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# ON THE Ψ - BOUNDEDNESS OF THE SOLUTIONS OF A LINEAR SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

BY

A. DIAMANDESCU

**Abstract.** In this paper, we provide necessary and sufficient conditions for Ψ - boundedness of the solutions of a linear homogeneous differential system and sufficient conditions for Ψ - boundedness of the solutions of a linear nonhomogeneous differential system.

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**1. Introduction.** The purpose of our paper is to provide sufficient conditions for Ψ - boundedness of the solutions of a differential system of the form

$$x' = A(t)x + f(t) \tag{1}$$

as a perturbed system of

$$y' = A(t)y. \tag{2}$$

We investigate conditions on a fundamental matrix Y(t) of the linear homogeneous differential system (2) and on the function f(t) under which the solutions of the systems (1) and (2) are Ψ - bounded on $R_+$. Here, Ψ is a continuous matrix function. The introduction of the matrix function Ψ permits to obtain a mixed asymptotic behavior of the solutions.

The problem of Ψ - boundedness of the solutions for systems of ordinary differential equations has been studied by many authors, as e.g.

Akinyele [1], [2], Constantin [4], [5], Avramescu [3], Hallam [11], Hara, Yoneyama, Itoh [12], Kuben [14], Morchalo [15]. In these papers, the function $\Psi$ is a scalar continuous function ( and increasing, differentiable and bounded in [1], [2], nondecreasing and such that $\Psi(t) \geq 1$ on $R_+$ in [4], [5] ).

**2. Definitions, notations and hypotheses.** Let $R^d$ denote the Euclidean d - space. For $x = ( x_1, x_2,... x_d )^T \in R^d$, let $\| x \| = \max \{ |x_1|, |x_2|,...|x_d| \}$ be the norm of x. For a matrix $A = ( a_{ij})$, we define the norm $| A |$ of A by $| A | = \sup_{\|x\| \leq 1} \|Ax\|$; it is well-known that

$$| A | = \max_{1 \leq i \leq d} \sum_{j=1}^{d} | a_{ij} | .$$

In the above systems, we suppose that:
- A(t) is a d × d continuous matrix on $R_+ = [0, \infty)$;
- $f : R_+ \longrightarrow R^d$ is a continuous function.

Let $\Psi_i : R_+ \to (0,\infty)$, i = 1, 2, ... d, be continuous functions and

$$\Psi = \text{diag } [\Psi_1,\Psi_2,...\Psi_d].$$

A matrix P is said to be a projection if $P^2 = P$. If P is a projection, then so is I - P. Two such projections, whose sum is I and hence whose product is 0, are said to be supplementary.

**Definition 1.** A function $\varphi : R_+ \to R^d$ is said to be $\Psi-$ bounded on $R_+$ if $\Psi(t)\varphi(t)$ is bounded on $R_+$.

**Definition 2.** The solutions of (1) are $\Psi-$ uniformly bounded on $R_+$ if for every $\alpha > 0$, there exists $H(\alpha) > 0$ such that any solution x(t) of (1) which satisfies the inequality $\| \Psi(t_0)x(t_0)\| < \alpha$ for some $t_0 \geq 0$, exists and satisfies the inequality $\| \Psi(t)x(t)\| < H(\alpha)$ for all $t \geq t_0$.

**Remarks.** 1. For $\Psi_i = 1$, i = 1, 2,...d, we obtain the notions of just boundedness respectively uniform boundedness.

2. If in the above Definition, we replace $\Psi$ with $\Psi^k$, $k \in Z \setminus \{0,1\}$, we generalize the notion of uniform boundedness of degree k with respect to a function $\varphi$ ( see [4] ).

3. It is easy to see that if the solutions of (1) are $\Psi-$ uniformly bounded on $R_+$, they are $\Psi-$ bounded on $R_+$.

4. It is easy to see that if $\Psi$ and $\Psi^{-1}$ are bounded on $R_+$, then the $\Psi$ - boundedness is equivalent with the just boundedness.

**3. $\Psi$ - boundedness of the solutions of the linear homogeneous system (2).** In this section we give necessary and sufficient conditions for the $\Psi$ - boundedness of the solutions of the linear homogeneous system (2).

**Theorem 1.** *Let Y(t) a fundamental matrix for linear system (2). Then, the solutions of linear system (2) are $\Psi-$ bounded on $R_+$ if and only if for every $s \geq 0$, there exists $K(s) > 0$ such that*

$$| \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) | \leq K(s), \text{ for all } t \geq s \geq 0.$$

**Proof.** The solution of (2) which takes the value $y \in R^d$ at $a \geq 0$ is $y(t) = Y(t)Y^{-1}(a)y$ for $t \geq 0$.

