ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLVIII, s.I a, Matematică, 2002, f.2.

ON THE FUNCTORIALITY OF A TORIC CONSTRUCTION

BY

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Abstract. In [3] Borel and Serre present the construction of some manifold with corners as the compactification of the quotient of the space X of maximal compact subgroups of a semi-simple $\mathbb{Q}$-algebraic group G by the action of a torsion free arithmetic subgroup Γ. In particular, a "corner" X(P) is associated to some fix parabolic subgroup P of G. In this note we prove the functoriality of Borel and Serre's construction of such a corner.

Mathematics Subject Classification: 20G30, 22F30, 14L40.

1. Borel and Serre's construction. In this section we recall the notations and the main steps of the construction in [3]. The properties of the linear algebraic groups and of their subgroups that are used in this construction can be found in [1], [2], [4].

Let k be a field of zero characteristic (usually a subfield of $\mathbb{R}$), G be an algebraic group over k and G^0 the connected component of the unity.

Let RG be the radical of G, R_uG it's unipotent radical and R_dG it's k-split radical. By definition, it is the maximal k-split connected subgroup of RG, it is normal and Zariski dense in G. A Levi subgroup of G is a maximal reductive k-subgroup. The Levi subgroups are R_uG-conjugated.

Definition 1.1. A $S-k$ space for G is a couple $(X,(L_x)_{x\in X})$ where X is a homogeneous G-space and $(L_x)_{x\in X})$ is a set of Levi subgroups of G such that:

(i) The isotropy subgroups H_x $(x \in X)$ of G in X are of the form $K \cdot S$, where S is a maximal torus in R_dG and K is a maximal compact subgroup in $G(\mathbb{R})$, contained in the normalizer of S.

(ii) $H_x \subset L_x$ and $L_{xg} = (L_x)^g$ for any $x \in X$ and $g \in G$.

Let S be a k-torus acting linearly on the vector space V. The weights of S in V are the characters $\chi \in X^*(S)$ such that $V_\chi = \{v \in V \ / \ s \cdot v = \chi(s)v \ , \ \forall s \in S\} \neq \{0\}$. A vector $v \in V_\chi$, $v \neq 0$ is called weight vector of S, with weight χ. Let T be a maximal torus of G and S a singular sub-torus of codimension 1. Taking $V = \mathfrak{g} = Lie(G)$, the weights $\chi \neq 0$ of T such that $S = (\ker\chi)^0$ and $Lie(Z_G S/R_u(Z_G S)) \cong Lie(Z_G T) \oplus kX_\chi \oplus kX_{-\chi}$, where X_χ and $X_{-\chi}$ are weight vectors for χ and $-\chi$ respectively, form the set of roots of G by respect to T, $R = R(G,T)$. This set is determined by G modulo isomorphisms, as the maximal torus are conjugated.

Let P be a parabolic $\mathbb{R}$-subgroup of G, Z the center of P/R_uP and $\gamma : X \times Z \to Z$, $(x,z) \mapsto x \circ z = x \cdot z_x$ the geodesic action of Z on the $S-k$ space X, where "$\cdot$" is the action of G on X and $z_x \in P$ (see [3]).

Let S be a maximal k-split torus of G^0/R_uG, ${}_k\phi = R(G^0/R_uG, S)$ the set of k-roots of G^0/R_uG by respect to S. Let Δ be a basis of $R(G^0/R_uG, S)$. There exists a bijection between the set of conjugacy classes in $\beta(G)$, the set of maximal parabolic k-subgroups of G, and the set of subsets of Δ (note also that $\beta(G)$ corresponds bijectively to $\beta(G/R_uG)$ and $\beta(G/R_dG)$ via the canonical projections). The conjugacy class corresponding to $J \subset \Delta$ is represented by the standard parabolic subgroup P_J, with the image P_J/R_uG in G/R_uG; we have $R_dP_J = S_J \cdot R_uP_J$, where $S_J = \left(\cap_{\alpha\in J} \ker\alpha\right)^0$.

