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AN APPLICATION OF PHRAGMEN-LINDELÖFF PRINCIPLE TO THE STABILITY THEORY

BY

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Abstract. We apply Phragmen-Lindelöff Principle in order to obtain a sufficient condition for asymptotic stability for bounded C_0-semigroups generated by bounded spectral operators.

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1. Introduction. Let X be a complex Banach space, let $\mathcal{L}(X)$ be the space of all linear bounded operators acting from X to X, endowed with the usual sup-norm, let $A \in \mathcal{L}(X)$ and let us consider the linear differential system

$$x'(t) = Ax(t). \tag{1.1}$$

Then, the unique solution of the associated Cauchy problem with the initial datum $x(t_0) = x_0$ is given by

$$x(t; t_0, x_0) = \exp{(t - t_0)}\, A.$$

In the case in which X is finite-dimensional, the stability of the null solution of (1.1) may be characterized by the spectral properties of A. The null solution is stable (equivalently, uniformly stable) iff all the eigenvalues of A have non-positive real parts and those with zero real parts have the corresponding Jordan blocks reduced to a single element. The null solution is asymptotically stable (equivalently uniformly asymptotic, or exponentially stable) iff all the eigenvalues of A have negative real parts. See for instance [3], Section 5.2, p. 94. These characterizations, very elegant and beautiful, use the canonical Jordan form for linear operators on complex finite-dimensional vector spaces.

The purpose of the present paper is to get a partial generalization of these results to complex Banach spaces (not necessarily finite-dimensional). In order to do that we restrict ourselves to a special class of linear bounded operators which have a canonical Jordan form, namely the class of Dunford spectral operators.

2. Preliminaries. Let $\mathcal{B} = \mathcal{B}(\mathbb{C})$ the σ-algebra of all borelian subsets of the complex plane and let X be a complex Banach space.

Definition 2.1. An application $E : \mathcal{B}(\mathbb{C}) \to \mathcal{L}(X)$ is called a *spectral measure* if the following conditions:

(i) $E(\emptyset) = 0, \quad E(\mathbb{C}) = I$

(ii) for any $\sigma_1, \sigma_2 \in B$, $E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2)$

(iii) for any $x \in X$ and any $x^* \in X^*$ the application $\sigma \mapsto x^*E(\sigma)x$ is countably additive

are satisfied.

We note that from (ii) it follows that $E(\sigma)^2 = E(\sigma)$ for any $\sigma \in B$ and therefore all values of the spectral measure are bounded projections on X. From (iii), by applying Pettis' Theorem (see [5], Theorem 1, p. 318) one gets that, for every $x \in X$, the mapping $\sigma \mapsto E(\sigma)x$ is countably additive. Moreover, $\sigma \mapsto x^*E(\sigma)x$ is a finite complex measure and consequently it is bounded. See [5], Corollary 2, p. 319. By applying the Uniformly Boundedness Principle, one obtains that $\{E(\sigma)\,;\ \sigma \in B\}$ is uniformly bounded, i.e. there exists $M > 0$ such that

$$\|E(\sigma)\| \leq M, \quad \text{for all } \sigma \in \mathcal{B}. \tag{2.1}$$

The property (2.1) allows us to define the integral of a bounded Borel-measurable function at a first stage for simple functions, i.e. Borel measurable functions having a finite number of values, and then for arbitrary bounded, Borel-measurable functions. We note that this is possible because we have

$$\left\| \int f(\lambda)(E(d\lambda) \right\| \leq 4M \sup_{\lambda} |f(\lambda)| \tag{2.2}$$

for each Borel-measurable simple function. Since any bounded, Borel-measurable function can be uniformly approximated by simple functions (on a measure space with finite measure - we recall that the space $(\mathbb{C}, \mathcal{B}(\mathbb{C}), x^*E(d\lambda)x)$ enjoys this property), it is possible to extend the definition of the integral, by using (2.2), to all bounded Borel-measurable functions.

