ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLVIII, s.I a, Matematică, 2002, f.2.

WEAKLY DECREASING SYSTEMS IN HILBERT SPACES

BY

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Abstract. We give contingent and proximal characterizations of the weak decreasing property of a function along trajectories of a differential inclusion in Hilbert spaces. To this end, we extend to infinite dimensions a result on the equality between the Fréchet and Dini subdifferentials. A Lyapunov stabilization result in a nonsmooth setting is given as application.

1. Introduction. Let H be a Hilbert space, let $f : H \to \mathbb{R} \cup \{\infty\}$ be a functional and F a multifunction from H to H. The system (f, F) is said to be *weakly decreasing* if for any $\xi \in H$ there exists a trajectory x for F (i.e., a solution to the differential inclusion $x'(t) \in F(x(t))$) on $\mathbb{R}_+$ with $x(0) = \xi$ which satisfies

$$f(x(t)) \leq f(x(0)) = f(\xi), \ \forall t \geq 0. \tag{1}$$

This is a basic tool in the study of asymptotic stability properties [11, p.208] as well as in the study of contingent or viscosity solutions to Bellman equations (see, for example [5], [12], [13]). It is closely related to the classical method of characteristics [4]. Our aim here is to present characterizations of the weakly decreasing property in Hilbert spaces in terms of contingent derivatives, proximal subdifferential and Fréchet subdifferential. It is obvious that the weak decreasing property (1) is equivalent to the weak invariance of the system $(\mathrm{epi}(f), F \times \{0\})$, so our approach is based on recent results on weak invariance in infinite dimensions [9], [6], [3]. We also discuss the decreasing of the function $t \mapsto f(x(t))$. This is based on weak invariance of preorders as in the finite dimensional setting [4].

2. Preliminaries. We present here several basic facts we shall use in the sequel. The reader is refer to [10] or [11] for more details.

Let H be a Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $|\cdot|$. For a given subset D in H, we denote by cl D, and cl co D the closure and closed convex hull of D, respectively. If the sequence $\{v_n\}$ converges weakly to v we write $v_n \xrightarrow{w} v$. We denote by B the closed unit ball in H. Finally, we use $\mathbb{R}_+$ for $[0,\infty)$. Let us introduce the *weak tangent cone* to the set S at $x \in S$,

$$T_S^w(x) = \{v \in H \ : \ \exists\, t_n \downarrow 0,\, v_n \xrightarrow{w} v;\ x + t_n v_n \in S\} \tag{2}$$

and the Bouligand tangent cone

$$T_S(x) = \{v \in H \ : \ \exists\, t_n \downarrow 0, v_n \to v;\ x + t_n v_n \in S\}.$$

It is easy to see that $v \in T_S^w(x)$ if and only if there exists $t_n \downarrow 0$ and $v_n \xrightarrow{w} v$ such that

$$\liminf_{n\to\infty} \frac{1}{t_n}\, d_S(x + t_n\, v_n) = 0,$$

where $d_S(\cdot)$ is the distance function

$$d_S(x) = \inf\{|x - s| \ : \ s \in S\}.$$

The (possible empty) set of closest points to x in A is denoted by

$$\mathrm{proj}_A(x) = \{s \in A \ : \ |x - s| = d_A(x)\}.$$

When $x \notin A$ and $s \in \mathrm{proj}_A(x)$ we say that the vector $x - s$ is a *perpendicular* to A at s. The set of all nonnegative multiples of such perpendiculars is denoted by $N_A^P(s)$ and is referred to as the *proximal normal cone* to A at s. If $x \in \mathrm{int} A$ or if no perpendicular to A exist at s, then by convention we set $N_A^P(s) = \{0\}$.

Let $f : H \to (-\infty, \infty]$ be a lower semicontinuous function. Recall the notions of *proximal subdifferential* $\partial_P f$, and *Fréchet subdifferential* $\partial_F f$. Namely, for $x \in \mathrm{dom}\, f := \{z : f(z) < \infty\}$,

$$\partial_P f(x) := \{\zeta \in H : \exists \sigma, \gamma > 0;\ f(y) - f(x) + \sigma|y-x|^2 \geq \langle \zeta, y - x\rangle\ \forall y \in x + \gamma B\},$$

and

$$\partial_F f(x) := \{\zeta \in H \ : \ \liminf_{p\to 0} \frac{f(x+p) - f(x) - \langle p, \zeta\rangle}{|p|} \geq 0.$$

We have the following connection between normals and subgradients:

$$\zeta \in \partial_P f(x) \Leftrightarrow (\zeta, -1) \in N^P_{\mathrm{epi}(f)}(x, f(x)),$$

where $\mathrm{epi}(f)$ means the epigraph of f, i.e.,

$$\mathrm{epi}\,(f) := \{(x, \alpha) \in H \times \mathbb{R} : f(x) \leq \alpha\}.$$

Consider now a multifunction F mapping H to the nonempty subsets of H. Let us state the basic hypothesis on F we shall use in the sequel.

