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CANCELLATION THEOREMS FOR HOLOMORPHIC VECTOR BUNDLES ON INFINITE-DIMENSIONAL COMPLEX SPACES

BY

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Abstract. Let X be an infinite-dimensional complex Banach manifold and E, F, G holomorphic vector bundles with finite rank on X such that $E \oplus G \cong F \oplus G$. Under what assumptions on X, G, E or F we may conclude that $E \cong F$? Here we study this question mainly when X is a finite-codimensional closed analytic subset of a projective space and/or some of the bundles are direct sums of line bundles.

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1. Introduction. Let Z be a compact complex manifold. By a theorem of Atiyah the classical Krull - Schmidt unique decomposition theorem holds for holomorphic vector bundles with finite rank on Z ([1], Th. 3). Hence if E, F and G are holomorphic vector bundles with finite rank on Z such that $E \oplus G \cong F \oplus G$, then $E \cong F$ (Cancellation Theorem). It is well-known that such Cancellation Theorem does not hold in general on non-compact complex manifolds; for instance by Theorem A and B of Cartan - Serre this Cancellation Theorem is false on every finite-dimensional Stein space with at least one non-trivial holomorphic vector bundle with finite rank. The aim of this paper is to study this Cancellation Property on infinite-dimensional complex spaces. Here the complex space cannot be compact and hence we need to assume other finiteness conditions. In section 3 we will prove the following results.

Theorem 1.1. *Let V be an infinite dimensional complex Banach space and $X \subset \mathbf{P}(V)$ a closed analytic subset with finite codimension. Fix positive*

integers s, r and integers $a_i, 1 \leq i \leq r$, and $b_j, 1 \leq j \leq s$. Let E be a rank r holomorphic vector bundle on X such that $E \oplus \mathbf{O}_X(b_1) \oplus ... \oplus \mathbf{O}_X(b_s) \cong \mathbf{O}_X(a_1) \oplus ... \oplus \mathbf{O}_X(a_r) \oplus \mathbf{O}_X(b_1) \oplus ... \oplus \mathbf{O}_X(b_s)$. Then $E \cong \mathbf{O}_X(a_1) \oplus ... \oplus \mathbf{O}_X(a_r)$.

Theorem 1.2. *Let V be an infinite dimensional complex Banach space and $X \subset \mathbf{P}(V)$ a closed analytic subset with finite codimension. Fix an integer $s \geq 2$ and holomorphic line bundles A_i, and $B_i, 1 \leq i \leq s$, on X such that $A_1 \oplus ... \oplus A_s \cong B_1 \oplus ... \oplus B_s$. Then there is a permutation α of the set $\{1, ..., s\}$ such that $A_i \cong B_{\alpha(i)}$ for every i.*

Remark 1.3. Let V be an infinite dimensional complex Banach space and $X \subset \mathbf{P}(V)$ a closed analytic subset with finite codimension. By [4], Prop. III.2.1.3, X is algebraic, i.e. it is the zero-set of finitely many continuos homogeneous polynomials in V. It is very easy to check that every holomorphic function on X is constant and in particular X is connected. We do not write down this well-known exercise because to prove Theorems 1.1 and 1.2 we will use a much stronger connectedness result ([4], Th. 1.3). In section two we collect a family of cancellation results of type $E \oplus G \cong F \oplus G$ implies $E \cong F$ when E and G are sufficiently skew (e.g. if G is spanned and $H^0(X, E) = 0$ (see Proposition 2.2)) or when $F \oplus G$ is trivial and $H^0(X, \mathbf{O}_X) = \mathbf{C}$ (Proposition 2.1). We will use some of these results in the proofs of Theorems 1.1 and 1.2. In section four we will prove a very weak form of unique factorization for holomorphic vector bundles with infinite rank (see Theorem 4.1) and discuss why we feel that not much more may be proven (Example 4.2).

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2. Spanned vector bundles.

