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SOME KINDS OF BEST APPROXIMATION PROBLEMS IN A LOCALLY CONVEX SPACE

BY

A.M. PRECUPANU and T. PRECUPANU

Abstract. In this paper we are concerned with the optimality of six kinds of best approximation problems with respect to a given family of seminorms. If the simultaneous best approximation problem has optimal solutions, we also establish several relationships between optimality properties of these problems.

0. Introduction. The best approximation problem in locally convex space with respect to a given family $\mathcal{P}$ of seminorms which generates the topology of the space can be considered in various ways. One of the possibilities is to define the best approximation problem with respect to each seminorm (equivalently with respect to Minkovski functional associated to a convex, absorbent and circled set). In this case using the associated separated space we obtain a best approximation problem in a normed linear space if their properties are considered separately for every seminorm and they are not globally regarded with respect to other seminorms of the family $\mathcal{P}$ (see for instance [9], [15], [18], [26-29], [31]). Another possibility arrives in the case of metrizable locally convex spaces when the best approximation problem can be formulated using cvasinormes or asymetrical norms (only positively homogeneous) ([1], [17], [32], [34], [35]). In [11] the problem is treated in normed modules over semifields while in [1], [12], [33] is considered the case of vectorial latices. Others generalizations can be obtained replacing the seminorms by some special functions (see [19], [20], [21], [27]). A treatment using Minkowski functional can be found in [2], [28] and a treatment using seminorms generated by equicontinuous set can be found

in [15], [31]. The Pareto (vectorial) optimization with respect to a family of seminorms also generates a special kind of best approximation problem in a locally convex space ([3], [6], [7]).

In this paper we shall define some types of best approximation problems with respect to a given family of seminorms on a linear space generalizing some known results in normed linear spaces.

1. Best approximation problems with respect to a family of seminorms. Let X be a real linear space and let $\mathcal{P} = \{p_i; i \in I\}$ be a family of seminorms on X. The family $\mathcal{P}$, consequently the index set I can be considered as on ordered set with respect to the usual order defined by $i \leq j$ if $p_i(x) \leq p_j(x)$ for all $x \in X$.

Let us consider a nonvoid subset $A \subset X$. For a fixed seminorm $p_i \in \mathcal{P}$, an element $\overline{x}_i \in A$ is called a p_i- best approximation (p_i -b.a.) of an element $x \in X$ through elements of A if

$$p_i(x - \overline{x}_i) \leq p_i(x - y), \text{ for all } y \in A, \tag{1}$$

i.e.

$$p_i(x - \overline{x}_i) = d_i(x; A), \tag{1'}$$

where

$$d_i(x; A) = \inf_{y \in A} p_i(x - y). \tag{2}$$

Denote by $P_i(x; A)$ the set of these elements $\overline{x}_i \in A$. The multivalued map $x \to P_i(x; A), x \in X$, is called the *projection map* on A with respect to the seminorm $p_i \in \mathcal{P}$. This map is *proper* if $P_i(x; A) \neq \emptyset$ for at least element $x \notin A$. If $X_i = X/\ker p_i$ is the associated separated space and $\widehat{p}_i$ is the corresponding norm, then by canonical map $Q_i : X \to X_i$ we obtain that $\overline{x}_i \in P_i(x; A)$ if and only if $\overline{x}_i \in A$ and $Q_i(\overline{x}_i) \in \widehat{P}_i(Q_i(x); Q_i(A))$, where $\widehat{P}_i$ corresponds to the best approximation problem in X_i.

Remark 1. If $x \in A + \ker p_i$ we obviously have

$$P_i(x; A) = (x + \ker p_i) \cap A, \ i \in I. \tag{3}$$

Hence, the p_i- b.a. problem is proper only for the elements $x \notin \overline{Q_i^{-1} \circ Q_i(A)}^{i}$.