Let $P \in \beta(G)$. The quotient

$$S_P = R_dP/(R_uP \cdot R_dG)$$

is a maximal k-split torus in $\mathcal{C}(P/(R_uP \cdot R_dG))$. Let $A_P = S_P^0$. If $P' \in \beta(G)$ is conjugated to P via G^0, then we have the canonical isomorphism $\sigma_{P',P} : S_{P'} \simeq S_P$. If we choose $P' = P_I$ standard, then $S_{P'} = S_I/S_\Delta$. The elements of $\Delta \setminus I$ induce a basis in $X^*(S_I/S_\Delta) \otimes \mathbb{Q}$. So we have the natural isomorphism [3]:

$$\rho : A_P \simeq (\mathbb{R}_+^*)^{\Delta\setminus I}$$

which we can also view as an open embedding of A_P in $\mathbb{R}^{\Delta\setminus I}$, whose closure is $\overline{A_P} = \mathbb{R}_+^{\Delta\setminus I}$.

Now let X be a homogeneous space of type $S-k$ for G. It is a principal A_P-bundle under the geodesic action. The corner associated to P is the total space of the associated bundle with the standard fibre $\overline{A_P}$ ([3]):

$$X(P) = X \times^{A_P} \overline{A_P}$$

So $X(P)$ is the quotient space of the product $X(P) = X \times \overline{A_P}$ by the equivalence relation: $(x,s) \sim (x',s')$ if and only if there exists $a \in A_P$ such that $x = x' \circ a$ and $z' = az$. The space $X(P)$ has a natural structure of manifold with corners induced by the one of the fibres.

2. The main result.

Theorem 2.1. *Let G and G' be two linear algebraic k-groups and $f : G \to G'$ be a group homomorphism. Let $P \subset G$, $P' \subset G'$ be parabolic k-subgroups of G and G' such that $f(P) \subseteq P'$. Let X and X' be $S-k$ homogeneous spaces under G and G' respectively and $F : X \to X'$ a function compatible with f and the structures described in the definition 1.1. Let*

$$X(P) = X \times^{A_P} \overline{A_P} \qquad (A_P \simeq (\mathbb{R}_+^*)^{\Delta \setminus I})$$

$$X'(P') = X' \times^{A_{P'}} \overline{A_{P'}} \qquad (A_{P'} \simeq (\mathbb{R}_+^*)^{\Delta' \setminus I'})$$

be the associated corners. Then there exists a function

$$Xf : X(P) \to X'(P')$$

defined by factorization from

$$F : X \to X' \quad , \quad \overline{F_A} : \overline{A_P} \to \overline{A_{P'}}$$

such that the association

$$(G, P, X) \quad \to \quad (G', P', X')$$

$$(f, F) \quad \to \quad Xf$$

is functorial.

Proof. We shall construct Xf following the steps of Borel and Serre's construction.

We can suppose that P and P' are standard parabolic subgroups, $P = P_I$ $(I \subset \Delta)$, $P' = P_{I'}$ $(I' \subset \Delta')$. The morphism f induces a group homomorphism $f^* : X^*(G') \to X^*(G)$ such that $f^*(\Delta') \subseteq \Delta$, as ${}_k\phi$, ${}_k\phi'$ are uniquely determined, modulo isomorphism, by G and G' respectively. As $f(P) \subseteq P'$, we have $f^*(I') \subseteq I$. Thus

$$f(S_I) = f\left((\cap_{\alpha \in J} \ker \alpha)^0 \right) \subset S_{I'} = (\cap_{\alpha \in J} \ker \alpha)^0$$

As $f(S_\Delta) \subseteq S_{\Delta'}$, it follows that f induces an morphism $\tilde{f} : S_P \to S_{P'}$. But $A_P = S_P^0$ and $f(S_P^0) \subseteq (f(S_P))^0$ and so we get the morphism $f_A : A_P \to A_{P'}$.

We have the canonical isomorphisms:

$$\begin{array}{cccc} \rho & : & A_P & \simeq & (\mathbb{R}_+^*)^{\Delta \setminus I} \\ & & f_A \downarrow & & \downarrow F_A \\ \rho' & : & A_{P'} & \simeq & (\mathbb{R}_+^*)^{\Delta' \setminus I'} \end{array}$$

and we obtain the morphism $F_A : (\mathbb{R}_+^*)^{\Delta \setminus I} \to (\mathbb{R}_+^*)^{\Delta' \setminus I'}$, $F_A = \rho' \circ f_A \circ \rho^{-1}$, that can be extended by continuity to $\overline{F_A} : \mathbb{R}_+^{\Delta \setminus I} \to \mathbb{R}_+^{\Delta' \setminus I'}$ which can be seen as $\overline{F_A} : \overline{A_P} \to \overline{A_{P'}}$.