Proposition 2.1. *Let X be a complex space such that $H^0(X, \mathbf{O}_X) = \mathbf{C}$ and E a holomorphic vector bundle with finite rank on X. Assume the existence of a holomorphic vector bundle G on X such that $E \oplus G \cong \mathbf{O}_{X^{\oplus m}}$ for some integer m. Then $E \cong \mathbf{O}_{X^{\oplus r}}$, where $r = rank(E)$.*

Proof. We have $rank(G) = m - r$. Since E and G are direct factors of the spanned vector bundle $\mathbf{O}_{X^{\oplus m}}, E$ and G are spanned by their global sections. Thus $h^0(X, E) \geq rank(E)$ and $h^0(X, G) \geq rank(G)$. Since

$h^0(X, E \oplus G) = h^0(X, \mathbf{O}_{X^{\oplus m}}) = m$, we obtain $h^0(X, E) = rank(E)$. Since E is spanned by its global sections, this implies the triviality of E.

Proposition 2.2. *Let X be a complex space such that $H^0(X, \mathbf{O}_X) = \mathbf{C}$ and E, F, G holomorphic vector bundles with finite rank on X such that $E \oplus G \cong F \oplus G$. Assume that G is spanned by its global sections and that $H^0(X, F) = 0$. Then $E \cong F$.*

Proof. By assumption $\{0\} \times G$ is the subsheaf, A, of $F \oplus G$ spanned by $H^0(X, F \oplus G)$. Hence the subsheaf, B, of $E \oplus G$ spanned by $H^0(X, E \oplus G)$ is a $rank(G)$ subbundle of $E \oplus G$ such that $(E \oplus G)/B \cong F$. To prove that $E \cong F$ it is sufficient to show that $B = \{0\} \times G$. Since $\{0\} \times G \subseteq B$ and $\{0\} \times G$ is a subbundle of $E \oplus G$, the inequality $B \neq \{0\} \times G$ would imply $rank(B) > rank(G)$, contradiction.

An immediate corollary of Proposition 2.2 is the following result which could be easily proved independently.

Corollary 2.3. *Let X be a complex space such that $H^0(X, \mathbf{O}_X) = \mathbf{C}$ and E, F holomorphic vector bundles with finite rank on X such that $E \oplus \mathbf{O}_{X^{\oplus s}} \cong F \oplus \mathbf{O}_{X^{\oplus s}}$ for some $s > 0$. Assume $H^0(X, E) = 0$. Then $E \cong F$.*

Definition 2.4. Let X be a complex space. Let $\Phi(X)$ be the set of all isomorphism classes of holomorphic vector bundles E on X with finite rank spanned by their global sections and such that $H^0(X, E)$ is finite-dimensional. Notice that if $\Phi(X) \neq \emptyset$, then $H^0(X, \mathbf{O}_X)$ is finite-dimensional. Conversely, if $H^0(X, \mathbf{O}_X)$ is finite-dimensional, then $\mathbf{O}_{X^{\oplus r}} \in \Phi(X)$ for every integer $r > 0$. Let $\Psi(X)$ be the set of all isomorphism classes of holomorphic vector bundles, E, on X such that there is $L \in Pic(X)$ with $E \otimes L \in \Psi(X)$. Notice that $Pic(X) \subseteq \Psi(X)$.

Remark 2.5. There are many examples of infinite-dimensional complex spaces X with large $\Phi(X)$. For instance take $X = \mathbf{P}(V) \times Y$ with V infinite-dimensional Banach space and Y any compact projective manifold. Let $f : X \to Y$ be the projection. There are many spanned vector bundles A with finite rank on Y. For any such bundle A the bundle $f^*(A)$ is spanned and $H0(X, f^*(A)) \cong H0(Y, A)$ is finite-dimensional. Thus $f^*(A) \in \Phi(X)$. There are many infinite-dimensional complex spaces X with $H^0(X, \mathbf{O}_X) =$

C and such that $\Phi(X) = \{\mathbf{O}_{X^{\oplus r}}\}_{r>0}$ (e.g. $\mathbf{P}(V)$ with V localizing infinite-dimensional Banach space by [3], Th. 8.5).