(*A*) For any $x \in H$, $F(x)$ is closed and convex and, for some positive c and for all $x \in H$, $F(x) \subset c(1 + |x|)B$.

An absolutely continuous function $x : [0, T] \to H$ is called a *solution* or *trajectory* for F if

$$x'(t) \in F(x(t)), \ a.e. \text{ for } t \in [0, T].$$

It is called an *ε-trajectory* for F if

$$x'(t) \in F(x(t) + \varepsilon B), \ a.e. \text{ for } t \in [0, T].$$

The system (S, F) is said to be *(exactly) weakly invariant* if for any $T > 0$ and any $\xi \in S$ there exists a trajectory $x(\cdot)$ of F on $[0, T]$ with $x(0) = \xi$ such that $x(t) \in$ for any $t \in [0, T]$. It is said to be *approximately weakly invariant* if for any $\varepsilon > 0$, $T > 0$ and $\xi \in S$ there exists an ε-trajectory $x(\cdot)$ of F on $[0, T]$ with $x(0) = \xi$ such that

$$d_S(x(t)) \leq \varepsilon, \ \forall t \in [0, T].$$

We say that F is *upper hemicontinuous* if for any $p \in H$ the function $x \mapsto \sigma(F(x), p)$ is upper semicontinuous, where

$$\sigma(F(x), p) := \sup_{y \in F(x)} \langle p, y \rangle.$$

In fact, if the multifunction F has closed convex and bounded values, it is upper hemicontinuous (or scalarly upper semicontinuous in other terminology) if and only if it is strongly-weakly upper semicontinuous, i.e., for any $x \in H$ and for any V, open neighbourhood of $F(x)$ in the weak

topology, there exists W, open neighbourhood of x in the strong topology, such that $F(y) \subset V$ for any $y \in W$. See, for example, [1, p. 62].

We say that F is *weak sequentially upper hemicontinuous* if for any $p \in H$ the functional $x \mapsto \sigma(F(x), p)$ is weak sequentially upper semicontinuous. See [6] for examples. They refer to superposition operators, in appropriate spaces, associated to upper semicontinuous mappings from $\mathbb{R}$ to $\mathbb{R}$. The weak sequentially upper semicontinuity condition of functionals appears to be important for the existence of maximum points. See, e.g., [15, p. 511] for such results and also for useful criteria.

3. The main result. For $x \in \operatorname{dom} f$ and $u \in H$, consider the extended real number

$$D f(x)(u) = \liminf_{\substack{t \downarrow 0 \\ p \to 0}} \frac{f(x + s(u + p)) - f(x)}{t}, \tag{3}$$

called the *Dini subderivative* of f at x in the direction u, and recall the equality [2, p. 226]

$$\operatorname{epi}(D f(x)) = T_{\operatorname{epi}(f)}(x, f(x)), \tag{4}$$

for any $x \in \operatorname{dom} f$. It is clear that

$$D f(x)(u) = \inf \left\{ \liminf_{n \to \infty} \frac{f(x + t_n u_n) - f(x)}{t_n} \, : \, t_n \downarrow 0, \, u_n \to u \right\}.$$

Moreover, the inf is in fact min in the relation above and, moreover, we may take lim instead of lim inf.

This suggests to define the *weak Dini subderivative* as

$$D^w f(x)(u) := \inf \left\{ \liminf_{n \to \infty} \frac{f(x + t_n u_n) - f(x)}{t_n} \, : \, t_n \downarrow 0, \, u_n \overset{w}{\to} u \right\},$$

for $x \in \operatorname{dom} f$. Here the inf is not necessarily a min but we also may take lim instead of lim inf. It is easy to see that

$$D^w f(x)(u) = \inf \left\{ \liminf_{\substack{t \downarrow 0 \\ n \to \infty}} \frac{f(x + t u_n) - f(x)}{t} \, : \, u_n \overset{w}{\to} u \right\},$$

in which form was introduced in [8]. Further, it follows directly from the definition of weak tangency that for any $x \in \operatorname{dom} f$ we have

$$T^w_{\operatorname{epi}(f)}(x, f(x)) \subseteq \operatorname{epi}(D^w f(x)) \subseteq \operatorname{cl} T^w_{\operatorname{epi}(f)}(x, f(x)). \tag{5}$$

