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Tomul XLVIII, s.I a, Matematică, 2002, f.2.

EPSILON SUBDIFFERENTIABILITY FOR VECTOR FUNCTIONS

BY

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This paper presents an approach for the ε-subdifferentials of vector functions. As usual, for a vector problem, we have two principal directions for study, the scalarization and the efficient points. In the second section we give some vector generalizations of the properties known for the real values functions using the scalarization method. In the third section we study the Pareto ε-subdifferential for a vector valued function by using a kind of ε-directional derivative defined with the aid of the approximative efficient points.

1. Preliminaries. In what follows the spaces under consideration are separated locally convex spaces if not stated otherwise. We will consider Z ordered by $Z_+ \subset Z$ a nonempty, convex cone ($\lambda Z_+ \subset Z_+, \forall \lambda \in R_+$, $Z_+ + Z_+ = Z_+$) which is pointed ($Z_+ \cap -Z_+ = \{0\}$). For $x_1, x_2 \in Z$, we will write $x_2 \leq_{Z_+} x_1$ or $x_1 \geq_{Z_+} x_2$ if $x_1 - x_2 \in Z_+$, $x_2 <_{Z_+} x_1$ or $x_1 >_{Z_+} x_2$ if $x_1 - x_2 \in Z_+ \setminus \{0\}$, $x_1 \nleq_{Z_+} x_2$ or $x_2 \ngeq_{Z_+} x_1$ if $x_2 - x_1 \notin Z_+$ and $x_1 \not>_{Z_+} x_2$ or $x_2 \not<_{Z_+} x_1$ if $x_1 - x_2 \notin Z_+ \setminus \{0\}$. If no confusion can happen, we will renounce to index the cone. We add to Z a smallest element denoted $-\infty$ and a biggest element denoted $+\infty$; we denote $\overline{Z} = Z \cup \{+\infty\} \cup \{-\infty\}$. We make the following conventions: $\lambda \cdot (\pm\infty) = \pm\infty$, $x + (+\infty) = +\infty$ for all $\lambda > 0$ and $x \in \overline{Z}$.

We recall the definitions of the different efficiencies which will be used in this paper.

Let $A \subset \overline{Z}, A \neq \emptyset, A \neq \{+\infty\}, \varepsilon \in Z_+$.

Definition 1.1.

1. We say that $p \in \overline{Z}$ is an $\varepsilon-$ infimum of A if there is no $a \in A$ such that $a <_{Z_+} p - \varepsilon$. The set of $\varepsilon-$infimum points of A will be denoted by ${}^{\varepsilon}INF\,(A \mid Z_+)$.

2. We say that $p \in \overline{Z}$ is an $\varepsilon-$minimum of A if p is $\varepsilon-$infimum of A and $p \in A$. The set of the $\varepsilon-$minimum points of A is denoted by ${}^{\varepsilon}MIN\,(A \mid Z_+)$.

3. We say that $p \in \overline{Z}$ is an $\varepsilon-$proximal infimum of A if p is $\varepsilon-$infimum of A and $\forall p' \in \overline{Z}, p' >_{Z_+} p$ there exists $a' \in A$ such that $a' <_{Z_+} p'$. The set of $\varepsilon-$proximal infimum points of A is denoted by ${}^{\varepsilon}INF_1(A \mid Z_+)$.

4. We say that $p \in A$ is ε-ideal minimum of A if $p \leq a$ for all $a \in A$. The set of ε-ideal minimum of A is denoted by ${}^{\varepsilon}IMIN\ (A \mid Z_+)$.

If the confusion is excluded, we will renounce to the cone index and we denote these sets simply ${}^{\varepsilon}INF\,A$, ${}^{\varepsilon}MIN\,A$, ${}^{\varepsilon}INF_1A$, ${}^{\varepsilon}IMINA$.
In a similar manner, for a set $A \subseteq \overline{Z}, A \neq \{-\infty\}$ we define the sets ${}^{\varepsilon}SUP\,A$, ${}^{\varepsilon}MAX\,A$, ${}^{\varepsilon}SUP_1A$ and ${}^{\varepsilon}IMAX\ A$.
For $\varepsilon = 0$, we meet the above notions to Isac and Postolică [11] and the minimum point for $\varepsilon = 0$ is the Pareto minimum points well-known in vectorial optimization. We mention that the notion SUP_1A is also used in the papers of S. Dolecki, C. Malivert and N. Boissard, defined with the aid of the superior closure of A.

