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LIPSCHITZ-CONTINUITY OF THE SOLUTION MAP OF SOME NONCONVEX HYPERBOLIC DIFFERENTIAL INCLUSIONS

BY

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1. Introduction. This paper is concerned with the Darboux problem for hyperbolic differential inclusions of the form

$$(1.1)\qquad \begin{aligned} &u_{xy}(x,y) \in F(x,y,u(x,y)),\\ &u(x,0)=\lambda(x,0), \quad u(0,y)=\lambda(0,y), \end{aligned}$$

where F is a multifunction from $\Pi \times R^n$ to nonempty subsets of R^n, $\Pi = I_1 \times I_2$, $I_i = [0,T_i], T_i > 0$, $i = 1,2$ and $\lambda(x,y) = \alpha(x) + \beta(y) - \alpha(0)$ with α and β two continuous functions from I_1 (resp. I_2) to R^n, satisfying $\alpha(0) = \beta(0)$.

Qualitative properties and structure of the set of solutions of the Darboux problem (1.1) have been studied by many authors ([1], [2], [3], [6], [7], [10], [11], [13], etc.). In [7] it is shown that the solution set of (1.1) with F single valued is an R_δ-set, in [3] it is proved that the solution set is a retract of a convex subset of a Banach space, in [11] a solution of (1.1) continuous with respect to λ is constructed and in [6] the arcwise connectedness of the solution set to (1.1) is shown.

In the present paper we study the properties of the map that associates to a given λ the set of solutions to (1.1) and the main purpose is to prove that this solution map depends Lipschitz-continuously on λ.

Our proof is based on an idea of Tallos ([12]), applying the contraction principle in the space of selections of the multifunction instead of the space of solutions.

We note that similar results are obtained in the case of "classical" differential inclusions by KANNAI and TALLOS ([8]) and in the case of mild solutions of semilinear differential inclusions in Banach spaces by author ([5]).

The paper is organized as follows: in Section 2 we recall some preliminary results that we use in the sequel and in Section 3 we prove our main result.

2. Preliminaries. Let $\mathcal{P}(R^n)$ be the family of all nonempty subsets of R^n and let Π be the rectangle $I_1 \times I_2$, where $I_1 = [0, T_1]$, $I_2 = [0, T_2]$. Given $a \in R$ we denote by L^1 the Banach space of all (equivalence classes of) Lebesgue measurable functions $\sigma : \Pi \to R^n$, endowed with the norm

$$||\sigma||_1 = \int\int_{\Pi} e^{-a(x+y)}||\sigma(x,y)||dxdy. \tag{2.1}$$

We denote by C the Banach space of all continuous functions $u : \Pi \to R^n$ endowed with the norm

$$||u||_C = \sup_{(x,y)\in\Pi} ||u(x,y)||.$$

By Λ we mean the linear subspace of C consisting of all $\lambda \in C$ such that there exist continuous functions $\alpha : I_1 \to R^n, \beta : I_2 \to R^n$, with $\alpha(0) = \beta(0)$, satisfying $\lambda(x,y) = \alpha(x) + \beta(y) - \alpha(0)$ for every $(x,y) \in \Pi$. Observe that Λ, equipped with the norm of C, is a separable Banach space.

In order to study problem (1.1) we introduce the following

Hypothesis 2.1. *Let $F : \Pi \times R^n \to \mathcal{P}(R^n)$ be a set -valued map with nonempty closed values, satisfying the following assumptions:*

i) The set-valued maps $(x,y) \to F(x,y,u)$ is measurable for all $u \in R^n$.

ii) There exist $k > 0$ such that, for almost all $(x,y) \in \Pi$ $F(x,y,.)$ is k-Lipschitz, i.e.

$$d_H(F(x,y,u), F(x,y,u')) \leq k||u - u'|| \quad \forall u, u' \in R^n.$$

where d_H is the Hausdorff-Pompeiu generalized metric on $\mathcal{P}(R^n)$ defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \quad d^*(A,B) = \sup\{d(a,B); a \in A\}.$$

iii) $d(0, F(x,y,0)) \leq k$ for almost all $(x,y) \in \Pi$.

