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A NOTE ON CERTAIN INTEGRAL AND DISCRETE INEQUALITIES

BY

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Abstract. In the present note we establish two new integral inequalities and their discrete analogues which can be used in the study of behaviour of solutions of certain differential and finite difference equations.

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1. Introduction. In 1984, Mate and Nevai [3] used the following inequalities to study the asymptotic behaviour of solutions of certain differential and difference equations.

Lemma 1. *Let $u(t)$ and $f(t)$ be real - valued nonnegative continuous functions defined for $t \in \mathbb{R}_+$ and $c \geq 0$ is a real constant. If*

$$u(t) \leq c + \int_t^\infty f(s)u(s)ds < \infty,$$

for $t \in \mathbb{R}_+$, then:

$$u(t) \leq c\ \exp\left(\int_t^\infty f(s)ds\right),$$

for $t \in \mathbb{R}_+$.

Lemma 2. *Let $u(n)$ and $f(n)$ be real - valued nonnegative functions defined for all integers n and $c \geq 0$ is a real constant. If*

$$u(n) \leq c + \sum_{s=n+1}^{\infty} f(s)u(s) < \infty,$$

for every integer n, then:

$$u(n) \leq c \exp\left(\sum_{s=n+1}^{\infty} f(s)\right),$$

for all integers n.

Integral and discrete inequalities of the above type, which provides explicit bounds on the unknown functions have proved to be very useful in the analysis of various problems in the theory of differential, integral and finite difference equations, see [1 - 8] and the references given therein. The main purpose of this note is to establish certain integral and discrete inequalities which claim their origin to the inequalities given in Lemmas 1 and 2, by using a fairly elementary analysis.

2. Main results. In what follows, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, \infty), \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ be subsets of $\mathbb{R}$. We use the usual conventions that empty sums and products are taken to be 0 and 1 respectively. We assume that all the functions which appear in the inequalities are real - valued and all the integrals, sums and products involved exists on the respective domains of their definitions.

A fairly general version of Lemma 1 is embodied in the following theorem.

Theorem 1. *Let $u(t), a(t), b(t), c(t)$ be nonnegative continuous functions defined for $t \in \mathbb{R}_+$. If*

$$u(t) \leq a(t) + b(t) \int_t^{\infty} c(s)u(s)ds, \tag{2.1}$$

for $t \in \mathbb{R}_+$, then:

$$u(t) \leq a(t) + b(t)A(t) \exp\left(\int_t^{\infty} c(s)b(s)ds\right), \tag{2.2}$$

for $t \in \mathbb{R}_+$, *where*

$$A(t) = \int_t^\infty c(s)a(s)ds, \tag{2.3}$$

for $t \in \mathbb{R}_+$

Proof. Define a function $z(t)$ by

$$z(t) = \int_t^\infty c(s)u(s)ds, \tag{2.4}$$

Then (2.1) can be written as

$$u(t) \leq a(t) + b(t)z(t). \tag{2.5}$$

From (2.4) and (2.5) we have:

$$z(t) \leq A(t) + \int_t^\infty c(s)b(s)z(s)ds, \tag{2.6}$$

where $A(t)$ is defined by (2.3). Clearly, $A(t)$ is nonnegative, continuous and nonincreasing function for $t \in \mathbb{R}_+$. First, we assume that $A(t) > 0$ for $t \in \mathbb{R}_+$. From (2.6) we observe that

$$\frac{z(t)}{A(t)} \leq 1 + \int_t^\infty c(s)b(s)\frac{z(s)}{A(s)}ds. \tag{2.7}$$

Now an application of Lemma 1 to (2.7) yields

$$z(t) \leq A(t) \exp\left(\int_t^\infty c(s)b(s)ds\right). \tag{2.8}$$

If $A(t)$ is nonnegative in (2.6), we carry out the above procedure with $A(t)+\epsilon$ instead of $A(t)$, where $\epsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\epsilon \to 0$ to obtain (2.8). The desired inequality in (2.2) follows from (2.5) and (2.8).

As an application of Theorem 1, we establish the following inequality which can be used in certain situations.

