparallel subspace F^m is characterized by each $H^a_{\alpha\beta} = 0$. From (2.4), (2.5), (2.16) and Lemma 1.1, we obtain

Theorem 2.4. *Let $b_i(x)$ be parallel with respect to $C\Gamma$ on F^n. Then a subspace F^m is totally h-autoparallel, if and only if the subspace $\bar{F}^m$ is totally h-autoparallel.*

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