ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLVIII, s.I a, Matematică, 2002, f.2.

ON A REVIEW BY M. HMISSI

BY

EUGEN POPA

The following is the review of a joint paper [3]:

Let $\Phi\colon \Omega \subset [0,\infty)\times X \mapsto X$ be an abstract local semidynamical system (ssd) without feebly critical points [O. Hajek, Dynamical systems in the plane, Academic Press, London, 1968; MR 39 #1767]. It is known that X endowed with the inherent topology is a locally compact metric space and (X,Φ) is a continuous ssd. The notion of tensor product of ssd was defined by Hajek, even for any family of ssd. On the other hand, the time change of ssd is also defined in the general case [D. H. Carlson, J. Differential Equations 11 (1972), 193–201; MR 44 #7078]. The authors consider some particular case of the preceding two notions in order to study the potential theory of ssd as introduced in the more general form by M. Hmissi [in Seminaire de Theorie du Potentiel, 135–144, Lecture Notes in Math., 1393, Springer, Berlin, 1989; Zbl 714.58029; Exposition. Math. 7 (1989), no. 3, 265–273; MR 90k:31010].

which was published as [4].

The purpose of this short note is to present a simple example which shows that the allegations of the reviewer are false.

For the sake of completeness, let us recall the relevant definitions from [2].

Definition 1. A *local dynamical system, or ld system* on X is a partial mapping $\Phi : \Omega \subseteq X \times (-\infty,+\infty) \to X$ which satisfies the following three conditions:

1. To each $x \in X$ there exists α_x, β_x in $[-\infty, +\infty]$ with $-\infty \leq \beta_x < 0 < \alpha_x \leq +\infty$ such that $(x,t) \in \Omega$ iff $\beta_x < t < \alpha_x$.

2. The initial value property $\Phi(x,0) = x$ obtains for all $x \in X$.

3. The group property $\Phi(\Phi(s,t_1),t_2) = \Phi(x,t_1+t_2)$ holds if both $(x,t_1) \in \Omega$ and $(x,t_1+t_2) \in \Omega$.

Definition 2. A *local semi–dynamical system (lsd system)* on X is a partial mapping $\Phi : \Omega \subseteq X \times [0,+\infty) \to X$ which satisfies the following three conditions:

1. To each $x \in X$ there is an α_x in $(0,+\infty]$ such that $(x,t) \in \Omega$ iff $0 \leq t < \alpha_x$.

2. and 3. as in the def. 1.

Definition 3. Let Φ be a given ld system on an abstract set X. A subset $D \subseteq X$ will be termed *t–open* if $x \in D$ implies $\Phi(x,t) \in D$, $\forall t \in (\varepsilon, \varepsilon)$, for some $\varepsilon > 0$. The system of t–open sets defines a topology on X, termed the *inherent* topology of Φ.

Let us stress that the inherent topology is defined in [2] **only** for ld systems (and not for lsd systems).

Definition 4. A topology τ on X is called compatible with a ld or lsd system Φ if Ω is open and Φ is continuous.

Let I be an infinite index set. Denote by

$$X := \{(-a,i) | a \in (0,+\infty), i \in I\} \cup [0,+\infty)$$

Define on a subset $\Omega \subset X \times (-\infty,+\infty)$ a map $\Phi : \Omega \to X$ as follows:

* for $a > 0$ and $i \in I$: $\Phi((-a,i),t) = (t-a,i)$ if $-\infty < t < a$;
$\Phi((-a,i),t) = t - a$ if $a \leq t$;
* for $x \geq 0$: $\Phi(x,t) = x + t$ if $-x < t$
(and not defined elsewhere).

Proposition 1. *(X,Φ) is a lsd system but not a ld system.*

Proof. 1. For $x = (-a,i)$ and $t < a$:
(i) if $s < a - t$ then:

$$\Phi(\Phi(x,t),s) = \Phi(\Phi((-a,i),t),s) = \Phi((t-a,i),s) = (s+t-a,i) =$$

$$= \Phi((-a,i),s+t) = \Phi(x,s+t)$$

since $s + t < a$.

(ii) if $s \geq a - t$ then:

$$\Phi\left(\Phi(x,t),s\right) = \Phi\left(\Phi((-a,i),t),s\right) = \Phi((t-a,i),s) = s + t - a =$$

$$= \Phi((-a,i), s+t) = \Phi(x, s+t)$$

since now $s + t \geq a$.