Remark 2.6. Let X be a complex space such that $H^0(X, \mathbf{O}_X) = \mathbf{C}$ and E, F holomorphic vector bundles with finite rank on X. $E \oplus F \in \Phi(X)$ if and only if $E \in \Phi(X)$ and $F \in \Phi(X)$.

Remark 2.7. Let X be a complex space and E, F holomorphic vector bundles with finite rank on X. If $E \oplus F \in \Psi(X)$, then $E \in \Psi(X)$ and $F \in \Psi(X)$. However, the converse is in general false, as shown by the following example. Let V be any infinite-dimensional complex topological vector space. Take $X = \mathbf{P}(V), E = \mathbf{O}_{\mathbf{P}(V)}$ and $F = \mathbf{O}_{\mathbf{P}(V)}(1)$. For any integer $d \geq 0 H^0(\mathbf{P}(V), \mathbf{O}_{\mathbf{P}(V)}(d))$ is the set of all continuous homogeneous polynomial of degree d on V. Hence if $d > 0$, then $H^0(\mathbf{P}(V), \mathbf{O}_{\mathbf{P}(V)}(d))$ is infinite dimensional. Thus $\mathbf{O}_{\mathbf{P}(V)} \oplus \mathbf{O}_{\mathbf{P}(V)}(1) \notin \Psi(X)$.

Theorem 2.8. *Let X be a complex space such that $H^0(X, \mathbf{O}_X) = \mathbf{C}$ and $E \in \Psi(X)$. Then there exist an integer $s \geq 1$ and indecomposable vector bundles $E_1, ..., E_s$ on X such that $E \cong E_1 \oplus ... \oplus E_s$ and such that for every integer $t \geq 1$ and every decomposition $E \cong F_1 \oplus ... \oplus F_t$ with each F_i indecomposable, we have $t = s$ and there is a permutation σ of the set $\{1, ..., s\}$ such that $F_i \cong E_{\sigma(i)}$ for every i. Furthermore, $E_i \in \Psi(X)$ for every i. If $E \cong A \oplus B$, then there is a subset I of $\{1, ..., s\}$ such that $A \cong \oplus_{i \in I} E_i$ and $B \cong \oplus_{i \in \{1,...,s\} \setminus I} E_i$.*

Proof. First assume $E \in \Phi(X)$. Since $rank(U \oplus V) = rank(U) + rank(V)$ for all vector bundles U, V on X, it is obvious that every holomorphic vector bundle with finite rank on any complex space has a decomposition into a finite direct sum of indecomposable holomorphic vector bundles. Furthermore, if $E \in \Phi(X)$ and $E \cong E_1 \oplus ... \oplus E_s$, then $E_i \in \Phi(X)$ by Remark 2.6. Only the other assertions are the content of the theorem when $E \in \Phi(X)$ and need to be proved. Assume $E \cong E_1 \oplus ... \oplus E_s$ with each E_i indecomposable. Set $v := h^0(X, E)$ and $r := rank(E)$. Since $E \in \Phi(X), H^0(X, E)$ spans E and induces a morphism $\psi : X \to G(r, v)$, where $G(r, v)$ is the Grassmannian of codimension r linear subspaces of $\mathbf{C}^r$. Let Y be the closed subscheme of $G(r, v)$ whose ideal sheaf $\mathbf{I}_{Y,G(r,v)}$ is defined in the following way. Fix $P \in G(r, v)$ and a germ $f \in \mathbf{O}_{G(r,v),P}; f \in \mathbf{I}_{Y,G(r,v),P}$ if and only if $\psi^*(f) \equiv 0$. The closed subscheme Y of $G(r, v)$ defined in this way is usually called the schematic image of ψ because ψ

