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SHAPE THEORY AN "APPROXIMATION" THEORY FOR TOPOLOGICAL SPACES

BY

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Abstract. We establish in term of an adjunction situation a necessary and sufficient condition for a subcategory to be dense in a category. So, the shape theory in sense of MARDEŠIC, S. and SEGAL, J. from [3] is an approximation theory for topological spaces in sense of category theory considered by CORDIER, J.M. and PORTER, T. in [5] and by MACLANE, S. in [2].

For $\mathcal{C}$ an arbitrary category we consider the category $pro\mathcal{C}$ [3,Ch. I]. Its objects are inverse systems over a directed set Λ, denoted $\underline{X} = (X_\lambda, p_\lambda^{\lambda'}, \Lambda)$ and its morphisms are classes $[f] = \left\{\underline{g} | \underline{g} \sim \underline{f}\right\}$, $\underline{f} = (f_\mu, \varphi) : \underline{X} \longrightarrow \underline{Y} = (Y_\mu, q_\mu^{\mu'}, M)$, $f_\mu : X_{\varphi(\mu)} \to Y_\mu$ morphism in $\mathcal{C}$, $\varphi : M \to \Lambda$ morphism of directed sets, such that $f_\mu p^\lambda_{\varphi(\mu)} = q_\mu^{\mu'} f_{\mu'} p^\lambda_{\varphi(\mu')}$ for $\lambda \geq \varphi(\mu), \varphi(\mu')$ and $\underline{f} \sim \underline{g} = (g_\mu, \psi) : \underline{X} \to \underline{Y}$ iff there exists an increasing function $\phi : M \longrightarrow \Lambda$ such that $\phi \geq \varphi, \psi$ and $f_\mu p^{\phi(\mu)}_{\varphi(\mu)} = g_\mu p^{\phi(\mu)}_{\psi(\mu)}$ for $\mu \in M$.

In [5, p.44] a $pro - object$ in $\mathcal{C}$, denoted (Λ, F), is a covariant functor $F : \Lambda \to \mathcal{C}$, $F(\lambda) = X_\lambda$, $\lambda \in \Lambda$ where Λ is a category cofinally small and cofiltered.

We can consider the bonding morphisms $p_\lambda^{\lambda'} = F(\nu) : X_{\lambda'} \to X_\lambda$, $lambda' \geq \lambda$, with $\nu : \lambda' \to \lambda$ morphism in Λ.

This $pro - object$ is called generalized inverse system in $\mathcal{C}$ and every such inverse system is isomorphic to an inverse system indexed by a cofinite directed ordered set.

From [3, p.22] we have the following definition for a dense subcategory:

Definition 1. Let $\mathcal{T}$ be a category and $\mathcal{P}$ a subcategory. The subcategory $\mathcal{P}$ is dense in the category $\mathcal{T}$ provided every object $X \in ob\,\mathcal{T}$ admits a $\mathcal{P}-extension$ $\underline{p} = (p_\lambda)_{\lambda\in\Lambda} : X \to \underline{X}$ with $\underline{X}$ in $pro\,\mathcal{P}$.

That is the following conditions are satisfied:

(i) $\underline{p}$ is a morphism in $pro\,\mathcal{T}$

(ii) for every inverse system $\underline{Y} = (Y_\mu, q_\mu^{\mu'}, M)$ in the subcategory $\mathcal{P}$ and any morphism $\underline{h} : X \to \underline{Y}$ in $pro\mathcal{T}$, there exists a unique morphism $\underline{f} : \underline{X} \to \underline{Y}$ in $pro\,\mathcal{P}$ such that $\underline{h} = \underline{f}\,\underline{p}$

or equivalent

(i) $\underline{p}$ is a morphism in $pro\,\mathcal{T}$

(AE1) for each $P \in ob\,\mathcal{P}$ and each $h : X \to P$ from $\mathcal{T}$ there is a $\lambda \in \Lambda$ and $f : X_\lambda \to P$ in $\mathcal{P}$ such that $f\,p_\lambda = h$.