If the solutions of (2) are $\Psi-$ bounded on $R_+$, then, for $y \in R^d$ and $s \geq 0$, there exists $K(y,s) > 0$ such that

$$|| \Psi(t)Y(t)Y^{-1}(s)y || \leq K(y,s), \text{ for all } t \geq s.$$

Let $y = \Psi^{-1}(s)e_i^T$, where $e_i = (0,...1,...0)$, for $i = 1,2,...d$. Then, there exists $K_s > 0$ such that

$$|| \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)e_i^T || \leq K_s, \text{ for all } t \geq s \text{ and } i = 1,2,...d.$$

It follows that the columns of the matrix $\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)$ are bounded for $t \geq s$. Thus, the elements of this matrix are bounded for $t \geq s$.

Hence, there exists $K(s) > 0$ such that

$$| \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) | \leq K(s), \text{ for all } t \geq s \geq 0.$$

Conversely, if the above condition is satisfied, then, for every solution $y(t)$ of (2), we have

$$\| \Psi(t)y(t) \| = \| \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)\Psi(s)y(s) \| \leq K(s) \| \Psi(s)y(s) \|,$$

for all $t \geq s$.

Thus, the solutions of (2) are $\Psi-$ bounded on $R_+$.

The proof is now complete.

**Remark.** We observe that $\Psi-$ boundedness of the solutions of the linear system (2) is equivalent with $\Psi$ - stability of the linear system (2) (see [7]).

**Theorem 2.** *Let Y(t) a fundamental matrix of the linear system (2). Then, the solutions of the linear system (2) are $\Psi-$ uniformly bounded on $R_+$ if and only if there exists a constant $K > 0$ such that*

$$| \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) | \leq K, \text{ for all } t \geq s \geq 0.$$

**Proof.** Suppose first that the solutions of the linear system (2) are $\Psi-$ uniformly bounded on $R_+$.

Let $u \in R^d$ be such that $\|u\| \leq 1$ and $s \geq 0$. From Definition 2, it follows that

$$\| \Psi(t)Y(t)Y^{-1}(s)(\frac{1}{2}\Psi^{-1}(s)u) \| \leq H(1), \text{ for all } t \geq s.$$

Thus,

$$| \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) | \leq 2H(1), \text{ for all } t \geq s \geq 0.$$

Suppose next that

$$| \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) | \leq K, \text{ for all } t \geq s \geq 0.$$

It is easy to see that, if the solution $y(t)$ of (2) satisfies the inequality

$$\| \Psi(s)y(s) \| < \alpha, \text{ for some } s \geq 0,$$

then,

$$\| \Psi(t)y(t) \| \leq K\alpha, \text{ for all } t \geq s.$$

This shows that the solutions of (2) are $\Psi-$ uniformly bounded on $R_+$.

The proof is now complete.

**Remarks.** 1. We observe that $\Psi-$ uniform boundedness of the solutions of the linear system ( 2 ) is equivalent with $\Psi$ - uniform stability of the linear system ( 2 ) ( see [7] ).

2. We specify that Ψ− uniform boundedness of the solutions of the linear system (2) is not equivalent with Ψ - boundedness of the solutions of the linear system (2). This is shown by the next example, transformed after an example due to PERRON ( [17] ).

**Example 1.** Let

$$\mathrm{A(t)} = \begin{pmatrix} 1 & 0 \\ 0 & \sin\ln(t+1) + \cos\ln(t+1) - 2 \end{pmatrix}.$$

Then, a fundamental matrix for system (2) is

$$\mathrm{Y(t)} = \begin{pmatrix} e^{t+1} & 0 \\ 0 & e^{(t+1)[\sin\ln(t+1)-2]} \end{pmatrix}.$$

Let

$$\Psi(\mathrm{t}) = \begin{pmatrix} e^{-(t+1)} & 0 \\ 0 & e^{t+1} \end{pmatrix}.$$

We have

$$\Psi(\mathrm{t})\mathrm{Y(t)} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\phi(t)} \end{pmatrix} \text{ and } \Psi(\mathrm{t})\mathrm{Y(t)Y^{-1}(s)}\Psi^{-1}(\mathrm{s}) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\phi(t)-\phi(s)} \end{pmatrix},$$

where $\phi(\mathrm{t}) = (\mathrm{t}+1)[\sin\ln(\mathrm{t}+1) - 1]$.