factor through the inclusion of Y in $G(r,v)$ and by its very definition it is the minimal closed subscheme Z of $G(r,v)$ such that ψ factors through the inclusion of Z in $G(r,v)$. Let Q be the universal quotient bundle on $G(r,v)$. Hence $rank(Q) = r$ and $h^0(G(r,v),Q) = v$. By the universal property of the Grassmannian and the definition of the map ψ we have $\psi^*(Q) = E$ and $\psi^*(H^0(G(r,v),Q)) = H^0(X,E)$. Set $E' := Q|Y$. Hence, seeing ψ as a morphism $\phi : X \to Y$, we have $\phi^*(E') = E$ and $\phi^*(H^0(Y,E')) = H^0(X,E)$. Now assume $E = A_1 \oplus A_2$. Hence $H^0(X,E) = H^0(X,A_1) \oplus H^0(X,A_2)$. Set $r_i := rank(A_i)$ and $v_i := h^0(X,A_i)$. Thus $r = r_1 + r_2$ and $v = v_1 + v_2$. Since A_i is spanned, it induces a morphism $\psi_i : X \to G(r_i,v_i)$. By construction ψ is obtained from (ψ_1,ψ_2) taking the natural inclusion $j : G(r_1,v_1) \times G(r_2,v_2) \to G(r,v)$ such that $j^*(Q) \cong (p_1^*(Q_1), p_2^*(Q_2))$, where Q_i is the universal quotient bundle on $G(r_i,v_i)$ and $p_i : G(r_1,v_1) \times G(r_2,v_2) \to G(r_i,v_i)$ is the projection. Hence we see that for any such decomposition $E = A_1 \oplus A_2$ we have a corresponding decomposition of Y and of E'. Thus if $E \cong E_1 \oplus ... \oplus E_s \cong F_1 \oplus ... \oplus F_t$, there are holomorphic vector bundles E_i' and F_j' on Y such that $E_i = \phi^*(E_i')$ and $F_j = \phi^*(F_j')$ for all i,j. Thus if E_i (resp. F_j) is irreducible, then E_i' (resp. F_j') is irreducible. Hence all the assertions of the statement of Theorem 2.8 (e.g. $s = t$) are true because they are true for the holomorphic vector bundle E' on the projective scheme Y ([1], Th. 3). Now we drop the assumption $E \in \Phi(X)$. Take $L \in Pic(X)$ such that $E \otimes L \in \Phi(X)$. Apply the part just proven to the vector bundle $E \otimes L$. We have $E \cong E_1 \oplus ... \oplus E_s \cong F_1 \oplus ... \oplus F_t$ if and only if $E \otimes L \cong E_1 \otimes L \oplus ... \oplus E_s \otimes L \cong F_1 \otimes L \oplus ... \oplus F_t \otimes L$. For any vector bundles A, B we have $A \otimes L \cong B \otimes L$ if and only if $A \cong B$ (just tensor with L^*). Hence all the assertions for E follow from the corresponding assertions for $E \otimes L$.

3. Proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We have $H^0(X, \mathbf{O}_X) = \mathbf{C}$ (Remark 1.3). We easily reduce to the case $s = 1$. Without loosing generality we may assume $a_1 \geq ... \geq a_r$. First assume $b_1 > a_1$. Twisting by the line bundle $\mathbf{O}_X(-b_1)$ we reduce to the case $b_1 = 0$. Thus $h^0(X, \mathbf{O}_X(a_1) \oplus ... \oplus \mathbf{O}_X(a_r) \oplus \mathbf{O}_X) = 1$ and $h^0(X,E) = 0$. Let D be the subsheaf of $E \oplus \mathbf{O}_X$ spanned by $H^0(X, E \oplus \mathbf{O}_X)$. By assumption $E \oplus \mathbf{O}_X/D$ is isomorphic both to E and to $\mathbf{O}_X(a_1) \oplus ... \oplus \mathbf{O}_X(a_r)$. Hence $E \cong \mathbf{O}_X(a_1) \oplus ... \oplus \mathbf{O}_X(a_r)$. Now assume $b_1 = a_1$. Twisting by the line bundle $\mathbf{O}_X(-b_1)$ we reduce to the case $b_1 = 0$. Let t be the maximal integer such that $1 \leq t \leq r$ and