(AE2) if $f, f' : X_\lambda \to P$ are morphisms in $\mathcal{P}$ and $fp_\lambda = f'p_\lambda$, then there is a bonding morphism $p_\lambda^{\lambda'} : X_{\lambda'} \to X_\lambda$ such that $fp_\lambda^{\lambda'} = f'\,p_\lambda^{\lambda'}$

From [5, p.53] we consider the following definition:

Definition 2. Let $K : \mathcal{P} \to \mathcal{T}$ be the inclusion functor and $X \in ob\,\mathcal{T}$.

A pro-object (Λ, F) in $\mathcal{P}$ is said to be $K-associated$ to X with pseudoprojections $(p_\lambda)_{\lambda\in|\Lambda|}$ if:

(B1) for all $\lambda \in |\Lambda|$, $p_\lambda \in \mathcal{T}(X, F(\lambda))$ and if $\nu \in \Lambda\,(\lambda', \lambda)\,, F(\nu)p_{\lambda'} = p_\lambda$

(B2) for all $P \in ob\mathcal{P}$ and $h \in \mathcal{T}(X, K(P)\,)$ there is some $\lambda \in |\Lambda|$ and $f \in \mathcal{T}(F(\lambda), K(P))$ such that $fp_\lambda = h$.

(B3) for all $P \in ob\mathcal{P}$ and $f, f\,' \in \mathcal{P}(F(\lambda), K(P))$ such that $f\,p_\lambda = f\,'\,p_\lambda$ there is a $\nu \in \Lambda\,(\lambda', \lambda)$ with $f\,F(\nu) = f\,'F(\nu)$.

Now we can prove the following result:

Proposition 1. *For $K : \mathcal{P} \longrightarrow \mathcal{T}$ the inclusion functor, the following conditions are equivalent:*

(i) $\mathcal{P}$ is a dense subcategory of $\mathcal{T}$

(ii) For every object $X \in ob\mathcal{T}$ there is $\underline{X}$ in $pro\mathcal{P}$, an object $K-$ associated with X.

Proof. We consider (Λ, F) an object in $pro\mathcal{P}, K-associated$ to $X \in ob\mathcal{T}$ with the pseudoprojections $p_\lambda : X \longrightarrow F(\lambda) = X_\lambda$, $p_\lambda \in \mathcal{T}(X, F(\lambda))$. From the condition (B1) we have for $F(\nu) = p_\lambda^{\lambda'}$, $\nu \in \Lambda(\lambda', \lambda)$, $p_\lambda^{\lambda'} p_{\lambda'} = p_\lambda$. So, $\underline{p}$ is a morphism in $pro\mathcal{T}$. The condition (B2) is equivalent with (AE1) for each $P \in ob\mathcal{P}$ and $h \in \mathcal{T}(X, P)$, $K(P) = P$, there is a $\lambda \in \Lambda$ and $f \in \mathcal{T}(X_\lambda, P)$ such that $fp_\lambda = h$. The condition (B3) is equivalent with the

condition (AE2): for all $P \in ob\mathcal{P}$ and $f, f' \in \mathcal{P}(X_\lambda, P)$ such that $fp_\lambda = f'p_\lambda$ there is $F(\nu) = p_\lambda^{\lambda'}$, $\nu \in \Lambda(\lambda', \lambda)$ with $fp_\lambda^{\lambda'} = f'p_\lambda^{\lambda'}$. For the pair $\left(\underline{p}, \underline{X}\right)$, $\underline{p} : X \longrightarrow \underline{X}$ morphism in $pro\,\mathcal{T}$, the conditions (AE1), (AE2) express that $\underline{p}$ is $\mathcal{P} - extension$ for X. This is true for all $X \in ob\mathcal{T}$. Then $\mathcal{P}$ is a dense subcategory of $\mathcal{T}$. ■

We want to characterize the dense subcategories with the condition of having a proadjoint for the inclusion functor $K : \mathcal{P} \longrightarrow \mathcal{T}$.