Definition 2.2. An operator $A \in \mathcal{L}(X)$ is called *spectral Dunford* if there exists a spectral measure $E : \mathcal{B}(\mathbb{C}) \to \mathcal{L}(X)$ such that

(i) $AE(\sigma) = E(\sigma)A$ for each $\sigma \in \mathcal{B}(\mathbb{C})$

(ii $\sigma(A|E(\sigma)X) \subseteq \overline{\sigma}$ for each $\sigma \in \mathcal{B}(\mathbb{C})$.

It is known that $E(\sigma(T)) = I$. See [6], Corollary 5, p. 1935. It is also known that a Dunford spectral operator A has a unique spectral measure which commutes with any operator commuting with A.

Definition 2.3. An operator $S \in \mathcal{L}(X)$ is called a *scalar operator* if it is spectral and

$$S = \int \lambda E(d\lambda).$$

Definition 2.4. An operator $N \in \mathcal{L}(X)$ is called *quasi-nilpotent* if

$$\lim_n \|N^n\|^{1/n} = 0 \text{ equivalently } \sigma(N) = \{0\}.$$

The key result we shall use is the following:

Theorem 2.1. *A linear bounded operator A is a spectral operator iff it has a representation $A = S + N$, where S is a scalar operator and N is a quasi-nilpotent operator commuting with S. This representation is unique and the operators A and S have the same spectrum and the same spectral measure.*

See [6], Theorem 5, p. 1939.

3. The results. If N is a nilpotent operator then $\{\exp(tN)\,;\ t \geq 0\}$ cannot be bounded except for $N = 0$. This remark plays an important role in the finite-dimensional case. If N is only quasi-nilpotent, then the family $\{\exp(tN)\,;\ t \geq 0\}$ may be bounded, even in Hilbert spaces.

Example 3.1. Let $V : L^2(0,1) \to L^2(0,1)$ be the Volterra operator

$$(Vx)(t) = \int_0^t x(s)\,ds, \quad x \in L^2(0,1).$$

The operator of interest will be $-V$. It is known that $\sigma(V) = \{0\}$ and consequently $\sigma(-V) = \{0\}$. See Problem 146, p. 93 in [7]. One may easily verify that the adjoint of V is given by

$$(V^* x)(t) = \int_t^1 x(s)\,ds, \quad x \in L^2(0,1).$$

By applying Problem 150, p. 95 in [7], we get for $A = (I+V)^{-1}$ that $\|A\| = 1$ and $\sigma(A) = \{1\}$. Indeed, we can write

$$\|(I+V)^{-1}x\| = (x,x) + (Vx,x) + (x,Vx) + (Vx,Vx) =$$

$$= \|x\|^2 + \|Vx\|^2 + ((V+V^*)x,x) \leq \|x\|^2$$

because

$$(V+V^*)x = \int_0^1 x(s)\,ds \ \text{ and } \ ((V+V^*)x,x) =$$

$$= \int_0^1 \left(\int_0^1 x(s)\,ds \right) \overline{x}(t)\,dt = \left| \int_0^1 x(s)\,ds \right|^2 \geq 0.$$

Therefore we have

$$\|(I+V)^{-1}x\| \geq \|x\|, \text{ for every } \ x \in L^2(0,1)$$

and consequently $\|(I+V)^{-1}\| \leq 1$. Following a suggestion of Zemánek [11] and applying Hille's exponential formula (see Theorem 8.3, p. 33 in [10]), we get

$$\|\exp(-tV)x\| = \lim_n \left\| \left[I - \frac{t}{n}(-V) \right]^{-n} \right\| =$$

$$= \lim_n \left\| \left(I + \frac{t}{n} V \right)^{-n} \right\| \leq \|x\|, \quad t \geq 0.$$

Thus $\{\exp[t(-V)],\ t \geq 0\}$ is a semigroup of contractions (hence it is bounded) and $-V \neq 0$. As a matter of fact, since $\sigma_p(-V^*) = \emptyset$, we have

$\sigma_p(-V^*) \cap i\mathbb{R} = \emptyset$ and $\sigma(-V^*) \cap i\mathbb{R} = \{0\}$. Consequently, by [1] and [9], $\lim_{t\to\infty} \exp(t(-V))x = 0$ for any $x \in L^2(0,1)$.

