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# COINCIDENCE OF GEODESIC CORRESPONDENCES IN REAL FINSLER MANIFOLDS

BY

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*Dedicated to Prof. Dr. Radu MIRON on the occasion of his 75th birthday*

**Abstract.** In this paper we present some results on the coincidence and fixed point properties of geodesic correspondences in Finsler manifolds. We follow the method of Frankel [5], which was generalized for the Finsler case first in [6] and [8].

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**1. Introduction.** T. Frankel ([5]) studied fixed points of correspondences of a Kähler manifold of positive bisectional curvature. His result was generalized by the second author to the case of Kähler Finsler manifolds [8]. In this paper we generalize Frankel's results to coincidence of correspondences of real Finsler manifold, not only for fixed points of a correspondence. The main result is the following.

**Theorem.** *Let $V$ and $W$ be two compact geodesic correspondence of a Finsler manifold $N$ of positive sectional curvature, one of them transversal. Then $V$ and $W$ have a coincidence.*

We also obtain results on coincidence of geodesic maps and consequently, fixed point properties of geodesic maps. We recall that examples of real Finsler manifolds of positive curvature have been obtained by R. Bryant [4] and Z. Shen [9].

**2. Preliminaries.** Let $M$ be a real manifold $M$ of dimension $n, (TM, \pi, M)$ the tangent bundle of $M$. The vertical bundle of the manifold

$M$ is the vector bundle $\overline{\pi} : \mathcal{V} \to TM$ given by $\mathcal{V} = \ker d\,\pi \subset T(TM)$. $(x^i)$ will denote local coordinates on an open subset $U$ of $M$, and $(x^i, y^i)$ the induced coordinates on $\pi^{-1}(U) \subset TM$. The radial vertical vector field $\iota$ is locally given by $\iota(u^a \frac{\partial}{\partial x^a}) = u^a \frac{\partial}{\partial y^a}|_u$.

A Finsler metric on $M$ is a function $F : TM \to \mathbb{R}_+$ satisfying the following properties:

(1) $F^2$ is smooth on $\widetilde{M}$, where $\widetilde{M} = TM \setminus \{0\}$,

(2) $F(u) > 0$, for all $u \in \widetilde{M}$,

(3) $F(\lambda u) = |\lambda| F(u)$ for all $u \in TM, \lambda \in \mathbb{R}$,

(4) For any $p \in M$ the indicatrix $I_x(p) = \{u \in T_pM | F(u) < 1\}$ is strongly convex.

A manifold endowed with a Finsler metric $F$ is called a Finsler manifold.

Condition (4) implies that the quantities $g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial x^i \partial x^j}$ means positive definite matrix, so a Riemannian metric $\langle , \rangle$ can be introduced in the vertical bundle $(\mathcal{V}, \overline{\pi}, TM)$.

On a Finsler manifold there does not exist, in general, a linear metrical connection. The analogue of the Levi - Civita connection lives just in the vertical bundle, however, there are several ones. In this paper we use the Cartan connection, which is a good vertical connection in $\mathcal{V}$, i.e. a $\mathbb{R}$ - linear map.

$$\nabla : \mathfrak{X}\widetilde{M} \times \mathfrak{X}(\mathcal{V}) \to \mathfrak{X}(\mathcal{V})$$

having the usual properties of a covariant derivations, metrical with respect to $g$, and 'good' in the sense that the bundle map $\Lambda : \widetilde{TM} \to \mathcal{V}$ defined by $\Lambda(X) = \nabla_X \iota$ is a bundle isomorphism when restricted to $\mathcal{V}$. The latter property induces the horizontal subspaces $H_u = \ker \Lambda$ for all $u \in \widetilde{M}$, which is direct summand of the vertical subspaces $V_u = \mathrm{Ker}(d\pi)_u$:

$$\tau_{\widetilde{M}} = \mathcal{H} \oplus \mathcal{V}$$

$\Theta : \mathcal{V} \to \mathcal{H}$ denotes the horizontal map associated to the horizontal bundle $\mathcal{H}$. For a tangent vector field $X$ on $M$ we have its vertical lift $X^V$ and its horizontal lift $X^H$ to $\widetilde{M}$.