For $\sigma \in L^1(\Pi)$, consider the following Darboux problem:

$$\begin{aligned} &u_{xy}(x,y) = \sigma(x,y), \\ &u(x,0) = \lambda(x,0), \quad u(0,y) = \lambda(0,y). \end{aligned} \tag{2.2}$$

Definition 2.2. Let $\lambda \in \Lambda$. The function $u \in C$ given by

$$u(x,y) = \lambda(x,y) + \int_0^x \int_0^y \sigma(\xi,\eta) d\xi d\eta \quad (x,y) \in \Pi$$

is said to be a solution of (2.2)

Obviously, problem (2.2) has a unique solution, which will be denoted by $u^{\lambda,\sigma}$.

Definition 2.3. Let Hypothesis 2.1 be satisfied and let $\lambda \in \Lambda$. A function $u \in C$ is said to be a solution of (1.1) if there exists a function $\sigma \in L^1$ such that

$$\sigma(x,y) \in F(x,y,u(x,y)) \quad a.e.(\Pi),$$

$$u(x,y) = \lambda(x,y) + \int_0^x \int_0^y \sigma(\xi,\eta) d\xi d\eta \quad (x,y) \in \Pi.$$

We denote by $S(\lambda)$ the solution set of (1.1).

In what follows we define

$$W := \{u \in C; \exists (\lambda,\sigma) \in \Lambda \times L^1 \quad \text{such that} \quad u = u^{\lambda,\sigma}\}$$

and

$$||u(.,.)||_W = ||u^{\lambda,\sigma}(.,.)||_W = ||\lambda(.,.)||_C + ||\sigma(.,.)||_1$$

It is easy to verify that $||.||_W$ is a norm and $(W, ||.||_W)$ is a Banach space.

Finally we recall some basic results concerning set valued contractions that we shall use in the sequel.

Let (Z,d) be a metric space and consider a set valued map T on Z with nonempty closed values in Z. T is said to be a λ-contraction if there exists $0 < \lambda < 1$ such that:

$$d(T(x),T(y)) \leq \lambda d(x,y) \quad \forall x,y \in Z$$

If Z is complete, then every set valued contraction has a fixed point, i.e. a point $z \in Z$ such that $z \in T(z)$ (see, for instance, [9]).

We denote by $Fix(T)$ the set of all fixed point of the multifunction T. Obviously, $Fix(T)$ is closed.

Proposition 2.4.([9]) *Let Z be a complete metric space and suppose that T_1, T_2 are λ-contractions with closed values in Y. Then*

$$d(Fix(T_1), Fix(T_2)) \leq \frac{1}{1-\lambda} \sup_{z \in Z} d(T_1(z), T_2(z)).$$

3. The main result. We are ready now to show that the set of solution of (1.1) depends Lipschitz-continuous on the initial condition.

Theorem 3.1. *Let Hypothesis 2.1 be satisfied and let $a > \sqrt{k}$.*

Then the map $\lambda \to S(\lambda)$ is Lipschitz-continuous on Λ with nonempty closed values in W.

Proof. Let us consider $\lambda \in \Lambda, \sigma \in L^1$ and define the following set valued maps:

$$M_{\lambda,\sigma}(x,y) := F(x,y,\lambda(x,y) + \int_0^x \int_0^y \sigma(\xi,\eta) d\xi d\eta),\ (x,y) \in \Pi, \tag{3.1}$$

$$T_\lambda(\sigma) := \{\phi(.) \in L^1; \phi(x,y) \in M_{\lambda,\sigma}(x,y)\ a.e.\ (\Pi)\} \tag{3.2}$$

We shall prove first that $T_\lambda(\sigma)$ is nonempty and closed for every $\sigma \in L^1$.

The fact that that the set valued map $M_{\lambda,\sigma}(.,.)$ is measurable is well known. For example the map $(x,y) \to F(x,y,\lambda(x,y) + \int_0^x \int_0^y \sigma(\xi,\eta) d\xi d\eta)$ can be approximated by step functions and we can apply Theorem III. 40 in [4]. Since the values of F are closed, with the measurable selection theorem (e.g. Theorem III.6 in [4]) we infer that $M_{\lambda,\sigma}(.,.)$ admits a measurable selection $\phi(.,.)$. According to Hypothesis 2.1, one has:

$$||\phi(x,y)|| \leq d(0, F(x,y,0)) + d_H(F(x,y,0), F(x,y,\lambda(x,y)+$$

$$+\int_0^x \int_0^y \sigma(\xi,\eta) d\xi d\eta)) \leq k(1 + ||\lambda||_C + \int_0^x \int_0^y ||\sigma(\xi,\eta)|| d\xi d\eta).$$