We define the following best approximation problems for a certain element $x \in X$ with respect to a nonvoid subset $A \subset X$:

I. there exists $\overline{x} \in A$ such that $\overline{x} \in P_i(x; A)$, for any $i \in I$;

II. for every $i \in I$ there exists at least a p_i - b.a. in A;

III. there exists $i \in I$ such that $P_i(x; A) \neq \emptyset$;

IV. there exist $i_0 \in I$ and $\overline{x} \in A$ such that $\overline{x} \in P_i(x; A)$ for any $i \geq i_0$;

V. there exists $\overline{x} \in A$ such that if $p_i(x - x_1) \leq p_i(x - \overline{x})$, for all $i \in I$, where x_1 is a certain element of A, then $p_i(x - x_1) = p_i(x - \overline{x})$ for all $i \in I$;

VI. there exist $\overline{x} \in A$ and $i_0 \in I$ such that

$$\max_{i \in I} \inf_{y \in A} p_i(x - y) = \min_{y \in A} \sup_{i \in I} p_i(x - y) = p_{i_0}(x - \overline{x}). \tag{4}$$

The solutions of the problem I are called the elements of *simultaneous best approximation* of x by elements of A and the solutions of the problem V are called the elements of *vectorial (Pareto) approximation* of x by elements of A. The problem VI is a best approximation problem formulated as a *minimax problem*. This problem is proper only for the elements x for which there exists $y \in A$ such that $\sup_{i \in I} p_i(x - y) < \infty$. Obviously, the set of solutions of the problem I is the set $\bigcap_{i \in I} P_i(x; A)$ and the problem II has as solutions the set $\prod_{i \in I} P_i(x; A)$. The problem III has solutions if $\bigcup_{i \in I} P_i(x; A) \neq 0$; the solutions are the pairs $(i_0, \overline{x}_i) \in I \times A$ such that $\overline{x}_i \in P_i(x; A)$.

According to Remark 1, the existence of the solutions of the problem IV is proper only for the elements $x \notin \bigcup_{i \in I} (Q_i^{-1} \circ Q_i)(A)$ when we must have $\bigcup_{i \in I} \bigcap_{j \geq i} P_j(x; A) \neq \emptyset$.

An element $\overline{x} \in A$ is a solution of the vectorial problem V if and only if do not exist elements $x' \in A$ such that $p_i(x - x') \leq p_i(x - \overline{x})$ for all $i \in I$, where the inequality is strict for at least index $i \in I$. A pair $(i_0, \overline{x}) \in I \times A$ is a solution of the problem VI if and only if we have

$$p_i(x - \overline{x}) \leq p_{i_0}(x - y) \text{ for all } (i, y) \in I \times A. \tag{4'}$$

Remark 2. If $x \in A$, then all problems have solutions and these

solutions are determined according to the relation (3). Moreover, if the family $\mathcal{P}$ is separated then the problems I, V have only the solution $x \in A$ while the problems IV, VI have as solutions any pair $(i, x), i \in I$.

2. Properties of the solutions of the best approximation problems. Since the problems I - IV are dependent of $P_i(x; A)$, we can obtain immediately various properties of them using the linear normed space X_i associated to each seminorm $p_i, i \in I$. Thus, using the duality theory by conjugation we can point out the dual characterization of the p_i- b.a. problem in the convex case.

Theorem 1. *If A is a convex set in X, then an element $\overline{x}_i \in I$ is a p_i-b.a. of $x \in X$ by elements of A if and only if there exists $x_i^* \in X^*$ such that the following properties are fulfilled:*

(i) $p_i(\overline{x}_i - x) = |\overline{x}_i^*|_i = \sup\{\overline{x}_i^*(u); p_i(u) \leq 1\}$,

(ii) $\overline{x}_i^*(y - x) \geq p_i^2(\overline{x}_i - x)$ *for all* $y \in A$.

Proof. Let us consider the linear normed space $X_i = X/\ker p_i$ and the corresponding best approximation problem for $Q(x)$ with respect to $Q(A)$. Next we use the well-known dual characterization of the best approximation elements in liniar normed spaces (see, for instance [13], [9], [4]).

Corollary. *If $\overline{x}_i$ is a p_i -b.a. element of $x \in X$ by elements of a convex set $A \subset X$, then the following minimax equality is fulfilled:*

$$\min_{y \in A} \sup_{|x_i^*|=1} x_i^*(y - x) = \max_{|x_i^*|=1} \inf_{y \in A} x_i^*(y - x) = p_i(x - \overline{x}_i). \tag{5}$$

As in the case of linear normed spaces the existence of solutions of the problem I can be assured by a property of *approximative compacity* of A with respect to the family of seminorms $\mathcal{P}$ (see [10], [30]).