As a matter of fact, the rightmost inclusion holds with the closure of vertical sections, i.e.,

$$(u,\lambda) \in \operatorname{epi}(D^w f(x)) \Rightarrow \exists \lambda_n \downarrow \lambda, (u,\lambda_n) \in T^w_{\operatorname{epi}(f)}(x, f(x)).$$

On the other hand, the weak Dini subderivative defines the *weak Dini subdifferential*

$$\partial^w_D f(x) := \{\zeta \in H : \langle \zeta, u\rangle \leq D^w f(x)(u)\ \forall u \in H\},$$

as the Dini subderivate does:

$$\partial_D f(x) := \{\zeta \in H : \langle \zeta, u\rangle \leq Df(x)(u)\ \forall u \in H\}.$$

A useful result in finite dimensions was given by Frankowska [13] and states that $\partial_F f(x) = \partial_D f(x)$ for every functional. We extend here this result to the infinite dimensional setting by proving that the Fréchet subdifferential coincides to the weak Dini subdifferential. Although the result holds in reflexive Banach spaces, we shall state and prove it in Hilbert spaces. It will be used in the proof of the main result of this paper.

Proposition 3.1. *Let $f : H \to (-\infty, \infty]$ be an extended function. Then, for any $x \in \operatorname{dom} f$, we have*

$$\partial_F f(x) = \{\zeta \in H : \langle \zeta, u\rangle \leq D^w f(x)(u)\ \forall u \in H\}.$$

Proof. Suppose that $\zeta \in \partial_F f(x)$. We shall show that, for any $u \in H$ and for any λ with

$$\lambda \in \left\{ \liminf_{n\to\infty} \frac{f(x + t_n u_n) - f(x)}{t_n} : t_n \downarrow 0,\ u_n \xrightarrow{w} u \right\},$$

we have $\langle \zeta, u\rangle \leq \lambda$, which clearly gives

$$\langle \zeta, u\rangle \leq D^w f(x)(u).$$

To this aim, let $t_n \downarrow 0$ $u_n \xrightarrow{w} u$ and

$$\lambda = \liminf_{n\to\infty} \frac{f(x + t_n u_n) - f(x)}{t_n}.$$

We can obtain subsequences (without relabeling) such that

$$f(x + t_n u_n) \leq f(x) + t_n(\lambda + 1/n).$$

Let us fix $\mu < 0$. Since

$$\mu < \liminf_{p \to 0} \frac{f(x+p) - f(x) - \langle p, \zeta \rangle}{|p|},$$

for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(x+p) - f(x) - \langle \zeta, p \rangle \geq (\mu - \varepsilon)|p| \ \forall p \in \delta B.$$

Therefore, for n sufficiently large, we have

$$(\mu - \varepsilon) t_n |u_n| + \langle \zeta, t_n u_n \rangle \leq f(x + t_n u_n) - f(x) \leq t_n(\lambda + 1/n).$$

We obtain $(\mu - \varepsilon)|u_n| + \langle \zeta, u_n \rangle \leq \lambda + 1/n$ and, since $|u_n| \leq \gamma$ for some $\gamma > 0$, passing to the limit we get $(\mu - \varepsilon)\gamma + \langle \zeta, u \rangle \leq \lambda$. Since this is true for each $\mu < 0$ and $\varepsilon > 0$, we obtain $\langle \zeta, u \rangle \leq \lambda$, as claimed.

Conversely, suppose that

$$\langle \zeta, u \rangle \leq D^w f(x)(u) \tag{6}$$

and suppose, by contradiction, that $\zeta \notin \partial_F f(x)$, that is,

$$\liminf_{p \to 0} \frac{f(x+p) - f(x) - \langle p, \zeta \rangle}{|p|} < 0.$$

It follows that there exist $\varepsilon > 0$ and a sequence $\{p_n\}$ with $|p_n| \leq 1/n$ such that

$$f(x + p_n) - f(x) - \langle \zeta, p_n \rangle < -\varepsilon |p_n|.$$

Denote $t_n := |p_n|$, observe that $t_n \neq 0$ and $t_n \to 0$, and set $u_n := p_n / t_n$. We have

$$f(x + t_n u_n) < f(x) + \langle \zeta, t_n u_n \rangle - \varepsilon t_n.$$

Since $|u_n| = 1$, along on a subsequence $u_n \xrightarrow{w} u$ for some $u \in H$, therefore, we obtain

$$D^w f(x)(u) \leq \langle \zeta, u \rangle - \varepsilon.$$

This contradicts (6), therefore $\zeta \in \partial_F f(x)$, which completes the proof.