Thus, integrating by parts, we obtain:

$$\int_0^{T_1}\int_0^{T_2} e^{-a(x+y)}||\phi(x,y)||dxdy \leq \int_0^{T_1}\int_0^{T_2} e^{-a(x+y)}k(1+||\lambda||_C+$$

$$+\int_0^x\int_0^y ||\sigma(\xi,\eta)||d\xi d\eta)dxdy = \frac{k}{a^2}(1+||\lambda||_C)(1-e^{-aT_1})(1-e^{-aT_2})+$$

$$+k\int_0^{T_1}\int_0^{T_2}(\int_\xi^{T_1}\int_\eta^{T_2} e^{-a(x+y)}dxdy)||\sigma(\xi,\eta)||d\xi d\eta \leq \frac{k}{a^2}(1+||\lambda||_C)+\frac{k}{a^2}||\sigma||_1,$$

therefore, if $\phi(.,.)$ is a measurable selection of $M_{\lambda,\sigma}(.,.)$ then $\phi(.,.) \in L^1$ and hence $T_\lambda(\sigma)$ is nonempty.

The set $T_\lambda(\sigma)$ is closed. Indeed, if $\phi_n \in T_\lambda(\sigma)$ and $||\phi_n - \phi||_1 \to 0$, then we can pass to a subsequence ϕ_{n_k} such that $\phi_{n_k}(x,y) \to \phi(x,y)$ for a.e. $(x,y) \in \Pi$ and we find that $\phi \in T_\lambda(\sigma)$.

The next step of the proof will show that $T_\lambda(.)$ is a contraction on L^1.

Let $\sigma_1, \sigma_2 \in L^1$ be given, $\phi_1 \in T_\lambda(\sigma_1)$ and let $\epsilon > 0$. Consider the following set valued map:

$$G(x,y) := M_{\lambda,\sigma_2}(x,y) \cap \{z \in R^n; \quad ||\phi_1(x,y) - z|| \leq$$

$$\leq k||\int_0^x\int_0^y (\sigma_1(\xi,\eta)-\sigma_2(\xi,\eta))d\xi d\eta)|| + \epsilon\}$$

Since

$$d(\phi_1(x,y), M_{\lambda,\sigma_2}(x,y)) \leq d_H(F(x,y,\lambda(x,y)+\int_0^x\int_0^y \sigma_1(\xi,\eta)d\xi d\eta),$$

$$F(x,y,\lambda(x,y)+\int_0^x\int_0^y \sigma_2(\xi,\eta)d\xi d\eta)) \leq k||\int_0^x\int_0^y(\sigma_1(\xi,\eta)-\sigma_2(\xi,\eta))d\xi d\eta)||,$$

we deduce that $G(.,.)$ has nonempty closed values.

Moreover, according to Proposition III.4 in [4], $G(.,.)$ is measurable.

Let $\phi_2(.,.)$ be a measurable selection of $G(.,.)$. It follows that $\phi_2 \in T_\lambda(\sigma_2)$ and

$$||\phi_1 - \phi_2||_1 = \int_0^{T_1}\int_0^{T_2} e^{-a(x+y)}||\phi_1(x,y)-\phi_2(x,y)||dxdy \leq$$

$$\leq \int_0^{T_1}\int_0^{T_2} e^{-a(x+y)}k(\int_0^x\int_0^y ||\sigma_1(\xi,\eta)-\sigma_2(\xi,\eta)||d\xi d\eta)dxdy+$$

$$+\epsilon\int_0^{T_1}\int_0^{T_2} e^{-a(x+y)}dxdy =$$

$$= k\int_0^{T_1}\int_0^{T_2}(\int_\xi^{T_1}\int_\eta^{T_2} e^{-a(x+y)}dxdy)||\sigma_1(\xi,\eta)-\sigma_2(\xi,\eta)||d\xi d\eta +$$

$$+\epsilon\int_0^{T_1}\int_0^{T_2} e^{-a(x+y)}dxdy.$$

Since ϵ is arbitrary, we deduce that

$$d(\phi_1, T_\lambda(\sigma_2)) \leq \frac{k}{a^2}||\sigma_1-\sigma_2||_1.$$

Replacing $\sigma_1(.,.)$ by $\sigma_2(.,.)$, we obtain

$$d_H(T_\lambda(\sigma_1), T_\lambda(\sigma_2)) \leq \frac{k}{a^2}||\sigma_1-\sigma_2||_1.$$

Hence $T_\lambda(.)$ is a contraction on L^1.