It is easy to see that $|\ \Psi(\mathrm{t})\mathrm{Y(t)}|$ is bounded on $\mathrm{R}_+$ and $|\ \Psi(\mathrm{t})\mathrm{Y(t)Y^{-1}(s)}\Psi^{-1}(\mathrm{s})\ |$ is unbounded for $\mathrm{t} \geq \mathrm{s} \geq 0$.

It follows that the solutions of (2) are Ψ - bounded on $\mathrm{R}_+$, but they are not Ψ - uniformly bounded on $\mathrm{R}_+$

**Theorem 3.** *Let Y(t) be a fundamental matrix of the linear system (2). Then, the linear system (2) has at least one, but not every, non-trivial Ψ - bounded solution on $R_+$ if and only if there exists a projection $P \neq 0$ such that*

*a). Ψ(t)Y(t)P is bounded on $R_+$,*

*b). Ψ(t)Y(t)(I-P) is unbounded on $R_+$.*

**Proof.** Suppose first that the linear system (2) has at least one, but not every, non-trivial $\Psi$ - bounded solution on $R_+$. Let y(t) be this solution.

Thus, there exists $K > 0$ such that

$$|| \Psi(t)Y(t)Y^{-1}(0)y(0) || \ \leq \ K, \text{ for all } t \ \geq \ 0.$$

Let $c = Y^{-1}(0)y(0) \neq 0$ be and $c = (c_1,...c_s,...c_d)^T$, where $|c_s| = \|c\|$. Let $P$ the null matrix in which the s - column is replaced with $\|c\|^{-1}c$. Thus, $P^2 = P$, $P \neq 0$ and $\Psi(t)Y(t)P$ is bounded on $R_+$.

But then, if $\Psi(t)Y(t)(I-P)$ is bounded on $R_+$, then $\Psi(t)Y(t)$ is bounded on $R_+$, which is contradictory.

Thus, the conditions a) and b) are satisfied.

Suppose next that the conditions a) and b) are satisfied.

It is easy to see that there exists $y_0 \in R^d$ , $y_0 \neq 0$ such that $(I-P)y_0 = 0$. Thus, $y(t) = Y(t)Py_0$ is a non-trivial $\Psi$ - bounded solution of (2).

If all the solutions of (2) are $\Psi$ - bounded on $R_+$, then $\Psi(t)Y(t)$ is bounded on $R_+$ ( Theorem 1 ). Then, $\Psi(t)Y(t)(I-P)$ is bounded on $R_+$, which is contradictory.

The proof is now complete.

**Remark.** We observe that the linear system (2) has at least one, but not every, non-trivial $\Psi$ - bounded solution on $R_+$ if and only if it is $\Psi$ - conditionally stable on $R_+$ ( see [10]).

A sufficient condition for $\Psi$ - boundedness of the solutions of the linear system (2) is given by

**Theorem 4.** *Let Y(t) be a fundamental matrix for (2).*

*Suppose that there exist a projection P and a positive constant K such that*

$$\int_0^t | \Psi(t)Y(t)PY^{-1}(s)\Psi^{-1}(s) | ds + \int_t^\infty | \Psi(t)Y(t)(I-P)Y^{-1}(s)\Psi^{-1}(s) | ds \leq K,$$

*for all $t \geq 0$.*

*Then:*

*1). if $P \neq 0$, $P \neq I$, the linear system (2) has at least one, but not every, non-trivial $\Psi$ - bounded solution on $R_+$; in addition, if y(t) is this solution, then $\lim_{t\to\infty} \| \Psi(t)y(t)\| = 0$.*

*2). if P = I, the solutions of the linear system (2) are Ψ - bounded on $R_+$; in addition, for any solution, $\lim_{t\to\infty} \| \Psi(t)y(t)\| = 0$.*

*3). if P = 0, the solutions of the linear system (2) (except null solution) are Ψ - unbounded on $R_+$.*

**Proof.** Suppose that $P \neq 0$. From Lemma 1, [9], it follows that there exists a constant $N > 0$ such that

$$| \Psi(t)Y(t)P | \leq Ne^{-K^{-1}t}, \text{ for all } t \geq 0.$$

**Case P ≠ I.** Since 0 is an eigenvalue of I - P, there exists a $v \in R^d$, $v \neq 0$, such that $( I - P )v = 0$. Let $y_0 = Y(t_0)v$ and $y(t)$ the solution of (2) with initial condition $y(t_0) = y_0$. Then, we have

$$\| \Psi(t)y(t) \| = \| \Psi(t)Y(t)Y^{-1}(t_0)y(t_0) \| =$$

$$= \| \Psi(t)Y(t)Pv \| \leq Ne^{-K^{-1}t} \| v \|, \text{ for all } t \geq 0.$$

Thus, $y(t)$ is a Ψ - bounded solution of (2) and, in addition, $\lim_{t\to\infty} \| \Psi(t)y(t)\| = 0$.