From [2] and [5] we have the following conditions equivalent to the definition of a proadjoint:

"A functor $F : \mathcal{C} \to \mathcal{C}'$ admits a proadjoint if and only if there is a functor

$g : pro\mathcal{C}' \to pro\mathcal{C}$ such that for every object $X \in ob\mathcal{C}$ and $\underline{Y'} \in ob\, pro\mathcal{C}'$, exists a natural isomorphism

$pro\mathcal{C}(g(\underline{Y'}), c(X)) \cong pro\,\mathcal{C}'(\underline{Y'}, c'F(X))$,

where $c : \mathcal{C} \to pro\mathcal{C}$, $c' : \mathcal{C}' \to pro\mathcal{C}'$ are the canonical embeddings".

Or equivalent, the following three conditions are satisfied:

(Ai) for any $Y \in ob\,\mathcal{C}'$, $X_1, X_2 \in ob\mathcal{C}$ and morphisms

$f_1 : Y \to F(X_1)$,

$f_2 : Y \to F(X_2)$ there is an $X \in ob\,\mathcal{C}$ with morphisms

$f : Y \to F(X)$,

$f'_1 : X \to X_1$,

$f'_2 : X \to X_2$ such that the diagram

$$\begin{array}{ccccc} & & Y & & \\ & f_1 \swarrow & | & \searrow f_2 & \\ F(X_1) & & f & & F(X_2) \\ & F(f'_1) \nwarrow & \downarrow & \nearrow F(f'_2) & \\ & & F(X) & & \end{array}$$

commutes: $F(f'_1)\, f = f_1$, $F(f'_2)\, f = f_2$.

(Aii) for any commutative diagram

$$Y \xrightarrow{f} F(X_1) \underset{\longrightarrow}{\overset{F(g_1)}{\longrightarrow}} F(X_2)$$

$$F(g_1)\, f = F(g_2)\, f$$ with Y in $\mathcal{C}'$, $X_1, X_2 \in ob\mathcal{C}$, there is an X in $\mathcal{C}$ and the morphisms $f_1 : Y \to F(X)$ in $\mathcal{C}'$, $f_2 : X \to X_1$ in $\mathcal{C}$ such that the

diagrams commute.

$$\begin{array}{ccccccccccccccc} & & & f & & & & F(g_1) & & & & & & g_1 & \\ Y & - & - & \to & F(X_1) & \to & & & F(X_2) & ,X & \to & X_1 & \to & & X_2 \\ & \searrow & & \nearrow & & \to & & & & & & & \to & & \\ & f_1 & & F(f_2) & & F(g_2) & & & & & f_2 & & g_2 & & \\ & & F(X) & & & & & & & & & & & & \end{array}$$

$g_1\, f_2 = g_2\, f_2$, $F(f_2)\, f_1 = f$.

(Aiii) for any Y in $\mathcal{C}\,'$, there is a small subcategory $\mathcal{C}(Y)$ of $\mathcal{C}$ such that every morphism $f : Y \to F(X)$, $X \in ob\,\mathcal{C}$ has a factorization

$$\begin{array}{ccccccc} & f_1 & & F(f_2) & & & \\ Y & \longrightarrow & F(X') & \longrightarrow & F(X) & \text{with} & X' \in |\mathcal{C}(Y)|\ ,\ f = F(f_2)f_1 \\ & \to & f & \to & \uparrow & & \end{array}$$

Now we can prove:

Proposition 2. *The inclusion functor $K : \mathcal{P} \hookrightarrow \mathcal{T}$ admits a proadjoint iff $\mathcal{P}$ is a dense subcategory of $\mathcal{T}$.*

Proof. For every pair of categories $(\mathcal{T}, \mathcal{P})$ with $\mathcal{P}$ a subcategory of $\mathcal{T}$ we consider the comma category of $X \in ob\,\mathcal{T}$ over $\mathcal{P}$ denoted X_P or $X \downarrow K$, with $K : \mathcal{P} \hookrightarrow \mathcal{T}$ the inclusion functor.