Definition. The set $A \subset X$ is called $\mathcal{P}$- *approximatively compact* if for each family of nets $(y_{ij})_{j \in J}, i \in I$, in A, such that the net $(p_i(x - y_{ij}))_{j \in J}$ converges to $d_i(x; A)$ for any $i \in I$, there exists a net $(y_k)_{k \in K}$ convergent to an element of A and it is a subnet of any net $(y_{ij})_{j \in J}, i \in I$.

Theorem 2. *The problem I has optimal solutions for every $x \in X$ (i.e. any element of the space X has simultaneously best approximation elements) with respect to every $\mathcal{P}$ - approximatively compact set $A \subset X$.*

Proof. Obviously, for each $i \in I$ and each $\varepsilon > 0$ there exists $x_{i,\varepsilon} \in A$ such that

$$p_i(x - x_{i,\varepsilon}) < d_i(x; A) + \varepsilon.$$

If $A \subset X$ is $\mathcal{P}$ - approximatively compact, the family of nets $(x_{i,\varepsilon})_{\varepsilon>0}, i \in I$, has a subnet $(x_k)_{k\in K}$ with the property mentioned in the last part of the above Definition. The limit of this subnet is a solution of the problem I. We mention the special case of the families of *hilbertian seminorms.* We recall that a seminorm p is hilbertian if it arries from an *inner- semiproduct* (see [22]), or equivalently the seminorm fulfils the *parallelogram law*, i.e.

$$p^2(x + y) + p^2(x - y) = 2[p^2(x) + p^2(y)] \text{ for all } x, y \in X \tag{6}$$

Consequently, we obtain the following generalization of a well-known result in Hilbert spaces.

Theorem 3. *Let X be a real linear space endowed with a family $\mathcal{P} = \{p_i; i \in I\}$ of hilbertian seminorms. A nonvoid convex and $\mathcal{P}$ - complete set $A \subset X$ has the property that every element of the space has a simultaneous best approximation if and only if for every $x \in X$ there exists a net $(x_j)_{j\in J} \subset A$ such that*

$$\lim_j p_i(x - x_j) = d_i(x; A) \tag{7}$$

for every $i \in I$.

Proof. Let $x \in X$ and $(x_j)_{j\in J} \subset A$ such that (7) is fulfilled. As in the case of Hilbert spaces, using (6) we can prove that $(x_j)_{j\in J}$ is a Cauchy net in A with respect to the locally convex topology generated by $\mathcal{P}$. Hence, according to the completeness hypothesis, it converges to a simultaneously best approximation of the element x by elements of A.

Conversely, if $\overline{x} \in A$ is a simultaneously best approximation of the element $x \in X$ by elements of the set A, then evidently the constant net $\overline{x}$ has the property (7).

The existence of solutions for the problem I leads in finding some simple properties concerning the existence of solutions for the other problems.

Theorem 4. *If the problem I has solutions, then:*
(i) all the problems II - V have solutions and

$$P_i(x;A) = (x - p_i^{-1}(d_i(x;A)) \cap A;$$

(ii) the solutions of the problem V coincide with the solutions of the problem I;
(iii) the problem VI has solutions if and only if there exists an $i_0 \in I$ such that

$$p_{i_0}(x - \overline{x}) = \sup_{i \in I}(x - \overline{x}) \tag{8}$$

where $\overline{x}$ is a solution of the problem I. Moreover, if (8) is fulfilled, then $(i_0, \overline{x})$ is a saddle point and $p_{i_0}(x - \overline{x})$ is the saddle value of the minimax problem VI.

Proof. (i), (ii) are immediately. In fact, (ii) is based on the general assertion that if the simultaneous approximation has solutions, also the vectorial (Pareto) approximation has solutions and these solutions are the same as the solutions of the simultaneous best approximation problem since we have a total order for the values of best approximations by seminorms $p_i, i \in I$.

Now, let us take $\overline{x}$, a solution of the problem I, i.e.

$$p_i(x - \overline{x}) \leq p_i(x - y) \text{ for all } y \in A \text{ and } i \in I. \tag{9}$$

If there exists $i_0 \in I$ for which (8) is fulfilled, using (9) for $i = i_0$ we obtain:

$$p_i(x - \overline{x}) \leq p_{i_0}(x - \overline{x}) \leq p_{i_0}(x - y), \text{ for all } i \in I, y \in A,$$

which says, according to (4'), that $(i_0, \overline{x})$ is a solution of the problem VI.