$a_t = 0$. Thus $h^0(X, E \oplus \mathbf{O}_X) = t + 1$. Hence $h^0(X, E) = t$. For every linear subspace W of $H^0(X, \mathbf{O}_X(a_1) \oplus ... \oplus \mathbf{O}_X(a_r) \oplus \mathbf{O}_X) \cong H^0(X, E \oplus \mathbf{O}_X)$ with $dim(W) = t$, the evaluation map $W \otimes \mathbf{O}_X \to \mathbf{O}_X(a_1) \oplus ... \oplus \mathbf{O}_X(a_r) \oplus \mathbf{O}_X$ induces an isomorphism between $W \otimes \mathbf{O}_X \cong \mathbf{O}_{X^{\oplus t}}$ and a direct factor of $\mathbf{O}_X(a_1) \oplus ... \oplus \mathbf{O}_X(a_r) \oplus \mathbf{O}_X$. In particular this is true if we take as W the image of $H^0(X, E)$ by the isomorphism induced by any isomorphism $E \oplus \mathbf{O}_X \cong \mathbf{O}_X(a_1) \oplus ... \oplus \mathbf{O}_X(a_r) \oplus \mathbf{O}_X$. Thus we have an inclusion $j : \mathbf{O}_{X^{\oplus t}} \to E \oplus \mathbf{O}_X$ such that $j(\mathbf{O}_{X^{\oplus t}}) \subset E$ and $j(E)$ is a direct factor of $E \oplus \mathbf{O}_X$, i.e. there is $f : E \oplus \mathbf{O}_X \to$ with $f \circ j$ the identity. Thus $f|(E \times 0)^\circ j$ is the identity, i.e. $\mathbf{O}_{X^{\oplus t}}$ is a direct factor of E. If $t = r$, we are done. If $t < r$, we reduce to a case with $r' := r - t$ and $s' := 1 + t$ and conclude by induction on the the integer r. Now assume $b_1 < a_1$. Twisting with the line bundle $\mathbf{O}_X(-a_1)$ we reduce to the case $a_1 = 0$. Fix an isomorphism $\phi : E \oplus \mathbf{O}_X(b_1) \to \mathbf{O}_X(a_1) \oplus ... \oplus \mathbf{O}_X(a_r) \oplus \mathbf{O}_X(b_1)$. Let z be the maximal integer such that $1 \leq z \leq r$ and $a_z = 0$. Thus $h^0(X, E \oplus \mathbf{O}_X(b_1)) = h^0(X, E) = z$ and the subsheaf J of $E \oplus \mathbf{O}_X(b_1)$ spanned by $H^0(X, E \oplus \mathbf{O}_X(b_1))$ is a subsheaf of J. We have $\phi(J) \cong \mathbf{O}_{X^{\oplus z}}$ and $\phi(J)$ is a direct factor of $\mathbf{O}_X(a_1) \oplus ... \oplus \mathbf{O}_X(a_r) \oplus \mathbf{O}_X(b_1)$. Hence $rank(J) = z$ and J is a direct factor of $E \oplus \mathbf{O}_X(b_1)$. First assume $z = r$. Hence $\mathbf{O}_X(a_1) \oplus ... \oplus \mathbf{O}_X(a_r) \oplus \mathbf{O}_X(b_1)/\phi(J) \cong \mathbf{O}_X(b_1)$. Thus $E \oplus \mathbf{O}_X(b_1)/J \cong \mathbf{O}_X(b_1)$. Since J is a subsheaf of E with $rank(J) = rank(E)$, this imply $J = E$ and hence $E \cong \mathbf{O}_{X^{\oplus r}}$, as wanted. Now assume $z < r$. As in the case $b_1 = a_1$ we see that J is a direct factor of E, i.e. there is a holomorphic vector bundle A such that $rank(A) = r - z$ and $A \oplus \mathbf{O}_{X^{\oplus z}} \cong E$. Thus $A \oplus \mathbf{O}_{X^{\oplus z}} \oplus \mathbf{O}_X(b_1) \cong \mathbf{O}_X(a_{z+1}) \oplus ... \oplus \mathbf{O}_X(a_r) \oplus \mathbf{O}_{X^{\oplus t}} \oplus \mathbf{O}_X(b_1)$. Apply to A the statement for the integers $r' := r - z$ and $s' := 1 + z$ and induction on the integer r.