The category X_P is cofinally small iff the category $\mathcal{P}$ admits a small subcategory $\mathcal{P}' = \mathcal{P}(X)$ such that for every morphism $f \in \mathcal{T}(X, P)$, $X \in ob\,\mathcal{T}$, $P \in ob\,\mathcal{P}$, has a factorisation with $f' \in \mathcal{T}(X, P')$, $P' \in ob\ \mathcal{P}'$, $u f\,' = f$, $u \in \mathcal{P}(P', P)$. If the category X_P is cofinally small we have that the functor $K : \mathcal{P} \to \mathcal{T}$ satisfies the condition (Aiii) with $f : X \to K(P) = P$, $f \in \mathcal{T}(X, P\,)$, $f_1 = f\,' \in \mathcal{T}(X, P\,')$, $f_2 = u \in \mathcal{P}(P', P)$, $f = u\, f\,'$, $u = K(u)$. The comma category X_P is cofiltered iff the following two conditions are satisfied: (Ci) for any two morphisms $f_1, f_2 \in \mathcal{T}(X, P_i), i = 1, 2$, $P_i \in ob\ \mathcal{P}, i = 1, 2$ there is a morphism $f \in \mathcal{T}(X, P)$, $P \in ob\ \mathcal{P}$, and there are the morphisms $u_i \in \mathcal{P}(P, P_i), i = 1, 2$ such that $u_i\, f = f_{\,i}\,,\ \ i = 1, 2$

$$\begin{array}{ccccccc} & & & X & & & \\ & f_1 & \swarrow & & \searrow & f_2 & \\ P_1 & & & | \ f & & & P_2 \\ & & & | & & & \\ & u_1 & \nwarrow & \downarrow & \nearrow & u_2 & \\ & & & P & & & \end{array}$$

(Cii) If $f \in \mathcal{T}(X, P\,')$, $u_1, u_2 \in \mathcal{P}(P\,', P)$, $P, P\,' \in ob\,\mathcal{P}$, are morphisms such that $u_1\, f = u_2\, f$, then there are the morphisms $f\,' \in \mathcal{T}(X, P\,'')$, $u \in \mathcal{P}(P'', P\,')$, $P'' \in ob\ \mathcal{P}$, such that $u f' = f$, $u_1 u = u_2 u$.

$$\begin{array}{ccccccc} & & f & & & & \\ X & - & - & \longrightarrow & P' & \underset{\overrightarrow{u_2}}{\overset{u_1}{\rightrightarrows}} & P \\ & \searrow & & \nearrow & & & \\ & f' & & u & & & \\ & & P'' & & & & \end{array}$$

For the inclusion functor $K : \mathcal{P} \longrightarrow \mathcal{T}$ the conditions (Ai), (Aii) are equivalent with the conditions (Ci) with $u_1 = f_1'$, $u_2 = f_2'$, $X_1 = P_1$, $X_2 = P_2$ and (Cii) with $X_1 = P'$, $X_2 = P$, $X = P''$ of $\mathcal{P}$, $g_1 = u_1$, $g_2 = u_2$, $f_2 = u$ morphisms in $\mathcal{P}$. We have that the inclusion functor $K : \mathcal{P} \hookrightarrow \mathcal{T}$ admits a proadjoint if and only if for every $X \in ob\ \mathcal{T}$ the comma category $X \downarrow K$ is cofiltered, cofinally small. From [3, p. 23] we have: "A subcategory $\mathcal{P} \subseteq \mathcal{T}$ is dense in a category $\mathcal{T}$ iff for each object $X \in ob\ \mathcal{T}$ the comma category X_P of X over $\mathcal{P}$ is cofiltered and cofinally small". Then, we have the conclusion of the proposition.

Finally, for the construction of a shape category, one of the following conditions must be fullfiled for the inclusion functor $K : \mathcal{P} \hookrightarrow \mathcal{T}$

(i) $\mathcal{P}$ is a dense subcategory in $\mathcal{T}$

(ii) for every object X in $\mathcal{T}$, the comma category $X \downarrow K$ is cofiltered, cofinally small

(iii) the functor K has a proadjoint

(iv) for every $X \in ob\ \mathcal{T}$, there is $\underline{X}$ in $ob\ pro\mathcal{P}$, $K - associated$ with X.