Like usually, $\operatorname{Int} A$ and $\operatorname{cl} A$ denote the interior and respectively the closure of A.

If the interior of Z_+ is nonempty, then we denote by $w\ {}^{\varepsilon}INF\,A$, $w\,{}^{\varepsilon}MIN\ A$, $w\,{}^{\varepsilon}INF_1A$, $w\,{}^{\varepsilon}SUP\,A$, $w\,{}^{\varepsilon}MAX\,A$, $w\,{}^{\varepsilon}SUP_1A$ the sets of efficient points corresponding to the cone $K = \operatorname{Int} Z_+ \cup \{0\}$. The strong infimum will be denoted $\inf A = \mathrm{IMAX}\{\alpha \in Z \mid \alpha \leq a,\ \forall a \in A\}$.

The notion of $\varepsilon-$minimum is also used in [14], where we find a general notion of $H-$efficiency with H a subset of Z_+.

Definition 1.2. [14] Let H be a nonvoid set of $Z_+\backslash\{0\}$. A point $p \in A$ is $H-$efficient point for A if $(p - H - K) \cap A = \emptyset$. The set of H-efficient points is denoted by ${}^{H}Eff\,A$.

For a function $f : X \to \overline{Z}$, we note $\operatorname{dom} f = \{x \in X \mid\ f(x) < +\infty\}$. We say that f is proper if $f(x) \neq -\infty, \forall x \in X$ and $\exists x \in X$ such that $f(x) \in Z$.

The function f is convex if
$\forall x, y \in X, \lambda \in [0,1]\ \ f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$
with the conventions $+\infty + (-\infty) = +\infty, 0(+\infty) = +\infty, 0(-\infty) = 0$.
The function f is lower semicontinuous (l.s.c.) at x_0 if:
i)$f(x_0) = -\infty$
or ii)$f(x_0) \neq -\infty$ and for all neighbourdhood V for $f(x_0)$, there exists a neighbourhood U for x_0 such that $f(U) \subseteq V + Z_+$. [if $f(x_0) = +\infty$, V is a neighbourhood for $f(x_0)$ if and only if $\exists \lambda \in Z$ such that $(\lambda + Z_+) \cup \{+\infty\} \subseteq V$].

We recall now some usual definitions which will be useful in what follows.

Definition 1.3.

1. An ordered linear topological space Z o.l.t.s. is called *order complete* if every nonempty, upper bounded subsets of Z has a superior bound.
2. The cone Z_+ is Daniell if every decreasing net which has strong infimum is convergent to its infimum.
3. The cone Z_+ is normal if there is a fundamental system of neighbourhoods for 0, $\mathcal{V}_Z(0)$ such that
$$(V - Z_+) \cap (V + Z_+) = V,\ \forall V \in \mathcal{V}_Z(0).$$
4. [12] Let $f : X \to \overline{Z}$, $\varepsilon \in Z_+,\ x_0 \in X$ such that $f(x_0) \in Z$. The set given by
$$\partial^{\varepsilon}_{\leq} f(x_0) = \{T \in \mathcal{L}(X, Z) \mid\ T(h) \leq f(x_0 + h) - f(x_0) + \varepsilon,\ \forall\ h \in X\}$$
is called the ε-subdifferential of f at x_0.
5. [11] Let $f : X \to \overline{Z},\ \varepsilon \in Z_+,\ x_0 \in X$ such that $f(x_0) \in Z$. The set given by
$$\partial^{\varepsilon}_{\not>} f(x_0) = \{T \in \mathcal{L}(X, Z),\ T(h) \not> f(x_0 + h) - f(x_0) + \varepsilon,\ \forall\ h \in X\}$$
is called the Pareto ε-subdifferential of f at x_0.

Obviously,
$$\partial^{\varepsilon}_{\not>} f(x_0) \supset \partial^{\varepsilon}_{\leq} f(x_0)$$
for all $\varepsilon \in Z_+$ and $x_0 \in \text{dom}\ f$.