Conversely, if $(i_0, \widetilde{x}) \in I \times A$ is a solution of the problem VI, using (9) and (4') we obtain

$$p_i(x - \overline{x}) \leq p_i(x - \widetilde{x}) \leq p_{i_0}(x - \overline{x}) \text{ for all } i \in I,$$

which yields to (8). Hence, (iii) is proved.

Remark 3. If $\overline{x}$ is a solution of the problem I, then $(i_0, \overline{x})$ is a solution of the problem VI whenever i_0 has the property (8). If $(i_1, x_1) \in I \times A$ is

an arbitrary solution of the problem VI, then $(i_1, \overline{x})$ is also a solution of the problem VI for every solution $\overline{x}$ of the problem I. Moreover, if (8) is fulfilled and $\overline{x} \in P_{i_0}(x; A)$, then $(i_0, \overline{x})$ is a solution of the problem VI.

Concerning the existence of the solutions for the problem II we shall use a general result established by the second author in [24] for the global optimality of a family of optimization problems having a lower - semicontinuous value function.

Let V be a topological space and let $F : U \times V \to \overline{R}$ be a function which generates the family of optimization problems

$$(\mathcal{P}_v) \qquad \min\{F(u, v); u \in U\},\ v \in V.$$

Denote $h : V \to \overline{R}$ the *value function* defined by

$$h(v) = \inf_{u \in U} F(u, v), \tag{10}$$

where U is a nonvoid abstract set.

Lemma A.(Lemma 2.3. in [24]). *The problems $(\mathcal{P}_v)_{v \in V}$ have optimal solutions whenever $h(v)$ is finite and h is lower - semicontinuous on V if and only if the set*

$$H = \{(v, a) \in V \times R;\ \text{there exists } u \in U \text{ such that } F(u, v) \leq a\} \tag{11}$$

is closed in $V \times R$.

In the case of problem II, if we suppose that the index set I is topologized, taking $U = X, V = I$, and

$$F(u, i) = p_i(x - u) + I_A(u),\ u \in X, i \in I,$$

where I_A is the indicator function of the set A (i.e. $I_A(u) = 0$ for $u \in A$ and $I_A(u) = \infty$ for $u \in X \setminus A$), we remark that $h(i) = d_i(x; A), i \in I$. Consequently, h is finite on I and the existence of the solutions for the problem II reduces to the existence of the solutions for the corresponding problems $(\mathcal{P}_i), i \in I$.

Theorem 5. *The problem II has solutions for a given element $x \in X$ and the function $i \to d_i(x; A), i \in I$, is lower - semicontinuous on I if and only if the set*

$$H = \{(i, a) \in I \times R \text{ there exists } u \in A \text{ such that } p_i(u - x) \leq a\} \tag{12}$$

is closed in $I \times R$.

Obviously, we have

$$H = \{(i,a) \in I \times R; A \cap \overline{S}_i(x;a) \neq \emptyset\} = \tag{12'}$$
$$= \{(i,a) \in I \times R; x \in A - \overline{S}_i(0;a)\},$$

where

$$\overline{S}_i(u;a) = \{y \in X; p_i(y-u) \leq a\}, u \in X, a > 0.$$

Using the same Lemma A we can give a characterization for the global optimality of the problem II and also with respect to $x \in X$. In this case we take $V = I \times X$.

Theorem 6. *The problem II has solutions for every* $x \in X$ *and the function* $(i,x) \to d_i(x;A), (i,x) \in I \times X$, *is lower - semicontinuous on* $I \times X$, *if and only if the set*

$$\widetilde{H} = \{(i,x,a) \in I \times X \times R; \text{ there exists } u \in A \text{ such that } p_i(u-x) \leq a\} \tag{13}$$

is closed in $I \times X \times R$.

In this case we have

$$\widetilde{H} = \{(i, y-z, a) \in I \times X \times R; y \in A, z \in \overline{S}_i(0;a)\}. \tag{13'}$$

The sets A which have the above property are p_i- proximal for any $i \in I$, i.e. every element of X has a p_i- b.a. in A. The optimality condition from the Theorem 6 also assures the lower - semicontinuity of the function $d_i(x;A); (i,x) \in I \times X$.