On the other hand, from Lemma 2, [9], it follows that $\| \Psi(t)Y(t)(I-P)x_0 \|$ is unbounded on $R_+$ for all $x_0 \in R^d$ such that $(I-P)x_0 \neq 0$.

Since 0 is an eigenvalue of P, there exists $u_0 \in R^d$, $u_0 \neq 0$ as an eigenvector: $Pu_0 = 0$. Let $z_0 = Y(t_0)u_0$ be.

For the solution $z(t)$ of (2) with initial condition $z(t_0) = z_0$, we have

$$\Psi(t)z(t) = \Psi(t)Y(t)Y^{-1}(t_0)z(t_0) = \Psi(t)Y(t)(I-P)u_0.$$

It follows that

$$\limsup_{t\to\infty} \| \Psi(t)z(t) \| = \infty.$$

Thus, the solution $z(t)$ is Ψ - unbounded on $R_+$.

**Case P = I.** In this case, we have that $| \Psi(t)Y(t)| \leq Ne^{-K^{-1}t}$, for all $t \geq 0$. From Theorem 1, the conclusion is obvious.

Now, suppose that $P = 0$. From Lemma 2, [9], it follows that

$$\limsup_{t\to\infty} \| \Psi(t)Y(t)x_0 \| = \infty,$$

for all $x_0 \in R^d$, $x_0 \neq 0$. This shows that the system (2) has no nontrivial Ψ - bounded solution on $R_+$.

The proof is now complete.

**Example 2.** Consider the linear system

$$x' = Ax \tag{3}$$

where A is a d × d real constant matrix.

Let $\alpha$, $\beta \in R$ be, $\alpha < \beta$. Suppose that the characteristic roots of A are divided into two sets $\Lambda_1$ and $\Lambda_2$ such that every root in the first set has real part less than $\alpha$ and every root in the second has real part greater than $\beta$. We can represent the vector space $R^d$ as the direct sum of two subspaces $X_1$, $X_2$ invariant under A such that all characteristic roots of the restriction of A in $X_i$ belong to $\Lambda_i$ (i = 1, 2). The corresponding projections $P_1$, $P_2$ of $R^d$ onto $X_1$, $X_2$ commute with A, are supplementary and there exists a constant $K > 0$ such that

$$| e^{tA} P_1 | \leq Ke^{\alpha t}, \text{ for } t \geq 0,$$

$$| e^{tA} P_2 | \leq Ke^{\beta t}, \text{ for } t \leq 0.$$

It will be noted that $P_i \neq 0$ if $\Lambda_i \neq \emptyset$ ( see [6], Chapter III ).

We consider two cases:

**Case I.** There exist characteristic roots of A with diferent real parts.

We consider 3 subcases:

i). we can choose $\alpha$, $\beta \in R$ as above such that all characteristic roots of A have real parts less than $\alpha$.

In this case, $\Lambda_2 = \emptyset$, $P_2 = 0$, $P_1 = I$. Let $\gamma_1 \geq \alpha$ be and $\Psi(t) = e^{-\gamma_1 t} I_d$. Then we have

$$| \Psi(t) e^{tA} e^{-sA} \Psi^{-1}(s) | \leq Ke^{(\alpha-\gamma_1)(t-s)}, \text{ for } t \geq s \geq 0.$$

From Theorem 2, it follows that the solutions of (3) are $\Psi$ -uniformly bounded on $R_+$.

If $\gamma_1 > \alpha$, from Theorem 4, it follows that the solutions of (3) are $\Psi$ - bounded on $R_+$ and, in addition, $\lim_{t\to\infty} \| \Psi(t) y(t) \| = 0$, for any solution y(t).

ii). we can choose $\alpha$, $\beta \in R$ as above such that all characteristic roots of A have real parts greater than $\beta$.