2. ε-subdifferentials via scalarization. Similarly to the real case, an existence result for ε-subdifferentials does holds for vector functions. Let recall the sandwich theorem established in [13] and which will be used for proving the existence theorem.

Theorem 2.1. *Let X and Z be two linear topological spaces such that Z is ordered by a convex, pointed, normal cone Z_+ such that Z is order complete. Let $-g : X \to \overline{Z}$ and $h : X \to \overline{Z}$ be two convex maps such that $g \leq h$, one of them finite and continuous at a point from the domain of the other. Under this hypothesis, there exists a continuous affine map f such that $g \leq f \leq h$.*

Proposition 2.1. *Let Z ordered by the normal cone Z_+ with nonempty interior which is order complete. If $f : X \to \overline{Z}$ is a convex, proper, lower semicontinuous function, then for all $\varepsilon \in \operatorname{Int} Z_+$ and $x_0 \in \operatorname{dom} f$ we have that $\partial^{\varepsilon}_{\leq} f(x_0) \neq \emptyset$.*

Proof. Since f is a convex, proper, lower semicontinuous function and X is a locally convex space, then for all $\varepsilon \in \operatorname{Int} Z_+$, there exists a convex, barrelled neighborhood of 0, $U \in \mathcal{V}_X(0)$ such that

$$f(x_0 + z) - f(x_0) + \varepsilon \geq 0, \; \forall\, z \in U.$$

Using the fact that Z is order complete, we derive that

$$\alpha = \inf\{f(x_0 + z) - f(x_0) + \varepsilon, z \in U\}$$

exists and α is positive.

Let define $p(z) = \begin{cases} \alpha & \text{if } z \in U \\ -\infty & \text{if } z \notin U \end{cases}$ and

$$q(z) = f(x_0 + z) - f(x_0) + \varepsilon, \; z \in X.$$

We observe that p is concave, p is finite and continuous at $0 \in \operatorname{dom} q$ and q is convex and

$$p(z) \leq q(z), \, \forall z \in X.$$

Using sandwich theorem (Theorem 2.1) we get $u \in \mathcal{L}(X, Z)$ and $c \in Z$ such that

$$p(z) \leq u(z) + c \leq q(z), \, \forall z \in X.$$

Particularly, for $z = 0$,

$$u(0) + c \geq p(0) = \alpha \geq 0$$

which implies $c \geq 0$ and $u(z) \leq q(z) - c \leq q(z)$. Thus, $u(z) \leq f(x_0 + z) - f(x_0) + \varepsilon$, $\forall z \in X$. Consequently, $u \in \partial^{\varepsilon}_{\leq} f(x_0)$. □

Proposition 2.2. *Let Z ordered by the closed cone Z_+ and let $f : X \to \overline{Z}$ be a proper function. Then, for all $\varepsilon \in Z_+$, $x_0 \in \operatorname{dom} f$ we have that $\partial^{\varepsilon}_{\leq} f(x_0)$ is a closed set in $\mathcal{L}_p(X, Z)$.*

Proof. We can rewrite the definition of the ε-subdifferential of f at x_0 like.

$$\partial^{\varepsilon}_{\leq} f(x_0) = \bigcap_{h \in X} \{T \in \mathcal{L}(X, Y),\, T(h) \leq f(x_0 + h) - f(x_0) + \varepsilon\}.$$

Since Z_+ is a closed cone we derive that the intersections sets are $\mathcal{L}_p(X, Z)$-closed and thus, $\partial^{\varepsilon}_{\leq} f(x_0)$ is a closed set in $\mathcal{L}_p(X, Z)$. □

For the real functions, we know that for a convex, proper, lower semicontinuous function, $\partial^{\varepsilon} f(x_0)$ is a equicontinuous set for $x_0 \in \operatorname{Int} \operatorname{dom} f$. A similar result does hold for the vector case.

Let recall a proposition due to Valadier [18] which will be used in what follows.

Proposition 2.3. *Let X be a Hausdorff locally convex space and Z be a linear topological space ordered by the convex, pointed, closed cone Z_+ such that the intervals $[x, z]_{Z_+}$ are topologically bounded. If $f : X \to \overline{Z}$ is a convex, proper function and if there exists a neighborhood for an interior point of* dom f *on which f is upper bounded then f is continuous on* Int dom f.