2. For $x = (-a, i)$ and $t \geq a$: the only case to be considered is: $a - t < s$ and we have

$$\Phi\left(\Phi(x,t),s\right) = \Phi\left(\Phi((-a,i),t),s\right) = \Phi(t-a,s) =$$
$$= s + t - a = \Phi((-a,i), s+t) = \Phi(x, s+t)$$

since $s + t > a$.

3. If $x \geq 0$ then, for $-x < t$ and $-x - t < s$ we have:

$$\Phi\left(\Phi(x,t),s\right) = \Phi(x+t,s) = s + t + x = \Phi(x, s+t)$$

Hence, except for the point $x = 0$, even the property 1. from the definition 1 of a ld system holds: if $x = (-a, i)$, then $\beta_x = -\infty$, $\alpha_x = +\infty$; if $x > 0$ then $\beta_x = -x$, $\alpha_x = +\infty$. Let us consider $x = 0$ and suppose that $\Phi(0, t)$ could be defined, for a $t < 0$. We prove that any choice produce a contradiction. Let us suppose first that $\Phi(0, t) = (-a, i)$. Let $j \neq i$. Then:

$$\Phi\left[\Phi((t,j),-t),t\right] = \Phi(0,t) = (-a,i)$$

On the other hand:

$$\Phi\left[\Phi((t,j),-t),t\right] = \Phi((t,j),0) = (t,j) \neq (-a,i)$$

Let us consider now the case $\Phi(0, t) = x \geq 0$. Then:

$$\Phi\left[\Phi(0,t),-t\right] = \Phi(x,-t) = x - t > x \geq 0$$

while:

$$\Phi\left[\Phi(0,t),-t\right] = \Phi(0,0) = 0$$

In the same manner, one shows that $\Phi(x, t)$ cannot be defined, for $x > 0$ and $t < -x$. Indeed, let us try first $\Phi(x, t) = (-a, i)$. For $s > a - t$, we get:

$$\Phi(\Phi(x,t),s) = \Phi((-a,i),s) = s - a$$

while:

$$\Phi(\Phi(x,t),s) = \Phi(x,t+s) = \Phi((-a,i),s+t) = s+t-a \neq s-a$$

The choice $\Phi(x,t) = y \geq 0$ leads also to a contradiction:

$$\Phi(\Phi(x,t),s) = \Phi(y,s) = y+s$$

and also:

$$\Phi(\Phi(x,t),s) = \Phi(x,t+s) = x+t+s \neq y+s$$

for $s \geq -x-t$.

Obviously, (X,Φ) is also a semi–dynamical system in the sense of Bucur–Bezzarga [1] and used in [3].

We are going to prove next that no "inherent topology" may be considered on this lsd system.

Proposition 2. *Let τ be a Hausdorff separated topology on X, such that Φ is a continuous mapping. Then each neighbourhood of a point $x \in X$, $x \geq 0$ contains an interval $(x-\varepsilon, x+\varepsilon)$, for some $\varepsilon > 0$; also, each neighbourhood of a point $(-a,i)$ contains an "interval" of the form: $\{(-b,i)|b \in (a-\varepsilon, a+\varepsilon)\}$. Finally, each neighbourhood of the point 0 contains the union of "intervals" of the form: $\{(-b,i)|0 < b < \varepsilon_i\}$ and $[0,\varepsilon)$, for some $\varepsilon_i > 0$, $\varepsilon > 0$.*

Proof. 1. Let us write the continuity of Φ at a point $((-a_0,i),t_0)$ with $0 < t_0 < a_0$:

$\forall V$ neighbourhood for $(t_0 - a_0, i) = \Phi((-a_0,i),t_0)$, $\exists W$ neighbourhood for $(-a_0,i)$ and $\delta > 0$ (with $\delta < t_0$ such that $\forall t \in (t_0-\delta, t_0+\delta)$ and $\forall x \in W$ we have $\Phi(x,t) \in V$.

Taking in particular $x = (-a_0,i)$, we get that each neigbourhood V contains an "interval" $\{(-a,i)|a \in (a_0-\varepsilon, a_0+\varepsilon)\}$ for some $\varepsilon > 0$ and $\varepsilon < a_0$.