In this case, $\Lambda_1 = \emptyset$, $P_1 = 0$, $P_2 = I$. Let $\gamma_2 < \beta$ be and $\Psi(t) = e^{-\gamma_2 t} I_d$. Then we have

$$| \Psi(t)e^{tA}e^{-sA}\Psi^{-1}(s) | \leq Ke^{(\beta-\gamma_2)(t-s)}, \text{ for } s \geq t \geq 0.$$

From Theorem 4, it follows that non-trivial solutions of (3) are Ψ - unbounded on $R_+$.

iii). we can choose $\alpha$, $\beta \in R$ as above such that $\Lambda_i \neq \emptyset$, i = 1, 2. Then, $P_i \neq 0$, $P_i \neq I$, i = 1, 2. Let $\gamma \in (\alpha, \beta)$ be and $\Psi(t) = e^{-\gamma t} I_d$. Then, we have

$$| \Psi(t)e^{tA}P_1e^{-sA}\Psi^{-1}(s) | \leq Ke^{(\alpha-\gamma)(t-s)}, \text{ for } t \geq s \geq 0$$

$$| \Psi(t)e^{tA}P_2e^{-sA}\Psi^{-1}(s) | \leq Ke^{(\beta-\gamma)(t-s)}, \text{ for } s \geq t \geq 0.$$

From Theorem 4, it follows that the system (3) has at least one, but not every, nontrivial Ψ - bounded solution y(t) on $R_+$; in addition, $\lim_{t\to\infty} \| \Psi(t)y(t)\| = 0$.

**Case II.** The characteristic roots of A have the same real part λ.

We consider 3 subcases:

a). Let $\gamma > \lambda$ and $\Psi(t) = e^{-\gamma t} I_d$. From Theorem 4, it follows that the solutions of (3) are Ψ - bounded on $R_+$ and, in addition, $\lim_{t\to\infty} \| \Psi(t)y(t)\| = 0$ for any solution y(t).

b). Let $\gamma < \lambda$ and $\Psi(t) = e^{-\gamma t} I_d$. From Theorem 4 it follows that nontrivial solutions of (3) are Ψ - unbounded on $R_+$.

c). Let $\gamma = \lambda$ and, e.g., $\Psi(t) = e^{-\gamma t}\varphi^{-1}(t) I_d$, where $\varphi : R_+ \longrightarrow (0,\infty)$ is a continuous function such that $t^{d-1} < \varphi(t)$, $t \geq 0$. Then, the solutions of (3) are Ψ - bounded on $R_+$.

**Remarks.** 1. In fact, the above results are true for $\gamma_1 \geq \alpha$, $\gamma_2 \leq \beta$ respectively $\gamma \in [\alpha, \beta]$.

2. We have a similar situation if A(t) is a d × d real continuous periodic matrix.

3. This Example generalizes a well-known result for boundedness of the solutions of (3).

**4. Ψ - boundedness of the solutions of the linear nonhomogeneous system (1).** In this section, we give sufficient conditions for Ψ - boundedness of the solutions of the linear nonhomogeneous system (1).

**Theorem 5.** *Suppose that:*

*1. the solutions of (2) are $\Psi$ - uniformly bounded on $R_+$,*

*2. the function f is such that*

$$\int_0^\infty \| \Psi(t)f(t) \| \, dt$$

*is convergent.*

*Then, the solutions of (1) are $\Psi$ - uniformly bounded on $R_+$.*

*If, in addition, the fundamental matrix Y(t) for (2) is such that $\lim_{t\to\infty} | \Psi(t)Y(t)| = 0$, then, the solutions of (1) are such that $\lim_{t\to\infty} \| \Psi(t)x(t)\| = 0$.*

**Proof.** Let Y(t) be a fundamental matrix for (2). From Theorem 2 it follows that there exists a constant $K > 0$ such that

$$| \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) | \ \leq \ K, \text{ for all } t \ \geq \ s \ \geq \ 0.$$

Let

$$L = \int_0^\infty \| \Psi(t)f(t) \| \, dt.$$

Let x(t) be the solution of (1) with the initial condition $x(t_0) = x_0$. Then, from the classical variation of parameters formula ( see [13] ), we obtain

$$x(t) = Y(t)Y^{-1}(t_0)x_0 + \int_{t_0}^t Y(t)Y^{-1}(s)f(s)ds,$$

for all $t \geq t_0$ ( in fact, for $t \geq 0$ ).

Thus, for $t \geq t_0$, we have

$$\| \Psi(t)x(t) \| \ \leq \ \| \Psi(t)Y(t)Y^{-1}(t_0)\Psi^{-1}(t_0)(\Psi(t_0)x_0) \| \ +$$

$$+ \int_{t_0}^t \| \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)(\Psi(s)f(s)) \| \, ds \ \leq \ K( \ \| \Psi(t_0)x_0 \| \ + L \ ).$$

It follows that the solutions of (1) are $\Psi$ - uniformly bounded on $R_+$.