Remark 2.1. If the cone Z_+ is normal then the intervals are topologically bounded sets.

Proposition 2.4. *Let Z ordered by the closed cone Z_+ such that the intervals $[a, b]_{Z_+}$ are topologically bounded. If $f : X \to \overline{Z}$ is a convex, proper function, $x_0 \in \operatorname{Int} \operatorname{dom} f$ such that f is upper bounded on a neighborhood of x_0 and $\varepsilon \in Z_+$, then there exists a neighborhood U of 0 such that $\bigcup\limits_{u \in U} \partial^{\varepsilon}_{\leq} f(x_0 + u)$ is a equicontinuous set in $\mathcal{L}(X, Z)$.*

Proof. Let V be a neighborhood of 0 in Z. By the hypothesis there exists a neighborhood U of 0 in X such that f is upper bounded on $x_0 + U$.

Let denote $\tilde{M}$ the upper bound. Let U_1 be a neighborhood of 0 such that $U_1 + U_1 \subset U$ and $T \in \partial^{\varepsilon}_{\leq} f(x_0 + u_1)$, $u_1 \in U_1$. We have

$$Tz \leq f(x_0 + u_1 + z) - f(x_0 + u_1) + \varepsilon, \forall\ z \in X.$$

For $z \in U_1$ we derive that $Tz \leq 2\widetilde{M} + \varepsilon$. Since the intervals are topologically bounded, there exists $\lambda < 1$ such that

$$\lambda[-2\widetilde{M} - \varepsilon,\, 2\widetilde{M} + \varepsilon] \subset V.$$

We derive $T(\lambda U_1) \subset V$ and thus $\bigcup\limits_{u \in U_1} \partial^{\varepsilon}_{\leq} f(x_0 + u)$ is a equicontinuous set in $\mathcal{L}(X,\, Z)$. □

Corollary 2.1. *Under the previous hypothesis, $\partial^{\varepsilon}_{\leq} f(x_0)$ is an equicontinuous set in $\mathcal{L}(X, Z)$.*

For real functions, the result was published in 1965 by Brøndsted and Rockafellar in [1].

Proposition 2.5. *Let $f : X \to \overline{Z}$ be a proper, convex function which is upper bounded on a neighborhood of $x_0 \in \operatorname{Int} \operatorname{dom} f$. If the intervals $[x,\, z]$ are weakly relatively compact then $\partial^{\varepsilon}_{\leq} f(x_0)$ is a weak relative compact set in $\mathcal{L}_p(X,\, Z_w)$.*
If Z_+ is closed, then $\partial^{\varepsilon}_{\leq} f(x_0)$ is a compact set in $\mathcal{L}_p(X,\, Z_w)$.

Proof. Similarly to the proof of the precedent proposition, we obtain that $\partial^{\varepsilon}_{\leq} f(x_0)$ is a equicontinuous set in $\mathcal{L}(X,\, Z_w)$ and for all $h \in U_1$ we have $Th \in [-M,\, M]$ if $T \in \partial^{\varepsilon}_{\leq} f(x_0)$. Since U_1 is absorbent we obtain that for all $h \in X$, Th belongs to an interval which is a weak relative compact set. Thus, the set $\{Th,\, T \in \partial^{\varepsilon}_{\leq} f(x_0)\}$ is closed in a weakly relatively compact set for all $h \in X$ and the conclusion follows. If Z_+ is closed, we have seen that $\partial^{\varepsilon}_{\leq} f(x_0)$ is a closed set and so, $\partial^{\varepsilon}_{\leq} f(x_0)$ is a compact set in $\mathcal{L}_p(X, Z_w)$. □ The links between $\partial_{\leq} f(x_0)$ and $\partial^{\varepsilon}_{\leq} f(x_0)$ and $\partial_{\not>} f(x_0)$ and $\partial^{\varepsilon}_{\not>} f(x_0)$ are given in the following proposition.