2. Let us write now the continuity of Φ at the point $((-a_0,i),a_0)$.

$\forall V$ neighbourhood for $0 = \Phi((-a_0,i),a_0)$, $\exists W$ neighbourhood for $(-a_0,i)$ and $\delta > 0$ (with $\delta < a_0$ such that $\forall t \in (a_0-\delta, a_0+\delta)$ and $\forall x \in W$ we have $\Phi(x,t) \in V$.

Taking in particular $x = (-a_0,i)$, we get that each neigbourhood V for 0 contains, for each $i \in I$, an "interval" $\{(-a,i)|a \in (0,\varepsilon_i)\}$ for some $\varepsilon_i > 0$ and also an interval $[0,\varepsilon_0)$.

3. Finally, let us write now the continuity of Φ at the point $((-a_0, i), t_0)$, with $t_0 > a_0$.

$\forall V$ neighbourhood for $t_0 - a_0$, $\exists W$ neighbourhood for $(-a_0, i)$ and $\delta > 0$ (with $\delta < t_0$ such that $\forall t \in (t_0 - \delta, t_0 + \delta)$ and $\forall x \in W$ we have $\Phi(x, t) \in V$.

Taking in particular $x = (-a_0, i)$, we get that each neigbourhood V for t_0 contains an interval $(t_0 - \varepsilon, t_0 + \varepsilon)$, for some $0 < \varepsilon < t_0$.

Hence, even if on this lsd system one cannot speak of an "inherent topology", the fact that each point in $X \setminus \{0\}$ satisfies the properties from the definition of a ld system suggest to consider the topologies on X, for which: each (or only one) point $x \in X$, $x > 0$ has as a fundamental system of neighbourhoods the intervals $(x - \varepsilon, x + \varepsilon)$, for $\varepsilon > 0$.

For such topologies, a fundamental system of neighbourhoods can be described. Let us write the continuity of Φ at $(0, t)$ with $t > 0$. From:

$\forall \varepsilon > 0 \exists V$ neighbourhood for 0 and $\delta > 0$ (and $\delta < t$) such that $\forall x \in V, \forall s \in (t - \delta, t + \delta) \Longrightarrow \Phi(x, s) \in (t - \varepsilon, t + \varepsilon)$.

For $x = (-a, i)$ this means that $s - a \in (t - \varepsilon, t + \varepsilon)$, hence $0 < a < \varepsilon - \delta$. For each $i \in I$ it follows that the intersection of V with the "branch" i is contained in an "interval" $\{(-a, i) | 0 < a < \varepsilon\}$ for some $\varepsilon > 0$.

For $x \geq 0$ the above continuity reads $x + s \in (t - \varepsilon, t + \varepsilon)$, hence $x < \varepsilon - \delta$, meaning again that $V \cap [0, +\infty)$ is contained in some interval $[0, \varepsilon)$.

In the same way, it follows that each point $(-a, i)$ has as a fundamental system of neighbourhoods the "intervals" of the form: $\{(-b, i) | b \in (a - \varepsilon, a + \varepsilon)\}$.

In order to describe the topologies of this form, let us consider the set $(0, +\infty)^{I \cup \{0\}}$, whose elements will be denoted as $\mathbf{e} := (\varepsilon_i)_{i \in I \cup \{0\}}$. For each such $\mathbf{e}$, let us denote $V_{\mathbf{e}} := \{(-a, i) | i \in I, 0 < a < \varepsilon_i\} \cup [0, \varepsilon_0)$. Each subset $\mathcal{V} \subseteq (0, +\infty)^{I \cup \{0\}}$, with the next two properties:

(i) $\forall i \forall \varepsilon > 0 \exists \mathbf{e} \in \mathcal{V}$ such that $\varepsilon_i < \varepsilon$

(ii) $\forall \mathbf{e}', \mathbf{e}'' \in \mathcal{V} \Longrightarrow \exists \mathbf{e} \in \mathcal{V}$ such that $\forall i : \varepsilon_i \leq \min(e'_i, e''_i)$

gives rise to a fundamental system of neighbourhoods for 0: $(V_{\mathbf{e}})_{\mathbf{e} \in \mathcal{V}}$.