Now, for $\varepsilon > 0$, there exists $t_1 \geq t_0$ such that

$$\int_{t_1}^\infty \| \Psi(s)f(s) \| \, ds < \frac{\varepsilon}{2K}.$$

Since $\lim_{t\to\infty} \mid \Psi(t)Y(t)\mid = 0$, there exists $t_2 \geq t_1$ such that

$$\mid \Psi(t)Y(t) \mid < \frac{\varepsilon}{2}\left[\parallel Y^{-1}(t_0)x_0 \parallel + \int_{t_0}^{t_1} \parallel Y^{-1}(s)f(s) \parallel ds\right]^{-1}, \text{ for } t \geq t_2.$$

Then, for $t \geq t_2$ we have

$$\parallel \Psi(t)x(t) \parallel \leq \mid \Psi(t)Y(t) \mid\parallel Y^{-1}(t_0)x_0 \parallel +$$

$$+ \mid \Psi(t)Y(t) \mid \int_{t_0}^{t_1} \parallel Y^{-1}(s)f(s) \parallel ds + K\int_{t_1}^{t} \parallel \Psi(s)f(s) \parallel ds < \varepsilon.$$

This shows that $\lim_{t\to\infty} \mid \Psi(t)x(t)\mid = 0$.

The proof is now complete.

**Remark.** Theorem 5 generalizes a similar result of A. Constantin [4] for $\Psi(t) = \varphi^k(t)I_d$.

The Theorem 5 is no longer true if we require that the function f is Ψ - bounded on $R_+$ instead of the condition

$$\text{" } \int_0^{\infty} \parallel \Psi(t)f(t) \parallel dt \text{ is convergent "}.$$

Even if the function f is such that $\lim_{t\to\infty} \parallel \Psi(t)f(t)\parallel = 0$, the Theorem 5 is no longer true.

This is shown by the next example.

**Example 3.** In system (2) we consider $A(t) = O_2$.

Then, a fundamental matrix for (2) is $Y(t) = I_d$.

Let

$$\Psi(t) = \begin{pmatrix} \frac{1}{t^2+1} & 0 \\ 0 & \frac{1}{t+1} \end{pmatrix}.$$

We have $\Psi(t)Y(t)=\Psi(t)$ and

$$\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) = \begin{pmatrix} \frac{s^2+1}{t^2+1} & 0 \\ 0 & \frac{s+1}{t+1} \end{pmatrix}, \text{ for } t \geq s \geq 0.$$

It follows that the solutions of (2) are Ψ - uniformly bounded on $R_+$ and $\lim_{t\to\infty} \mid \Psi(t)Y(t) \mid = 0$.

If we take $f(t) = ((t+1)\ln(t+1),\ \ln(t+1))^T$, then, we have

$$\Psi(t)f(t) = \begin{pmatrix} \frac{(t+1)\ln(t+1)}{t^2+1} \\ \\ \frac{\ln(t+1)}{t+1} \end{pmatrix}.$$

Therefore, $\lim\limits_{t\to\infty} \| \Psi(t)f(t)\| = 0$.

On the other hand, the solutions of the system (1) are

$$x(t) = \begin{pmatrix} C_1 + \frac{(t+1)^2}{4}\left[2\ln(t+1) - 1\right] \\ \\ C_2 + (t+1)\left[\ln(t+1) - 1\right] \end{pmatrix}.$$

Since $\lim\limits_{t\to\infty} \| \Psi(t)x(t)\| = \infty$, it follows that the solutions of the system (3) are $\Psi$ -unbounded on $R_+$ .

**Theorem 6.** *Suppose that:*

*1). the fundamental matrix Y(t) of (2) satisfies the condition*

$$\int_0^t \varphi(s) \mid \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) \mid ds \ \leq \ K, \text{ for all } t \ \geq \ 0,$$

*where K is a positive constant and $\varphi$ is a continuous positive function on $R_+$ such that $\int_0^\infty \varphi(s)ds=\infty$;*

*2). the function f is $\varphi^{-1}\Psi$ - bounded on $R_+$.*

*Then, the solutions of (1) are $\Psi$ - bounded on $R_+$ .*

*If, in addition, $\lim\limits_{t\to\infty} \| \varphi^{-1}(t)\Psi(t)f(t)\| = 0$, then $\lim\limits_{t\to\infty} \| \Psi(t)x(t)\| = 0$ for any solution x(t) of (1).*

**Proof.** From Theorem 2, [8], it follows that $\lim\limits_{t\to\infty} \mid \Psi(t)Y(t) \mid = 0$.

As in the proof of the above Theorem, if x(t) is a solution of (1), then, for all $t \geq t_0 \geq 0$, we have

$$\| \Psi(t)x(t) \| \ \leq \ \mid \Psi(t)Y(t) \mid\mid\mid Y^{-1}(t_0)x(t_0) \| +$$

$$+ \int_{t_0}^t \varphi(s) \mid \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) \mid\mid\mid \varphi^{-1}(s)\Psi(s)f(s) \| ds \ \leq$$

$$\leq \ \Psi(t)Y(t) \mid\mid\mid Y^{-1}(t_0)x(t_0) \| + K \sup_{s\geq 0} \| \varphi^{-1}(s)\Psi(s)f(s) \| .$$

Thus, the solution x(t) is Ψ - bounded on $R_+$ .