Taking into account the Remark 1, it is useful to see for the problem III the elements $x \notin \bigcup_{i \in I} Q_i^{-1} \circ Q_i(A)$ for which we have solutions or equivalently, the elements $x \notin \bigcup_{i \in \overline{I}} Q_i^{-1} \circ Q_i(A)$ for which $P_i(x;A) = \emptyset$ for all $i \in I$. In this case we shall request that the family of optimization problems $(\mathcal{P}_v)_{v \in V}$ has not optimal solutions. With this aim in mind we shall use a general result established in an earlier paper [20] in which we request the upper - semicontinuity of the value function.

Lemma B. (Theorem 1.1. in [20].) *The problems* $(\mathcal{P}_v)_{v \in V}$ *have not optimal solutions for every* $v \in V$ *such that* $a_0 < h(v) < \infty$ *and the value*

function h is upper - semicontinuous on V if and only if the set

$$H_0 = \{(v,a) \in V \times R; a > a_0 \text{ and there exists } u \in U \text{ such that } F(u,v) \leq a\} \tag{14}$$

is open in $V \times R$, where

$$a_0 = \inf\{h(v); v \in V\}. \tag{15}$$

In the case of problem III, for a fixed element $x \in X$ we have

$$a_0 = \inf\{d_i(x;A); i \in I\}. \tag{16}$$

If we denote $R_0 = (a_0, x)$ we obtain the following characterization of the optimality for problem III if the index set I is topologized.

Theorem 7. *If for a given element $x \in X$, the function $i \to d_i(X,A)$, $i \in I$, is upper - semicontinuous on I and $d_i(x;A) > a_0$ for all $i \in I$, then the problem III has at least a solution if and only if the set $H \cap (I \times R_0)$ is not an open set in $I \times R$.*

The hypothesis concerning the existence of a topology on the index set I (equivalently the existence of a topology on the space of seminorms $\mathcal{P}$) is satisfied in many usual locally convex spaces. On the other hand, taking into account several minimax results, a such as hypothesis is very useful also in the study of the problem VI. In what follows we shall analyze two significant examples.

Example 1(the weak topology). Let X be a locally convex space endowed with the weak topology, i.e. the family of seminorms is the following:

$$\mathcal{P} = \{p_{x^*}; x^* \in X^*\}, \text{ where } p_{x^*}(x) = |x^*(x)|, x \in X. \tag{17}$$

In this case, the best approximation problems have solutions for too special sets A. For instance, if X is a Banach space, the problem II has solutions for bounded weakly closed sets only in the case in which the set A is weakly compact (see James Theorem [14], [13], [25]). Consequently, the problem II has solutions for every bounded weakly closed set only in reflexive Banach spaces. On the other hand, we observe that

$$\sup_{x^* \in X^*} p_{x^*}(x-y) = \infty \text{ if } x \neq y.$$

Hence, the problem VI is proper only in the trivial situation when A is singleton. But, the problem VI can be proper not only for singletons if the family $\mathcal{P}$ is replaced by subfamilies which also generate the weak topology. Thus, if X is a linear normed space considering $\{p_{x^*}; ||x^*|| \leq 1\}$, then the index set $I = \overline{S}_{X^*}(0;1)$ is just w^* - compact. Consequently, the problem VI has solutions if and only if the element x has a best approximation element with respect to the norm of the Banach space X (by virtue of the dual characterization of the best approximation elements, given in minimax form; see for instance [4], [15]).

Example 2. Let K be a topological space and let $C(K)$ be the linear space of all scalar continuous functions on K, endowed with pointwise convergence topology, that is the topology generated by the family of seminorms

$$\mathcal{P} = \{p_t; t \in K\}, \text{ where } p_t(x) = |x(t)|, x \in C(K). \tag{18}$$

In the special case when K is compact we see that the problem VI is related to the uniform approximation problem on K. The last situation presented in the previous example can be reduced to the best approximation in a closed linear subspace of $C(K)$, taking $K = \overline{S}_{X^*}(0;1)$ with respect to the w^* topology in X^*.