Consequently, $T_\lambda(.)$ admits a fixed point $\sigma \in L$. We define

$$u(x,y) = \lambda(x,y) + \int_0^x\int_0^y \sigma(\xi,\eta)d\xi d\eta \quad (x,y)\in \Pi$$

and we have that $u(.,.) \in S(\lambda)$ is solution to (1.1). Since $Fix(T_\lambda(.))$ is closed, we also get that $S(\lambda)$ is a closed set of W.

We prove next the following inequality:

$$d_H(T_{\lambda_1}(\sigma), T_{\lambda_2}(\sigma)) \leq \frac{k}{a^2}||\lambda_1-\lambda_2||_C \quad \forall \sigma\in L^1, \lambda_1,\lambda_2\in C \tag{3.3}$$

Let us consider the set valued map

$$G_1(x,y) = M_{\lambda_2,\sigma}(x,y)\cap\{z\in R^n;\ ||\phi(x,y)-z||\leq k||\lambda_1-\lambda_2||_C+\epsilon\},$$

where $\phi(.)$ is a measurable selection of $M_{\lambda_1,\sigma}(.)$ and $\epsilon > 0$.

With the same arguments used for the set valued map $G(.,.)$, we deduce that $G_1(.,.)$ is measurable with nonempty closed values. Let $\psi(.,.)$ be a measurable selection of $G_1(.,.)$.

Since ϵ is arbitrary, we deduce that

$$d(\phi, T_{\lambda_2}(\sigma)) \leq \frac{k}{a^2}||\lambda_1-\lambda_2||_C.$$

Replacing $\lambda_2(.)$ by $\lambda_1(.)$ we obtain (3.3).

From (3.3) and Proposition 2.4 we obtain:

$$d_H(Fix(T_{\lambda_1}), Fix(T_{\lambda_2})) \leq \frac{k}{a^2-k}||\lambda_1 - \lambda_2||_C.$$

Let $\lambda_1, \lambda_2 \in \Lambda$ and $u \in S(\lambda_1)$. In particular there exists $\sigma_1 \in Fix(T_{\lambda_1})$ and thus, for every $\epsilon > 0$ there exists $\sigma_2(.,.) \in Fix(T_{\lambda_2})$ such that

$$||\sigma_1 - \sigma_2||_1 \leq \frac{k}{a^2-k}||\lambda_1 - \lambda_2||_C + \epsilon. \tag{3.4}$$

Put $w(x,y) = \lambda_2(x,y) + \int_0^x \int_0^y \sigma_2(\xi,\eta)d\xi d\eta$; then $w(.,.) \in S(\lambda_2)$ and

$$||u - w||_W = ||\lambda_1 - \lambda_2||_C + ||\sigma_1 - \sigma_2||_1 \leq$$

$$||\lambda_1 - \lambda_2||_C + \frac{k}{a^2-k}||\lambda_1 - \lambda_2||_C + \epsilon =$$

$$= \frac{a^2}{a^2-k}||\lambda_1 - \lambda_2||_C + \epsilon.$$

We infer that

$$d(u, S(\lambda_2)) \leq \frac{a^2}{a^2-k}||\lambda_1 - \lambda_2||_C + \epsilon$$

for every $\epsilon > 0$, which means

$$d(u, S(\lambda_2)) \leq \frac{a^2}{a^2-k}||\lambda_1 - \lambda_2||_C \tag{3.5}$$

and by interchanging λ_1 with λ_2, if we denote $L := \frac{a^2}{a^2-k} > 0$ we obtain

$$d(S(\lambda_1), S(\lambda_2)) \leq L||\lambda_1 - \lambda_2||_C$$

and the proof is complete.

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