Now, for a given $\varepsilon > 0$, there exists $t_1 \geq t_0$ such that

$$\| \varphi^{-1}(t)\Psi(t)f(t) \| < \frac{\varepsilon}{2K}, \text{ for all } t \geq t_1.$$

Moreover, there exists $t_2 > t_1$ such that

$$| \Psi(t)Y(t) | < \frac{\varepsilon}{2}\left[\| Y^{-1}(t_0)x(t_0) \| + \int_{t_0}^{t_1} \| Y^{-1}(s)f(s) \| ds\right]^{-1}, \text{ for all } t \geq t_2.$$

Then, for $t \geq t_2$ we have

$$\| \Psi(t)x(t) \| \leq | \Psi(t)Y(t) | \| Y^{-1}(t_0)x(t_0) \| + \int_{t_0}^{t_1} \| \Psi(t)Y(t)Y^{-1}(s)f(s) \| ds +$$

$$+ \int_{t_1}^{t} \varphi(s) \, | \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) | \| \varphi^{-1}(s)\Psi(s)f(s) \| ds \leq$$

$$\leq | \Psi(t)Y(t) | \left[\| Y^{-1}(t_0)x(t_0) \| + \int_{t_0}^{t_1} \| Y^{-1}(s)f(s) \| ds\right] +$$

$$+ \frac{\varepsilon}{2K} \int_{t_1}^{t} \varphi(s) \, | \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) | \, ds < \varepsilon.$$

This shows that $\lim_{t\to\infty} \| \Psi(t)x(t)\| = 0$.

The proof is now complete.

**Remarks.** 1. Theorem 6 generalizes a similar result of Constantin, [4], Theorem 3.1, for $\Psi(t) = \varphi^k(t)I_d$ and $A(t) = A$.

2. Theorem 6 generalizes a result of O. Perron, [16], in conection with the linear equation $x' = a(t)x + b(t)$.

3. Hallam introduced in [11] two positive scalar continuous functions $\varphi$ and $\psi$ for to determine bounds on the solution vectors of (1). It is clear that the function $\varphi$ does not influence the conclusions of the theorem. The function $\varphi$ can only serve to weaken the required hypotheses on f.

Instead, the introduction of the matrix function Ψ permits to obtain a mixed asymptotic behavior of the solutions.

We give an example which shows some advantages that are obtained by introducing the scalar function $\varphi$ and the matrix function $\Psi$.

**Example 4.** Consider the linear system of ordinary differential equations

$$x' = \begin{pmatrix} -2a(t+1) - \frac{1}{t+1} & 0 \\ 0 & -2a(t+1) - \frac{2}{t+1} \end{pmatrix} x + b(t) \tag{4}$$

where $a > 0$, $x = (x_1, x_2)^T$ and $b(t) = (b_1(t), b_2(t))^T$ is a continuous vector function on $R_+$.

A fundamental matrix for the homogeneous system is

$$Y(t) = \begin{pmatrix} \frac{1}{t+1}e^{-a(t+1)^2} & 0 \\ 0 & \frac{1}{(t+1)^2}e^{-a(t+1)^2} \end{pmatrix}.$$

We have

$$Y(t)Y^{-1}(s) = \begin{pmatrix} \frac{s+1}{t+1}e^{-a(t+1)^2+a(s+1)^2} & 0 \\ 0 & \frac{(s+1)^2}{(t+1)^2}e^{-a(t+1)^2+a(s+1)^2} \end{pmatrix}.$$

Consider 4 cases:

**Case 1.** $\Psi(t) = I_2$ and $\varphi(t) = 1$.

The condition 1 of Theorem 6 is satisfied:

$$\int_0^t | \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) | ds \leq \frac{1}{2a}, \text{ for all } t \geq 0.$$

Then, if $b(t)$ is a bounded function on $R_+$, then the solutions of (4) are bounded on $R_+$.

We remark that, in fact, the above conditions for $A(t)$ and $b(t)$ are Perron's conditions for the linear equation $x' = a(t)x + b(t)$ ( see [16] ).