As particular choices for $\mathcal{V}$, let us mention:

(i) $\mathcal{V} = (0, +\infty)^{I \cup \{0\}}$ (the finest topology);

(ii) $\mathcal{V} = \{\mathbf{e} | \varepsilon_i = \varepsilon, \forall i\}$ (the coarsest topology);

(ii) for each sequence of positive, real numbers $a_n \to 0$ and any partition (I_n) for I, let us define $\varepsilon_0 = \varepsilon$ and $\varepsilon_i = a_n . \varepsilon$ if $i \in I_n$.

In this way, we get examples of non–comparable topologies, for which even the borelian σ–algebra are different (some being countable generated, and others not).

This example is probably the simplest with these properties. There are classical examples, in which the useful topologies are not locally compact or metrizable. Let us recall the semi–groups of operators on (infinite dimensional) Banach spaces, with the norm or weak topologies.

Theorem 1. *There is no topology on X, which is separated Hausdorff, compatible with Φ, locally compact and such that the trace on $X \setminus \{0\}$ be the inherent topology.*

Proof. Let us suppose, on the contrary, that τ is such a topology. Let K be a compact neighbourhood for 0. Consider the following open covering $(D_i)_{i \in I \cup \{0\}}$. Choose D_0 an open set, such that $0 \in D_0 \subset K$ and such that, for any $i \in I$, there exists at least a point $(-a, i) \in K \setminus D_0$. For each $i \in I$, let D_i denote an "interval" of the form $\{(-a, i) | 0 < \alpha < a < \beta\}$, such that $K \subset \bigcup_{i \in I \cup \{0\}} D_i$.

Obviously, such a covering exists and it admits no finite sub–covering.

As a **conclusion**:

The first sentence of the review is correct (and this is the only one).

"It is known that X endowed with the inherent topology is a locally compact metric space and (X, Φ) is a continuous ssd." This is the key false statement: it was shown above that for an abstract local sd, there is no definition for an "inherent topology" and in fact there may well not exists acceptable, locally compact, metrizable topologies at all.

"The notion of tensor product of ssd was defined by Hajek, even for any family of ssd." This is another confusion, such a general definition being possible only when Φ is everywhere defined. If the life–times on factors are arbitrarily small, such a definition is not possible.

"On the other hand, the time change of ssd is also defined in the general case [D. H. Carlson, J. Differential Equations 11 (1972), 193–201; MR 44 #7078]." Again, this notion is not explicitly stated, being considered only in connection with a homeomorphism between two lsd (in the topological setting).

"The authors consider some particular case of the preceding two notions in order to study the potential theory of ssd as introduced in the more general form by M. Hmissi [in Seminaire de Theorie du Potentiel, 135–144, Lecture Notes in Math., 1393, Springer, Berlin, 1989; Zbl 714.58029; Exposition. Math. 7 (1989), no. 3, 265–273; MR 90k:31010]." Of course,

just the converse is true, the framework considered by M. Hmissi in his own cited papers is strictly particular to the one introduced in [1] and considered in [3].

The reviewer voluntarily mistakes "locally semi–dynamical system"(lsd) (where one cannot speak of "inherent topology") for "locally dynamical system"; this point was clearly explained to him (less the above example) during 1994 Kouty International Conference on Potential Theory.

For a correct and professional review of the same paper, see [5].

The Guide for reviewers in Mathematical Reviews says: "the reviewer should bear in mind that the author cannot reply to his criticism in the pages of MATHEMATICAL REVIEWS, thus the reviewer has but one opportunity to be right and fair."

It seems to me that M. Hmissi missed this opportunity.

REFERENCES

1. Bucur, G. and Bezzarga, M. – *Théorie du potentiel pour les systèmes semi–dynamiques,* Rev. Roum. Math. Pures Appl. t.XXXIX,no.5 (1994),pg.439–456 (also preprint 11/1993, IMAR Bucureşti).
2. Hajek, O. – *Dynamical systems in the plane,* Academic Press, London, 1968
3. Popa, E. and Popa, L. – *An example in the axiomatic potential theory: the semi-dynamical systems,* An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat. 40 (1994), no. 4, 383–396.
4. MR 97c:31017 by Mohamed Hmissi
5. Zbl. 0847.31012

Received: 15.III.2002

Faculty of Mathematics,
"Al.I. Cuza" University
6600 - Iaşi,
ROMÂNIA