The problem I of simultaneous best approximation with respect to family $\mathcal{P}$ of seminorms is just the pointwise best approximation problem, which can be trivial in simple situations (e.g. if A is a linear subspace [12]).

The problem IV is a problem of type I for a section $\mathcal{P}_{i_0}$ of $\mathcal{P}$. The theorem 4 addapted to the family of seminorms $\mathcal{P}_{i_0}$ will generate similar best approximation problems in a weaker form. A special case appears if we consider directed families of seminorms when every section $\mathcal{P}_i$ generates the same locally convex topology on the space X.

For the problem V we shall use methods of vectorial optimization of Pareto type. In the case of closed convex sets using polarity calculus we can establish results of optimality by scalarization generated by duality theory (see [4], [15]).

In the following we shall point out some properties in the case of directed families of seminorms.

Theorem 9. *If the pair $(i_0, \overline{x}) \in I \times A$ is a solution of the problem VI, then it is also a solution of the problem IV and we have*

$$p_i(x - \overline{x}) = p_{i_0}(x - \overline{x}). \tag{19}$$

Conversely, if $(i_0, \overline{x}) \in I \times A$ is a solution of the problem IV and (19) is fulfilled, then $(i_0, \overline{x}) \in I \times A$ is a solution of problem VI for the section $\mathcal{P}_{i_0}$ of the family $\mathcal{P}$ of seminorms.

Proof. Indeed, if $(i_0, \overline{x})$ is a solution of problem VI, that is (4') is fulfilled, for $i \geq i_0$ we obtain

$$p_i(x - \overline{x}) \leq p_{i_0}(x - y) \leq p_i(x - y) \text{ for all } y \in A, \tag{20}$$

which shows that $\overline{x} \in P_i(x; A)$, for all $i \geq i_0$; hence $(i_0, \overline{x})$ is a solution of problem IV. Also, from (20) for $y = \overline{x}$ we have $p_i(x - \overline{x}) \leq p_{i_0}(x - \overline{x})$ which together with hypothesis $i \geq i_0$ leads to the equality (19).

Conversely, if $(i_0, \overline{x})$ is a solution of the problem IV we have $\overline{x} \in P_{i_0}(x; A)$ and according to equality (19) we obtain

$$p_i(x - \overline{x}) = p_{i_0}(x - \overline{x}) \leq p_{i_0}(x - y) \text{ for all } y \in A \text{ and } i \geq i_0,$$

which assures the second part of the theorem considering (4') for $\mathcal{P}_{i_0}$.

Remark 4. As we have seen in the above proof, if (19) is fulfilled and $\overline{x} \in P_{i_0}(x; A)$, then $(i_0, \overline{x})$ is a solution of the problem VI for $\mathcal{P}_{i_0}$. Therefore, $\overline{x} \in P_i(x; A)$ for all $i \geq i_0$ and consequently $(i_0, \overline{x})$ is a solution of the problem IV. Moreover, if $(i_1, \overline{x})$ is a solution of the problem VI for $\mathcal{P}_{i_0}$, then (19) is fulfilled for i_1 and so, every element $(i, \overline{x}) \in I \times A$ with $i \geq i_1$ is also a solution of the problem VI for $\mathcal{P}_{i_0}$. The problem VI have the same value for every section $\mathcal{P}_i, i \geq i_0$.

Remarks 5. Taking into account that the optimality of best approximation problem is not conserved if we take a comparable seminorm, the equality (19) is natural. But, if the best approximation value is the same, the elements of best approximation are conserved. Thus, if $d_i(x; A) = d_j(x; A)$, then every element $\overline{x}$ of p_j- b.a. is also a p_i- b.a. if $i \leq j$. Conversely, if $\overline{x}$ is a p_j- b.a. and $p_i(x - \overline{x}) = p_j(x - \overline{x}), j \leq i$, then $\overline{x}$ is also a p_i- b.a. Consequently, by virtue of the equality (19), if $\overline{x}$ is a p_{i_0}- b.a. then it

is also a p_i- b.a. for every $i \geq i_0$. In previous considerations, the conditions $d_i(x; A) = d_j(x; A)$, and $p_i(x - \overline{x}) = p_j(x - \overline{x})$ are not comparable.

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Faculty of Mathematics,
"Al.I. Cuza" University,
Bd. Carol I, nr. 11,
6600, Iaşi
ROMANIA