**Case 2.** $\Psi(t) = I_2$ and $\varphi(t) = t + 1$.

It is easy to see that the condition 1 of Theorem 6 is satisfied:

$$\int_0^t (s+1) | \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) | ds \leq \frac{1}{2a}, \text{ for all } t \geq 0.$$

Now, the condition 2). of Theorem 6 about b(t) is satisfied if b(t) is $\frac{1}{t+1}$ - bounded, that is if $\frac{b(t)}{t+1}$ is bounded on $R_+$.

Then, the solutions of (4) are bounded on $R_+$.

This is a weaker condition than the Perron's condition " b(t) is bounded on $R_+$".

**Case 3.** $\Psi(t) = (t+1)I_2$ and $\varphi(t) = t + 1$.

The condition of the Theorem 6 about Y(t) is satisfied.

Now, the condition 2). of Theorem 6 about b(t) is satisfied if b(t) is bounded on $R_+$.

Then, every solution x(t) of the equation (4) is such that $(t+1)x_1(t)$ and $(t+1)x_2(t)$ are bounded on $R_+$.

It is easy to see that for $x_1(t)$ and $x_2(t)$, this result is stronger than the result of PERRON, [16].

**Case 4.** $\Psi(t) = \begin{pmatrix} t+1 & 0 \\ 0 & (t+1)^2 \end{pmatrix}$ and $\varphi(t) = t + 1$.

It is easy to see that the condition 1 of Theorem 6 is satisfied:

$$\int_0^t (s+1) \mid \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) \mid ds \leq \frac{1}{2a}, \text{ for all } t \geq 0.$$

Now, the condition 2. of Theorem 6 about b(t) is satisfied if b(t) is $\begin{pmatrix} 1 & 0 \\ 0 & t+1 \end{pmatrix}$ - bounded on $R_+$, that is if $b_1(t)$ and $(t+1)b_2(t)$ are bounded on $R_+$.

Then, every solution x(t) of the equation (4) is such that $(t+1)x_1(t)$ and $(t+1)^2x_2(t)$ are bounded on $R_+$.

It is easy to see that for $x_1(t)$, this result is stronger than the O. Perron's result contained in [16].

On the other hand, we see that the asymptotic properties of the solutions $x_1(t)$ and $x_2(t)$ are not the same.

This is an advantage that is obtained by introducing the matrix function Ψ.

In the same manner as above, we can discuss about the second part of the Theorem 6.

As a Corollary of our Theorems 5 and 6, we have the following

**Corollary 1.** *Consider the linear equation*

$$x' = Ax + f(t) \tag{5}$$

*where A is a $d \times d$ real constant matrix.*

*Suppose that all characteristic roots of A have real parts less than $\alpha$, where $\alpha$ is a given real number. Let $\gamma \geq \alpha$ be and $\Psi(t) = e^{-\gamma t} I_d$.*

*i). Suppose that f is $\Psi$ - bounded on $R_+$.*

*Then, the solutions of (5) are $\Psi$ - bounded on $R_+$.*

*If, in addition, $\lim_{t\to\infty} e^{-\gamma t} \|f(t)\| = 0$, then $\lim_{t\to\infty} e^{-\gamma t} \| x(t) \| = 0$, for any solution x(t) of (5).*

*ii). Suppose that f is such that*

$$\int_0^\infty e^{-\gamma t} \| f(t) \| \, dt$$

*is convergent.*

*Then, the solutions of the system (5) are $\Psi$ - uniformly bounded on $R_+$ and $\lim_{t\to\infty} e^{-\gamma t} \| x(t) \| = 0$, for any solution x(t) of (5).*

The Theorem 6 is no longer true if we require that the solutions of linear system (2) are $\Psi$ - uniformly bounded on $R_+$ and $\lim_{t\to\infty} | \Psi(t)Y(t)| = 0$, instead of the condition

$$\int_0^t | \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) | \, ds \leq K, \text{ for all } t \geq 0.$$

This is shown by the next example.

**Example 5.** In the Example 3, we take $f(t) = (3t^2+1, 2t+1)^T$. We have $\| \Psi(t)f(t)\| \leq 3$ for all $t \geq 0$.

On the other hand, the solutions of (1) are $x(t) = (c_1 + t^3 + t, c_2 + t^2 + t)^T$, $t \geq 0$.

It is easy to see that all solutions of (1) are $\Psi$ - unbounded on $R_+$.

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*Department of Applied Mathematics,*
*University of Craiova*
*Fax: +(00)40-(0)51/435255*
*13, "Al.I. Cuza" str.,*
*1100, Craiova*
*ROMÂNIA*