Application. We consider $\mathcal{T} = H\,Top$ the homotopy category of topological spaces and $\mathcal{P} = H\,Pol$ the homotopy category of spaces having the homotopy type of polyhedra.

Let be $K : H\,Pol \longrightarrow H\,Top$ the inclusion functor. In [3,p.46] the author prove that $\mathcal{P}$ is a dense subcategory of $\mathcal{T}$ and construct the shape category $Sh_{(H\,Top,\,H\,Pol)} = Sh$ with the objects-all topological spaces and the morphisms $F = (\underline{p}, \underline{q}, [\underline{f}]) : X \longrightarrow Y$ the equivalences classes of morphisms $\underline{f} : \underline{X} \longrightarrow \underline{Y}$ of $pro - H\,Pol$, where $\underline{p} : X \longrightarrow \underline{X}$, $\underline{q} : Y \longrightarrow \underline{Y}$ are $H\,Pol - extensions$ for X and Y respectively and $\underline{f} : \underline{X} \longrightarrow \underline{Y}$,

$\underline{f\,'} : \underline{X\,'} \longrightarrow \underline{Y\,'}$, $\underline{f} \backsim \underline{f\,'}$ iff $\underline{f\,'}\underline{i} = \underline{j}\,\underline{f}$, $\underline{i} : \underline{X} \cong \underline{X\,'}$, $\underline{j} : \underline{Y} \cong \underline{Y\,'}$ are the natural isomorphisms between two different extensions of the same topological space.

We can consider the functor $g : H\,Top \longrightarrow pro\,H\,Pol$

$g(X) = \underline{X}$, $g[f] = [\underline{f}]$ for every $f : X \longrightarrow \underline{Y}$ and $\underline{f} : \underline{X} \longrightarrow \underline{Y}$ such that

$$\begin{array}{ccccc} & & f & & \\ & X & \longrightarrow & Y & \\ \underline{p} & \downarrow & & \downarrow & \underline{q} \\ & \underline{X} & \longrightarrow & \underline{Y} & \\ & & \underline{f} & & \end{array}$$

and we have that $pro\,H\,Pol\,(g(Y), c(X)) \cong H\,Top\,(Y, K(X))$ is a bijection. Here,

$g(Y) = \underline{Y}$ is $K - associated$ with $Y \in ob\,H\,Top$ and $X \in ob\,H\,Pol$,

$c : H\,Pol \longrightarrow pro\,H\,Pol$ is the canonical embedding.

For the shape category $Sh_{(Top,\,HPol)}$ or simply Sh we consider the shape functor $S : HTop \to Sh$ which fixes objects and for $[f] \in HTop(X, Y)$, $S([f]) = F, F = (\underline{p}, \underline{q}, [\underline{f}])$. The functor $G : Sh \to proHTop$, $G(X) = (\underline{X}, \underline{p})$, and for $F \in Sh(X, Y)$, $G(F) = [\underline{f}]$ is a proadjoint for S,

$proHTop(c(X), G(Y)) \cong Sh(S(X), Y)$

(a bijection between the shape morphisms $F \in Sh(X, Y)$ and the morphisms

$\underline{h} : X \to \underline{Y}$ of $proHTop$) for $X \in obHTop$, $Y \in obSh$, $c : HTop \to proHTop$ the canonical embedding.

Another type of approximation theory (strong shape approximation) for topological spaces is constructed if instead of $proHTop$ is considered $H(proTop)$ and the coherent homotopy category $CH(proTop)$,as in [4]. The strong shape category Ssh and the strong shape functor $S_1 : HTop \to Ssh$ is obtained. Using the telescope functor

$CH(proTop) \to H(proTop)$ the author defines an adjoint for the strong shape functor S_1. In [1] is given a factorization for the shape functor $S = S_2S_1$ with $S_2 : Ssh \to Sh$.

So, for the shape functor we have only a proadjoint functor and for the strong shape functor we have just an adjoint functor.

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