Using $\Theta$ first we get the radial horizontal vector field $\chi = \Theta \circ \iota$. Secondly we can extend the covariant derivation $\nabla$ of the vertical bundle to the whole tangent bundle of $\widetilde{M}$: Denoting it with the same letter, for horizontal vector fields $H$ we have:

$$\nabla_X H = \Theta(\nabla_X(\Theta^{-1}(H)))\ \forall X \in \mathfrak{X}\widetilde{M}$$

and then, an arbitrary vector field $Y \in \mathfrak{X}\widetilde{M}$ is decomposed into vertical and horizontal parts, so

$$\nabla_X Y = \nabla_X Y^V + \nabla_X Y^H.$$

Thus $\nabla : \mathfrak{X}\widetilde{M} \times \mathfrak{X}\widetilde{M} \to \mathfrak{X}\widetilde{M}$ is linear connection on $\widetilde{M}$ induced by a good vertical connections, its torsion and curvature $\Omega$ are defined as usual. Specially the sectional curvature of $\nabla$ along a curve $\sigma$ is given as follows:

$$R_{\dot\sigma}(U^H, U^H) = \langle \Omega(\dot\sigma^H, U^H)U^H, \dot\sigma^H \rangle$$

for any $U \in \mathfrak{X}(M)$. This is called the horizontal flag curvature in [1]. A curve $\sigma : [0,1] \to M$ is called geodesic if $\nabla_{T^H} T^H = 0$, where $T = \dot\sigma$.

**3. Product of Finsler manifolds**. In this section we construct the product of Finsler manifolds. Let $(M_1, F_1), (M_2, F_2)$ be two Finsler manifolds with the Cartan connection. Consider the product manifold $M_1 \times M_2$ with the metric

$$F(v_1, v_2) = \sqrt{F_1^2(v_1) + F_2^2(v_2)},\ \forall (v_1, v_2) \in TM_1 \times TM_2$$

This is homogeneous, smooth and positive definite on $\widetilde{M_1} \times \widetilde{M_2}$ because $F_1, F_2$ have these properties on $\widetilde{M_1}, \widetilde{M_2}$. The Levi matrix of $F$ is positive definite on $\widetilde{M_1} \times \widetilde{M_2}$ because it is of the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ where $A, B$ are the Levi matrix of $F_1, F_2$.

Let $\mathcal{H}_1, \mathcal{H}_2$ be the horizontal bundles of the manifolds $(M_1, F_1), (M_2, F_2)$ and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

The metrics $F_1, F_2$ induce the Riemannian structures $\langle , \rangle_1$ and $\langle , \rangle_2$ on the horizontal bundles. It follows that on the bundle $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ we have the metric

$$\langle X + U, Y + V \rangle = \langle X, Y \rangle_1 + \langle U, V \rangle_2$$

The Cartan connection of the product manifold is related to the Cartan connections of $M_1$ and $M_2$ as follows:

$$\nabla_{X+U}(Y + V) = \nabla_X Y + \nabla_U V, \forall X, Y \in \mathfrak{X}(\mathcal{H}_1), U, V \in (\mathcal{H}_2).$$

The sectional curvature of $M_1 \times M_2$ satisfies the relation:

$$R(X+U, Y+V) = R(X,Y)+R(U,V),\ \forall X, Y \in \mathfrak{X}(\mathcal{H}_1),\ \text{and } U, V \in \mathfrak{X}(\mathcal{H}_2).$$

These results come from the characterization of warped product of Finsler manifolds, proved by the authors in [7].

It follows from [7] that a curve $\sigma = (\alpha, \beta)$ in the product manifold is a geodesic if and only of the components $\alpha$ and $\beta$ are geodesics in the factors.