However, if N is quasi-nilpotent and the powers N^n tend to zero fast enough, then $\{\exp(tN),\ t \geq 0\}$ cannot be bounded except for $N = 0$.

Proposition 3.1. *Let $N \in \mathcal{L}(X)$ be such that $\{\exp(tN),\ t \geq 0\}$ is bounded. If*

$$\lim_n n\,\|N^n\|^{1/n} = 0, \quad \text{then} \quad N = 0.$$

Proof. Following an idea in [2], we will apply the Phragmen-Lindelöff principle. Let us consider the entire function $\varphi(z) = \exp(zN)$, $x \in \mathbb{C}$. Since N is quasi-nilpotent, φ is of order ≤ 1 and minimal type (this means that for each $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that $\|\varphi(z)\| \leq C_\varepsilon \exp(\varepsilon|z|)$, $z \in \mathbb{C}$). Since $\lim_n n\|N^n\|^{1/n} = 0$, we can prove that the entire function $\psi(z) = \exp(z^2 N) = \varphi(z^2)$, $z \in \mathbb{C}$ is of order ≤ 1 and minimal type. Indeed, for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$, we have

$$n\|N^n\|^{1/n} < \varepsilon, \quad \text{whence} \quad \|N^n\| < \frac{\varepsilon^n}{n^n}.$$

Denoting

$$M_\varepsilon = \max_{0\leq n\leq n_0} \frac{n^n\|N^n\|}{\varepsilon^n} \quad \text{and} \quad C_\varepsilon = \max\{1, M_\varepsilon\},$$

we obtain:

$$\|N^n\| \leq C_\varepsilon \frac{\varepsilon^n}{n^n}, \quad \text{for each} \quad n \in \mathbb{N}. \tag{3.1}$$

From (3.1) we get

$$\|\varphi(z)\| = \|\exp(zN)\| \leq C_\varepsilon \sum_{n=0}^{\infty} \frac{|z|^n}{n!} \frac{\varepsilon^n}{n^n} =$$

$$= C_\varepsilon \sum_0^\infty \frac{\left(\varepsilon^{1/2}|z|^{1/2}\right)^{2n}}{(2n)!} \frac{(2n)!}{n!n^n} =$$

$$= C_\varepsilon \sum_0^\infty \frac{\left(\varepsilon^{1/2}|z|^{1/2}\right)^{2n}}{(2n)!} \frac{(n+1)\ldots(n+2)}{n^n} \leq$$

$$\leq C_\varepsilon \sum_0^\infty \frac{\left(\varepsilon^{1/2}|z|^{1/2}\right)^{2n}}{(2n)!} \exp\left(\frac{1}{n} + \cdots + \frac{n}{n}\right) =$$

$$\leq C_\varepsilon \sum_0^\infty \frac{\left(\varepsilon^{1/2}|z|^{1/2}\right)^{2n}}{(2n)!} \exp\left(\frac{n+1}{2}\right) =$$

$$\leq C_\varepsilon e^{1/2} \sum_0^n \frac{\left(\varepsilon^{1/2}|z|^{1/2}e^{1/4}\right)^{2n}}{(2n)!} \leq$$

$$\leq C_\varepsilon e^{1/2} \exp\left(\varepsilon^{1/2}e^{1/4}|z|^{1/2}\right), \quad z \in \mathbb{C}.$$

Consequently,

$$\|\psi(z)\| = \|\varphi(z^2)\| \leq C_\varepsilon e^{1/2} \exp\left(\varepsilon^{1/2}e^{1/4}|z|\right), \quad \text{for each} \quad z \in \mathbb{C}$$

which shows that ψ has order ≤ 1 and minimal type. On the other hand

$$\{\psi(t),\ t \in \mathbb{R}\} = \{\exp(t^2N),\ t \in \mathbb{R}\} = \{\exp(tN),\ t \geq 0\}$$

is bounded by hypothesis. Thus, by applying the Phragmen-Lindelöff principle, i.e. Theorem 3.13.8 in [8], we deduce that ψ must be constant. So $\psi(z) = I$ for each $z \in \mathbb{C}$ and therefore $N = 0$. $\square$