Let us state now another result which will be used in the sequel. It refers to the so called *admissibility* of the system (P, F), where P is a preorder on H, i.e., a reflexive and transitive binary relation on H. It is convenient to identify the preorder $\preceq$ on H with the multifunction defined by

$$P(\xi) := \{\eta \in H \,:\, \xi \preceq \eta\}$$

for all $\xi \in H$.

We say that (P, F) is admissible if for every $\xi \in H$ and for every $T > 0$ there exists a trajectory $x(\cdot)$ for F on $[0, T]$ with $x(0) = \xi$ such that for every $s \in [0, T]$ and for every $t \in [s, T]$ we have $x(t) \in P(x(s))$. A basic result in finite dimensions [4] asserts that if the graph of P is closed and F satisfies hypothesis (A) and has closed convex values, then (P, F) is admissible if and only if, for every $x \in H$, the system $(P(x), F)$ is weakly invariant. This result was extended to infinite dimensions in [7] and allows us to state the following proposition.

Proposition 3.2. *Suppose that the graph of P is weakly-weakly sequentially closed. Suppose further that F satisfies hypothesis (A), has closed convex values and is weak sequentially upper hemicontinuous. Then, (P, F) is admissible if and only if, for any $x \in H$, $(P(x), F)$ is weakly invariant.*

We are now in a position to state and prove our main result.

Theorem 3.1. *Suppose that F satisfies Hypothesis (A) and is upper hemicontinuous. Let f be lower semicontinuous. Then the following conditions are equivalent to each other.*

(i) For any $x \in \operatorname{dom} f$ there exists $u \in F(x)$ such that $D^w f(x)(u) \leq 0$;

(ii) $\inf\{\langle \zeta, u\rangle \,:\, u \in F(x)\} \leq 0 \ \forall x \in H,\ \zeta \in \partial_P f(x)$;

(iii) $\inf\{\langle \zeta, u\rangle \,:\, u \in F(x)\} \leq 0 \ \forall x \in H,\ \zeta \in \partial_F f(x)$.

Suppose, in addition, that f is weak sequentially lower semicontinuous and F is weak sequentially upper hemicontinuous. Then, the conditions (i), (ii) and (iii) are equivalent to the conditions (iv) below.

(iv) For any $\xi \in H$ there exists a trajectory $x(\cdot)$ for F on $\mathbb{R}_+$ with $x(0) = \xi$ which satisfies (1), i.e.

$$f(x(t)) \leq f(\xi) \ \forall t \geq 0;$$

Suppose, in addition, that $f : H \to \mathbb{R}$ is weak sequentially continuous. Then, the conditions (i) to (iv) are equivalent to the condition (v) below.

(v) *For any $\xi \in H$ there exists a trajectory $x(\cdot)$ for F on $\mathbb{R}_+$ with $x(0) = \xi$ such that the function $t \mapsto f(x(t)$ is decreasing on*

Proof. By (5), condition (i) implies

$$(F(x) \times \{0\}) \cap \operatorname{clco} T^w_{\operatorname{epi}(f)}(x, f(x)) \neq \emptyset \ \forall x \in H.$$

Clearly this condition is equivalent to

(i') $(F(x) \times \{0\}) \cap \operatorname{cl}\operatorname{co} T^w_{\operatorname{epi}(f)}(x,t)) \neq \emptyset \ \forall (x,t) \in \operatorname{epi}(f)$.

On the other hand, condition (ii) is equivalent to

(ii') $\inf\{\langle(\zeta,\theta),(u,0)\rangle : u \in F(x)\} \leq 0$,
$\forall (x,t) \in \operatorname{epi}(f)$, $(\zeta,\theta) \in N^P_{\operatorname{epi}(f)}(x,t)$.

The proof of this claim goes exactly as in the finite dimensional setting [11, p. 212]. We have to note that a basic ingredient used there is the Horizontality Theorem of Rockafellar, which is true in Hilbert spaces too [14].

We can apply now a result in [9] which says that the conditions (i') and (ii') are equivalent to the condition

(i'') $(F(x) \times \{0\}) \cap T^w_{\operatorname{epi}(f)}(x,t)) \neq \emptyset \ \forall (x,t) \in \operatorname{epi}(f)$

and to the approximate weak invariance of the system $(\operatorname{epi}(f), F \times \{0\})$. Note that in [9] F is assumed upper semicontinuous but the result also holds if F is upper hemicontinuous [3]. Therefore, using (5) again, we obtain $(i) \Leftrightarrow (ii)$. To complete the proof of the first part of our theorem, we apply Proposition 3.1. and the observation that $\partial_P f(x) \subseteq \partial_F f(x)$.