**4. The main result.** Let $N$ be an $n$ - dimensional real differentiable manifold. A *correspondence* $V$ of a manifold $N$ with itself is an $n$ - dimensional submanifold of $N \times N$. Two correspondences, $V, W$ are said to have a *coincidence* iff $V \cap W \neq \emptyset$. A correspondence $V \subset N \times N$ is called *transversal* if $T_{(p,q)}V \oplus T_{(p,q)}(\{p\} \times N) = T_{(p,q)}(N \times N)$ and $T_{(p,q)}V \oplus T_{(p,q)}(N \times \{q\}) = T_{(p,q)}(N \times N)$ hold for all $(p,q) \in V$. Since $T_{(p,q)}(\{p\} \times N)$ and $T_{(p,q)}(N \times \{q\})$ are orthogonal, it follows that any vector orthogonal to $V$ at $(p,q)$ cannot be tangent to $\{p\} \times N$ or $N \times \{q\}$.

A correspondence $V \subset N \times N$ is said to be *geodesic* iff $V$ is a totally geodesic submanifold in $N \times N$.

Now we can prove the main Theorem stated in the Introduction.

**Proof.** We denote $M = N \times N$. We assume that $V$ and $W$ do not intersect each other. Then there is a shortest geodesic $\sigma(t)$ from $V$ to $W$ with the endpoints $\sigma(a) \in V$ and $\sigma(b) \in W$ resp., for $V$ and $W$ are compact. By the Gauss lemma this geodesic strikes the manifolds $V$ and $W$ orthogonally.

We horizontally lift all the quantities from the tangent level to the second tangent level $TTM$ (in fact to $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$, where $\mathcal{H}_1$ is the horizontal bundle associated to $N$). The horizontal lift from $T_xM$ to $\mathcal{H}$ will be denoted by superscript $H$.

We consider $P \in H_{\dot\sigma(a)}\widetilde{M}$ the parallel translated of $T^H_{\sigma(a)}V$ with respect to the Cartan connection along $\dot\sigma$ to the point $\dot\sigma(b)$. The parallel transport defined by the Cartan connection maps horizontal vectors into horizontal one, $\dot\sigma^H(a)$ into $\dot\sigma^H(b)$, and it is angle preserving and dimension preserving. It follows that $\dot\sigma^H(b) \perp P$ and that implies dim $(P + T^H_{\sigma(b)}W) \leq$ dim $M - 1$. It follows that

$$\begin{aligned}
&\dim(P \cap T^H_{\sigma(b)}(W)) = \\
&= \dim P + \dim T^H_{\sigma(b)}(W) - \dim(P + T^H_{\sigma(b)}(W)) \geq \\
&\geq \dim V + \dim W - (\dim M - 1) = n + n - (2n-1) = 1.
\end{aligned}$$

We can choose a unit vector $w^H \in P \cap T^H_{\sigma(b)}M$. There exists a unit vector $v^H \in T^H_{\sigma(a)}V$ such that $w^H$ is obtained from $v^H$ by parallel translation. The parallel translation of $v^H$ along $\dot\sigma$ generates a vector field $U^H$. We consider the variation $\sum : (-\epsilon, \epsilon) \times [0,1] \to M$ of $\sigma$ with transversal vector field $X = d\pi(U^H)$. Then, by the second variation formula (cf. [1], p. 38) we have

$$\frac{d^2\ell_{\sum}}{ds^2}(0) = \langle \nabla_{U^H} U^H, T^H \rangle_{\dot\sigma}\Big|_a^b + \\ + \int_a^b \left[ \|\nabla_{T^H} U^H\|^2_{\dot\sigma} - \langle \Omega(T^H, U^H)U^H, T^H\rangle_{\dot\sigma} - \Big|\frac{\partial}{\partial t}\langle U^H, T^H\rangle_{\dot\sigma}\Big|^2 \right] dt$$