Proof of Theorem 1.2. We have $H^0(X, \mathbf{O}_X) = \mathbf{C}$ (Remark 1.3). By [2], Th. 1.3, there is a family t of lines contained in X and with the following properties:

(i) for all $P, Q \in X$ there is an integer $t > 0$ and lines $D_1, ..., D_t \in \tau$ such that $P \in D_1, Q \in D_t$ and $D_1 \cap D_{i+1} \neq \emptyset$ for $1 \leq i \leq t$.

(ii) $deg(L|D) = deg(L|D')$ for all $D, D' \in \tau$.

By (ii) there are integers a_i and $b_i, 1 \leq i \leq s$, such that $A_i|D \cong \mathbf{O}_D(a_i)$ and $B_i|D \cong \mathbf{O}_D(b_i)$ for every $D \in \tau$. Without loosing generality we may assume $a_1 \geq ... \geq a_s, b_1 \geq ... \geq b_s$ and $a_1 \geq b_1$. By twisting with A_1* we reduce to the case $A_1 \cong \mathbf{O}_X$. Fix $L \in Pic(X)$ and set $x := deg(L|D)$ for

any $D \in \tau$. Since τ covers X we see that $h^0(X, L) = 0$ if $x < 0$.
Claim: Assume $x = 0$. We have $h^0(X, L) > 0$ if and only if $L \cong \mathbf{O}_X$.
Proof of the Claim: Fix $\sigma \in H^0(X, L)$ and assume L not trivial. If σ has no zero, then it induces a trivialization of L . Hence we may assume the existence of $P \in X$ such that σ vanishes at P. Fix any $Q \in X$. By (i) there are lines $D_1, ..., D_t \in \tau$ such that $P \in D_1, Q \in D_t$ and $D_1 \cap D_{i+1} \neq \emptyset$ for $1 \leq i \leq t$. Since $x = 0, L|D_i$ is trivial for every i. Since $\sigma(P) = 0, \sigma|D_1 \equiv 0$. If $t = 1$ we obtain $\sigma(Q) = 0$. Assume $t \geq 2$ and $D_2 \neq D_1$. Since σ vanishes at the point $D_1 \cap D_2$, we have $\sigma|D_2 \equiv 0$; we have $\sigma|D_2 \equiv 0$ even if $D_2 = D_1$. After $t - 2$ similar steps we obtain $\sigma|D_t \equiv 0$. Hence $\sigma(Q) = 0$. Since Q was an arbitrary point of X, we have $\sigma \equiv 0$. Thus $H^0(X, L) = 0$.