Theorem 2. *Let* $u(t), v(t), a(t), b(t), p(t), h_i(t)$ $(i = 1, 2, 3, 4)$ *be nonnegative continuous functions defined for* $t \in \mathbb{R}_+$ *and* $\mu \geq 0$ *is a real constant. If*

$$u(t) \leq a(t) + p(t)\left[\int_t^\infty h_1(s)u(s)ds + \int_t^\infty h_2(s)e^{\mu s}v(s)ds\right], \tag{2.9}$$

$$v(t) \leq b(t) + p(t)\left[\int_t^\infty h_3(s)e^{-\mu s}u(s)ds + \int_t^\infty h_4(s)v(s)ds\right], \tag{2.10}$$

for $t \in \mathbb{R}_+$, *then:*

$$u(t) \leq Q(t),\ \ v(t) \leq e^{-\mu t}Q(t), \tag{2.11}$$

for $t \in \mathbb{R}_+$, *where*

$$Q(t) = f(t) + p(t)\overline{A}(t)\exp\left(\int_t^\infty h(s)p(s)ds\right), \tag{2.12}$$

$$f(t) = a(t) + b(t)e^{\mu t}, \tag{2.13}$$

$$h(t) = \max\left[(h_1(t) + h_3(t)), (h_2(t) + h_4(t))\right], \tag{2.14}$$

$$\overline{A}(t) = \int_t^\infty h(s)f(s)ds, \tag{2.15}$$

for $t \in \mathbb{R}_+$.

Proof. From (2.10) we observe that

$$e^{\mu t}v(t) \leq e^{\mu t}b(t) + p(t)\left[\int_t^\infty h_3(s)u(s)ds + \int_t^\infty h_4(s)e^{\mu s}v(s)ds\right]. \tag{2.16}$$

Define a function $w(t)$ by

$$w(t) = u(t) + e^{\mu t} v(t). \tag{2.17}$$

From (2.9), (2.16) and (2.17) we observe that

$$w(t) \leq f(t) + p(t) \int_t^{\infty} h(s) w(s) ds. \tag{2.18}$$

The bounds in (2.11) follows from an application of Theorem 1 to (2.18) and splitting.

3. Discrete analogues. In this section, we establish the discrete versions of Theorem 1 and 2 which can be used in the study of certain finite difference and sum - difference equations.

Theorem 3. *Let $u(n), a(n), b(n), c(n)$ be nonnegative functions defined for $n \in \mathbb{N}_0$. If*

$$u(n) \leq a(n) + b(n) \sum_{s=n+1}^{\infty} c(s) u(s), \tag{3.1}$$

for $n \in \mathbb{N}_0$, then

$$u(n) \leq a(n) + b(n) B(n) \prod_{s=n+1}^{\infty} [1 + c(s) b(s)], \tag{3.2}$$

for $n \in \mathbb{N}_0$, where

$$B(n) = \sum_{s=n+1}^{\infty} c(s) a(s), \tag{3.3}$$

for $n \in \mathbb{N}_0$.

Proof. Define a function $z(n)$ by

$$z(n) = \sum_{s=n+1}^{\infty} c(s) u(s). \tag{3.4}$$

Then (3.1) can be written as

$$u(n) \leq a(n) + b(n)z(n). \tag{3.5}$$

From (3.4) and (3.5) we have

$$z(n) \leq B(n) + \sum_{s=n+1}^{\infty} c(s)b(s)z(s), \tag{3.6}$$

where $B(n)$ is defined by (3.3). Clearly $B(n)$ is nonnegative and nonincreasing function for $n \in \mathbb{N}_0$. First we assume that $B(n) > 0$ for $n \in \mathbb{N}_0$. From (3.6) we observe that

$$\frac{z(n)}{B(n)} \leq 1 + \sum_{s=n+1}^{\infty} c(s)b(s)\frac{z(s)}{B(s)}. \tag{3.7}$$

Define a function $v(n)$ by the right hand side of (3.7), then $\frac{z(n)}{B(n)} \leq v(n)$ and

$$v(n) - v(n+1) = c(n+1)b(n+1)\frac{z(n+1)}{B(n+1)} \leq c(n+1)b(n+1)v(n+1),$$

i.e.