$U^H$ is parallel along $\dot\sigma$ and $T^H \circ \dot\sigma = \dot\sigma^H$, so $\nabla_{T^H} U^H|_{\dot\sigma} = \nabla_{\dot\sigma^H} = 0$. Thus the first term of the integral vanishes. So does the last term, for $U^H \perp T^H$ holds along $\dot\sigma$. The end terms can be omitted, since we can choose such variation where all transversal curves are geodesic, therefore $\nabla_{U^H} U^H = 0$. Since $V$ and $W$ are totally geodesic submanifolds in $M = N \times N$, this property ensures that the variation $\sum$ is proper, ie. the endpoints of the variational curves move in $V$, and $W$, resp. Summarizing we have

$$\frac{d^2\ell_{\sum}}{ds^2}(0) = -\int_a^b \langle \Omega(T^H, U^H)U^H, T^H\rangle_{\dot\sigma} dt = -\int_a^b R_{\dot\sigma}(U^H, U^H) dt < 0,$$

thus contradicting the minimality of $\sigma$. $\square$

It is clear that this Theorem yields that any correspondence $V$ of a Finsler manifold of positive sectional curvature has a fixed point. In fact, take the diagonal of $N \times N$ for $W$.

**5. Coincidence of geodesic maps.** A smooth map $f : N \to N$ gives rise to a correspondence, the graph $G(f)$ of $f; G(f) = \{(p, f(p)) | p \in N\}$. $G(f)$ is a special type of correspondence since $f$ is single valued. Let $\Delta = \{(p,p) | p \in N\}$ be the diagonal of $N \times N$. It is clear that a map $f$ has a fixed point whenever $G(f)$ intersects the diagonal $\Delta$. A correspondence will be said to have a fixed point if it intersects the diagonal.

Now we want to apply this theorem for geodesic maps. A smooth map $f : N \to N$ is called *geodesic* if for every geodesic curve $\alpha$ of $N$ the image $f(\alpha)$ is geodesic. For a detailed study on geodesic maps see Bácsó's works ([2,3]).

**Lemma.** *If $f$ is a geodesic map, then its graph $G(f)$ is a geodesic correspondence.*

**Proof.** If $\sigma = (\alpha, \beta) \subset G(f)$ is a geodesic curve of the graph $G(f)$, then $\alpha$ and $\beta$ are geodesic of $N$ by Theorem 9 of [7]. In this case $\beta$ is just the image $f(\alpha)$. $\square$

**Lemma.** *If $f : N \to N$ is a one-to-one map and $G(f)$ is a geodesic correspondence, then $f$ is a geodesic map.*

**Proof.** Let $\alpha$ be a geodesic curve in $N$, and $\beta = f(\alpha)$ its image, $\sigma = (\alpha, \beta)$. Let $\overline{\sigma} = (\overline{\alpha}, \overline{\beta})$ be a geodesic curve with $\dot{\overline{\sigma}}(0) = \dot{\sigma}(0)$. $\overline{\sigma}$ is in $G(f)$, for $G(f)$ is a totally geodesic submanifold. $\dot{\overline{\sigma}}(0) = \dot{\sigma}(0)$ implies $\dot{\overline{\alpha}}(0) = \dot{\alpha}(0)$ and $\dot{\overline{\beta}}(0) = \dot{\beta}(0)$. Therefore $\overline{\alpha} = \alpha$. The injectivity of $f$ implies that $\overline{\beta} = \beta = f(\alpha)$. Thus, again by Theorem 9 of [7] we get that $\beta$ is geodesic, which was to be proved. $\square$

These two lemmas show that the notion of geodesic correspondence is a natural extension of the notion of geodesic map. Let us consider now two geodesic maps $f, g : N \to N$. Then $G(f)$ and $G(g)$ give geodesic correspondence, each of them is transversal. Therefore from our Theorem we can deduce the following.

**Corollary.** *If $f, g : N \to N$ are geodesic maps of a Finsler manifold $N$ with positive sectional curvature, then they have a coincidence. Specially, $f$ always has a fixed point.*

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