By hypothesis the function F conserves the geodesic action: if $x_1 = x \circ a$,

$$x_1' = F(x_1) = F(x \circ a) = F(x \cdot a_x) = F(x) \cdot f(a_x) = F(x) \circ F_A(a)$$

We must prove now that by factorization we obtain a function $X(P) \to X'(P')$. Let $(x, z) \sim (x_1, s_1)$. This implies that there exists $a \in A_P$ such that $x = x_1 \circ a$ and $z_1 = az$, so we have $F(x) = F(x_1) \circ F_A(a)$ and $\overline{F_A}(z_1) = F_A(a)\overline{F_A}(z)$ and thus $(F(x), \overline{F_A}(z)) \sim (F(x_1), \overline{F_A}(z_1))$. It follows that we have a function $Xf : X(P) \to X'(P')$.

Consider now the sets of data (G, P, X), (G', P', X') and $(G", P", X")$ and the morphisms $f : G \to G'$, $f' : G' \to G"$, $F : X \to X'$, $F' : X' \to X"$ which satisfy compatibility conditions like those in the statement of the theorem. Let $g = f' \circ f$, $G = F' \circ F$. It follows that $g(P) \subseteq P"$. We must show that $\overline{G_A} = \overline{F_A'} \circ \overline{F_A}$.

We have

$$g^* : X^*(G") \longrightarrow X^*(G) \quad , \quad g^* = f^* \circ f'^*$$

Supposing that $P = P_I$, $P' = P_{I'}$, $P" = P_{I"}$ are standard parabolic subgroups, we have $f^*(I') \subseteq I$, $f'^*(I") \subseteq I'$ and so $g^*(I") \subseteq I$ and $g(S_I) \subseteq S_{I"}$. It follows that g induces in a natural way, by factorization, a morphism $\tilde{g} : S_P \to S_{P"}$, where $S_P = S_I/S_\Delta$, $S_{P"} = S_{I"}/S_{\Delta"}$. As $g = f' \circ f$, it follows that $\tilde{g} = \tilde{f'} \circ \tilde{f}$ and $g{=}f_A' \circ f_A$ for the restrictions to $A_P = S_P^0$, $A_{P'} = S_{P'}^0$.

We obtain the diagram

$$\begin{array}{ccccc} \rho & : & A_P & \simeq & (\mathbb{R}_+^*)^{\Delta\setminus I} \\ & & f_A \downarrow & & \downarrow F_A \\ \rho' & : & A_{P'} & \simeq & (\mathbb{R}_+^*)^{\Delta'\setminus I'} \\ & & f'_A \downarrow & & \downarrow F'_A \\ \rho" & : & A_{P"} & \simeq & (\mathbb{R}_+^*)^{\Delta"\setminus I"} \end{array}$$

and obviously

$$G_A = \rho" \circ g_A \circ \rho^{-1} = \left(\rho" \circ f'_A \circ (\rho')^{-1}\right)\left(\rho' \circ f_A \circ \rho^{-1}\right) = F'_A \circ F_A$$

and, extending by continuity, $\overline{G_A} = \overline{F'_A} \circ \overline{F_A}$.

As for the couples $(F, \overline{F_A})$, $(F', \overline{F'_A})$ the equivalences used for defining the varieties $X(P)$, $X'(P')$, $X"(P")$, the relations $G = F' \circ F$, $\overline{G_A} = \overline{F'_A} \circ \overline{F_A}$ imply $X(f' \circ f) = X(f') \circ X(f)$.

The equality $X(id_G) = id_{X(P)}$ is obvious, and so the correspondence

$$\begin{array}{rcl} (G, P, X) & \to & (G', P', X') \\ (f, F) & \to & Xf \end{array}$$

is functorial. ■

Acknowledgements. This note is a result of the research work realized by the author when he was undergraduate student at the University “Al. I. Cuza” of Iaşi. I express my gratitude to Prof. Liliana Răileanu for proposing the subject and for continuous support.

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Received: 23.IX.2002

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