The equivalence of (i') and (ii') to (iii) follows by [3], where it is proved that, in case F is weak sequentially upper hemicontinuous and $\operatorname{epi}(f)$ is weak sequentially closed, the conditions (i') and (ii') are equivalent to the exact weak invariance of $(\operatorname{epi}(f), F \times \{0\})$.

To conclude the proof, we have to show that, if f is weak sequentially continuous, then $(iii) \Leftrightarrow (iv)$. To this aim, consider the preorder P defined by

$$P(x) = \{y \,:\, f(y) \leq f(x)\}$$

and apply Proposition 3.2. to the system (P, F). □

Remark 3.1. The equivalence between (ii) and (iii) follows also by the following extension to Hilbert spaces of [11, Proposition 3.4.5]. The proof is the same as in [11, p. 138] with the remark that the Subbotin' s Theorem used there was extended to infinite dimensions in [8].

Proposition 3.3. *Let $\zeta \in \partial_D^w f(x) = \partial_F f(x)$. Then, for any $\varepsilon > 0$ there exist $z \in x + \varepsilon B$ and $\xi \in \partial_P f(z)$ such that $|f(x) - f(z)| < \varepsilon$ and $|\zeta - \xi| < \varepsilon$.*

Indeed, by this proposition, it follows that, if $\zeta \in \partial_D^w f(x) = \partial_F f(x)$ (see Proposition 3.1.) there exist sequences $z_n \to x$ and $\xi_n \to \zeta$ such that $\xi_n \in \partial_P f(z_n)$. This fact, together with the upper hemicontinuity of F, allow us to obtain the implication $(ii) \Rightarrow (iii)$. The converse is obvious since $\partial_P f(x) \subseteq \partial_F f(x)$.

4. Lyapunov stabilization. A direct application of the results of Section 2 refers to the Lyapunov theory of stabilization. See [11, p. 208] for a development in the finite dimensional nonsmooth setting. Our aim here is to extend it in the context of a Hilbert space.

Let $W : H \to \mathbb{R}$ be a weak sequentially continuous function which satisfies a linear growth condition, $Q : H \to (-\infty, \infty]$ a weak sequentially lower semicontinuous function and $x^* \in H$ such that $0 \in F(x^*)$. The pair of functions (Q, W) are said to be a *Lyapunov pair* for x^* if they satisfy he following properties:

- Positive definiteness: $Q,\ W \geq 0;\ W(x) = 0$ if and only if $x = x^*$.
- Growth: the level sets $\{x \in H : Q(x) \leq q\}$ are closed and bounded $\forall q \in \mathbb{R}$.
- Infinitesimal decrease: $\inf\{\langle \zeta, u \rangle : u \in F(x)\} \leq -W(x)\ \forall x \in H,\ \zeta \in \partial_P Q(x)$.

Theorem 4.1. *Assume that F satisfies Hypothesis (A) and is weak sequentially upper hemicontinuous. Let (Q, W) be a Lyapunov pair for x^*. Then for any $\xi \in dom Q$ there exists a trajectory for F on $[0, \infty)$ with $x(0) = \xi$ such that $\lim_{t\to\infty} x(t) = x^*$.*

The proof of this theorem follows the same lines as [11, Theorem 5.5] and is based on Theorem 3.1. applied on $H \times \mathbb{R}$ to the pair $(\tilde{Q}, \tilde{F})$, where $\tilde{Q}(x, t) := Q(x) + t$ and $\tilde{F}(x, t) := F(x) \times \{W(x)\}$.

Taking into account the first part of Theorem 3.1., we can express the infinitesimal decrease in equivalent forms (compare [11, Proposition 5.3]).

Proposition 4.1. *Suppose that Q is lower semicontinuous, W is continuous and F satisfies Hypothesis (A) and is upper hemicontinuous. Then the following conditions are equivalent.*

(i) *For any $x \in domQ$ there exists $u \in F(x)$ such that $D^wQ(x)(u) \leq -W(x)$;*

(ii) $\inf\{\langle \zeta, u\rangle \,:\, u \in F(x)\} \leq -W(x)\ \forall x \in H,\ \zeta \in \partial_P Q(x)$;

(iii) $\inf\{\langle \zeta, u\rangle \,:\, u \in F(x)\} \leq -W(x)\ \forall x \in H,\ \zeta \in \partial_F Q(x)$.

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Received: 25.IX.2002

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