Remark 3.1. The hypothesis $\lim_n n\|N^n\|^{1/n} = 0$ may be replaced by the apparently weaker form $\lim_n n\|N^n x\|^{1/n} = 0$ for each $x \in X$, or even by $\lim_n n|x^* N^n x|^{1/n} = 0$ for each $x \in X$ and $x^* \in X^*$.

Theorem 3.1. *Let $A \in \mathcal{L}(X)$ be a Dunford spectral operator and let $A = S + N$ be its canonical decomposition, where S is a Dunford scalar operator and N is quasi-nilpotent and $NS = SN$. If $\{\exp(tA),\ t \geq 0\}$ is bounded and*

$$\lim_n n\|N^n x\|^{1/n} = 0 \quad \text{for any} \quad x \in E(\sigma(A) \cap i\mathbb{R})X,$$

then

$$N|E(\sigma(A) \cap i\mathbb{R}) = 0 \tag{3.2}$$

and

$$\lim_n \exp(tA)x = 0 \quad \text{for any} \quad x \in E(\sigma(A) \cap \{\lambda \in \mathbb{C},\ \mathrm{Re}\lambda < 0\}), \tag{3.3}$$

where E is the spectral measure of A.

Proof. The first part is a direct consequence of Proposition 3.1. applied to $N|E(\sigma(A)\cap i\mathbb{R})X$ by observing that

$$\exp(tN) = \exp(t(A-S)) = \exp(tA)\exp(-tS)$$

is bounded on $E(\sigma(A)\cap i\mathbb{R})$. Indeed, we have

$$\exp(-tS) = \int_{\sigma(S)} \exp(-\lambda t)E(d\lambda),$$

whence

$$\exp(-tS|E(\sigma(A)\cap i\mathbb{R})X = \int_{\sigma(S)\cap i\mathbb{R}} \exp(-\lambda t)E(d\lambda)\,|E(\sigma(A)\cap i\mathbb{R}X)\ .$$

As for the second part, by virtue of the Riesz-Dunford analytic functional calculus (see [4]), we remark first that (3.3) holds for every $x\in E(\sigma(A)\cap\{\lambda\in\mathbb{C},\ \mathrm{Re}\lambda<-\alpha\}$, $\alpha>0$. Let $x\in E(\sigma(A)\cap\{\lambda\in\mathbb{C},\ \mathrm{Re}\lambda<0\}$. Then, by the countable additivity of the spectral measure E, we have $x=\lim_n x_n$, where $x_n\in E(\sigma(A)\cap\{\lambda\in\mathbb{C},\ -\frac{1}{n+1}<\mathrm{Re}\lambda\le-\frac{1}{n}\}$. Now, we can write

$$\exp(tA)x = \exp(tA)(x-x_n)+\exp(tA)x_n.$$

Since $\{\exp(tA),\ t\ge 0\}$ is bounded, we deduce

$$\|\exp(tA)(x-x_n)\| \le M\|x-x_n\| \le \frac{\varepsilon}{2} \ \text{ for any } \ n\ge n_0.$$

On the other hand, $\lim_n \exp(tA)x_n = 0$ for each $n\in\mathbb{N}$ and therefore, for a fixed $n\ge n_0$ we can find $t_0\ge 0$ such that, for every $t\ge t_0$,

$$\|\exp(tA)x_n\| \le \frac{\varepsilon}{2}$$

and this achieves the proof. □

Problem. If $\lim_{t\to\infty}\exp(tA)x=0$ for every $x\in E(\sigma(A)\cap i\mathbb{R})$, does it follows that $E(\sigma(A)\cap i\mathbb{R})=0$ and thus $S|E(\sigma(A)\cap i\mathbb{R})=0$?

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