By the Claim if $x = 0$ either $h^0(X, L) = 0$ or $h^0(X, L) = 1$ and in this case we have $L \cong \mathbf{O}_X$. Set $y := h^0(X, A_1 \oplus ... \oplus A_s)$. By the Claim, the inequalities $b_i \leq 0$ and $a_i \leq 0$ for all i and the normalization $A_1 \cong \mathbf{O}_X$, we have $1 \leq y \leq s$ and y is both the number of trivial line bundles in the sequence $A_1, ..., A_s$ and the number of trivial line bundles in the sequence $B_1, ..., B_s$. Permuting the indices we may assume $A_i \cong \mathbf{O}_X$ if and only if $1 \leq i \leq y$ and $B_i \cong \mathbf{O}_X$ if and only if $1 \leq i \leq y$. If $y = s$ we obtain $A_i \cong B_i$ for all i. Assume $y < s$. Since $h^0(X, A_{y+1} \oplus ... \oplus A_s) = h^0(X, B_{y+1} \oplus ... \oplus B_s) = 0$, we have $A_{y+1} \oplus ... \oplus A_s \cong B_{y+1} \oplus ... \oplus B_s$ by Corollary 2.3. If $y = s - 1$, we are done. If $y \leq s - 2$ we may apply induction on the integer s and conclude the proof.

4. Holomorphic bundles with infinite rank. For holomorphic vector bundles with as fibers infinite-dimensional topological vector spaces, one cannot expect similar results (see Example 4.2) and we are able to prove only the following result.

Theorem 4.1. *Let X be a complex space such that $H^0(X, \mathbf{O}_X) = \mathbf{C}$ and E, F, A, B holomorphic vector bundles on X with fibers isomorphic to an arbitrary topological vector space. Assume $E \oplus F \cong A \oplus B$ and that E is simple (i.e. assume $H^0(X, End(E)) \cong \mathbf{C}$). Then either there is a holomorphic vector bundle A_1 on X such that $A \cong E \oplus A_1$ or there is a holomorphic vector bundle B_1 on X such that $B \cong E \oplus B_1$.*

Proof. Let $i_1 : E \cong E \times \{0\} \to E \oplus F, j_1 : A \cong A \times \{0\} \to A \oplus B$ and $j_2 : B \cong B \times \{0\} \to A \oplus B$ the natural inclusions. Let $\pi_1 : E \oplus F \to E, p_1 : A \oplus B \to A$ and $p_2 : A \oplus B \to B$ be the natural projections. Fix an

isomorphism $u : E \oplus F \to A \oplus B$. Set $\lambda := p_1 \circ u \circ i_1 : E \to A, \mu := p_2 \circ u \circ i_2 : E \to B, \phi := \pi_1 \circ j_1 \circ \lambda$ and $\psi := \pi_1 \circ j_2 \circ \mu$. Since E is simple, ϕ (resp. ψ) is induced by the multiplication by some scalar α (resp. β). If $\alpha \neq 0$ (resp. $\beta \neq 0$), ϕ (resp. ψ) induces a left inverse of the map λ (resp. μ). Thus if $\alpha \neq 0$ (resp. $\beta \neq 0$), then there is a holomorphic vector bundle A_1 (resp. B_1) such that $A \cong E \oplus A_1$ (resp. $B \cong E \oplus B_1$). Since i_1 is injective, we cannot have $(\alpha, \beta) = (0, 0)$ and hence we are done. Unfortunately, Theorem 4.1 does not imply in general a cancellation theorem with respect to simple bundles as shown by the following example.

Example 4.2. Let X be any complex space such that $H^0(X, \mathbf{O}_X) = \mathbf{C}$. Notice that $\mathbf{O}_X$ is simple. Let V be a complex topological vector space such that $V \cong V \oplus \mathbf{C}$ and hence $V \oplus C \cong V \oplus C^{\oplus 2}$; for instance take as V any infinite-dimensional Hilbert space. Let ${}^V\mathbf{O}_X$ be the trivial vector bundle on X with fiber isomorphic to X. We have ${}^V\mathbf{O}_X \oplus \mathbf{O}_X \cong^V \mathbf{O}_X \oplus \{0\}$ and ${}^V\mathbf{O}_X \oplus \mathbf{O}_X \cong {}^V\mathbf{O}_X \oplus \mathbf{O}_{X^{\oplus 2}}$.

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