$$v(n) \leq [1 + c(n+1)b(n+1)]v(n+1). \tag{3.8}$$

By setting $n = s$ and substituting $s = n, n+1, \ldots, m-1 (m \geq n+1$ is arbitrary in $\mathbb{N}_0)$, successively we get

$$v(n) \leq v(m) \prod_{s=n+1}^{m} [1 + c(s)b(s)]. \tag{3.9}$$

Noting that $\lim\limits_{m\to\infty} v(m) = 1$ and letting $m \to \infty$ in (3.9) we get

$$v(n) \leq \prod_{s=n+1}^{\infty} [1 + c(s)b(s)]. \tag{3.10}$$

Using (3.10) in $\frac{z(n)}{B(n)} \leq v(n)$ we get

$$z(n) \leq B(n) \prod_{s=n+1}^{\infty} [1 + c(s)b(s)]. \tag{3.11}$$

The proof of the case when $B(n) \geq 0$ can be completed as mentioned in Theorem 1. The desired inequality in (3.2) follows from (3.5) and (3.11).

Theorem 4. *Let $u(n), v(n), a(n), b(n), p(n), h_i(n)$ $(i = 1, 2, 3, 4)$ be non-negative functions defined for $n \in \mathbb{N}_0$ and $\mu \geq 0$ is a real constant. If*

$$u(n) \leq a(n) + p(n) \left[\sum_{s=n+1}^{\infty} h_1(s)u(s) + \sum_{s=n+1}^{\infty} h_2(s)e^{\mu s}v(s) \right], \tag{3.12}$$

$$v(n) \leq b(n) + p(n) \left[\sum_{s=n+1}^{\infty} h_3(s)e^{-\mu s}u(s) + \sum_{s=n+1}^{\infty} h_4(s)v(s) \right], \tag{3.13}$$

for $n \in \mathbb{N}_0$, then

$$u(n) \leq E(n), v(n) \leq E(n)e^{-\mu n}, \tag{3.14}$$

where

$$E(n) = g(n) + p(n)\overline{B}(n) \prod_{s=n+1}^{\infty} [1 + h(s)p(s)], \tag{3.15}$$

$$g(n) = a(n) + b(n)e^{\mu n}, \tag{3.16}$$

$$h(n) = \max[(h_1(n) + h_3(n)), (h_2(n) + h_4(n))], \tag{3.17}$$

$$\overline{B}(n) = \sum_{s=n+1}^{\infty} h(s)g(s), \tag{3.18}$$

for $n \in \mathbb{N}_0$.

The proof of this theorem follows by the same argument as in the proof of Theorem 2 and by making use of Theorem 3. Here we omit the details.

Finally, we note that there is no essential difficulty in obtaining variants of Theorems 2.6.2 (first part, p. 138.) and Theorems 2.6.4., 2.6.5 (pp. 142 - 144) given in [4] in the framework of Theorems 1 and 2 and also their discrete versions in the framework of Theorems 3 and 4 given above. Since this translation is quite straightforward and requires no fresh insight, we do not discuss it here.

REFERENCES

1. Bainov, D. and Simeonov, P. – *Integral Inequalities and Applications,* Kluwer Academic Publishers, Dordrecht, 1992.

2. DeBlasi, F.S. and Schines, J. – *On the stability of difference equations in Banach spaces,*An. Şti. Univ. "Al.I.Cuza", Iaşi, Sect. I, 20(1974), 65 - 80.

3. Mate, M. and Nevai, P. – *Sublinear perturbations of the differential equation* $y^{(n)} = 0$ *and of the analogus difference equation,* J. Differential Equations 53(1984), 234 - 257.

4. Pachpatte, B.G. – *Inequalities for Differential and Integral Equations,* Academic Press, New York and London, 1998.

5. Pachpatte, B.G. – *On some new discrete inequalities and their applications to a class of sum - difference equations,* An. Şti. Univ. "Al.I. Cuza" Iaşi, Sect. I, 24(1978), 315 - 326.

6. Pachpatte, B.G. – *On certain new finite difference inequalities,* Indian J. Pure Appl. Math. 24(1993), 373 - 384.

7. Pachpatte, B.G. – *Some new finite difference inequalities,*Computers Math. Applic. 28(1994), 227 - 241.

8. Papaschinopoulos, G. – *On the summable manifold for discrete systems,* Math. Japonica 33(1988), 457 - 468.

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