Proposition 2.6. *If Z_+ is a superior closed cone (i.e. $-Z_+ = \bigcap\limits_{\varepsilon > 0} (-Z_+ + \varepsilon)$) then, for $x_0 \in \operatorname{dom} f$ such that $f(x_0) \in Z$ we have:*

1. $\bigcap\limits_{\varepsilon \in Z_+ \setminus \{0\}} \partial^{\varepsilon}_{\leq} f(x_0) = \partial_{\leq} f(x_0)$*;*

2. $\bigcap_{\varepsilon \in Z_+ \setminus \{0\}} \partial^{\varepsilon}_{\not>} f(x_0) = \partial_{\not>} f(x_0)$.

Proof.

1. Obviously, $\partial_{\leq} f(x_0) \subset \bigcap_{\varepsilon \in Z_+ \setminus \{0\}} \partial^{\varepsilon}_{\leq} f(x_0)$.

Let $T \in \bigcap_{\varepsilon \in Z_+ \setminus \{0\}} \partial^{\varepsilon}_{\leq} f(x_0)$. This relation implies that for all $\varepsilon \in Z_+ \setminus \{0\}$ we have

$$T(h) \leq f(x_0 + h) - f(x_0) + \varepsilon, \, \forall h \in X.$$

It follows $T(h) - f(x_0 + h) + f(x_0) \in \bigcap_{\varepsilon > 0} (-Z_+ + \varepsilon) = -Z_+$ which means that $T \in \partial_{\leq} f(x_0)$.

2. Obviously, $\partial_{\not>} f(x_0) \subset \bigcap_{\varepsilon \in Z_+ \setminus \{0\}} \partial^{\varepsilon}_{\not>} f(x_0)$.

For $T \in \bigcap_{\varepsilon \in Z_+ \setminus \{0\}} \partial^{\varepsilon}_{\not>} f(x_0)$ we have

$$T(h) \not> f(x_0 + h) - f(x_0) + \varepsilon, \, \forall h \in X, \, \forall \varepsilon \in Z_+ \setminus \{0\}$$

We derive

$$T(h) - f(x_0 + h) + f(x_0) \in C\Big(\bigcup_{\varepsilon \in Z_+ \setminus \{0\}} (Z_+ + \varepsilon) \Big) \cup \{0\} = C(Z_+) \cup \{0\}$$

and thus $T \in \partial_{\not>} f(x_0)$. □

Let's observe that if Z_+ is closed and has a nonempty interior, then

$$w\partial_{\not>} f(x_0) = \bigcap_{\mu \in \mathrm{Int}\, Z_+} w\partial^{\mu}_{\not>} f(x_0)$$

for all $x_0 \in \mathrm{dom}\, f$ with $f(x_0) \in Z$.

For the real functions, Asplund & Rockafellar, Borwein, etc. give other links between the subdifferentials and the ε-subdifferentials of a function based on the Ekeland variational principle.

For the vector case, some generalizations for the variational principle were proposed by Loridan (1984), Németh (1986), Tammer (1992), Dentcheva & Helbig (1994), Henkel & Tammer (1995), Göpfert & Tammer & Zălinescu (1998).

We recall here the variational principle given by Göpfert & Tammer & Zălinescu [7] which generalize the Tammer principle given in [17].

Theorem 2.2. *Let (X, d) be a complete metric space, Y a Hausdorff locally convex space ordered by a convex, closed cone K and $k_0 \in K \setminus (-K)$. Suppose that $f : X \to Y \cup \{+\infty\}$ is a proper function, $x_0 \in \operatorname{dom} f$ and for each $r \in \mathbb{R}$, the set $\{x \in X \mid f(x) \leq_K f(x_0) + rk_0\}$ is closed. If for some $\varepsilon > 0$ we have that $f(X) \bigcap (f(x_0) - \varepsilon k_0 - K \setminus \{0\}) = \emptyset$, then there exists $\overline{x} \in \operatorname{dom} f$ such that*

$$f(\overline{x}) + \sqrt{\varepsilon} k_0 d(x, x_0) \leq_K f(x_0), \quad d(x, x_0) \leq \sqrt{\varepsilon}$$

and

$$f(x) + \sqrt{\varepsilon} k_0 d(x, \overline{x}) \leq_K f(\overline{x}) \Rightarrow x = \overline{x}.$$

This result proves that Theorem 4.1 [17] is valid without the boundedness from below of $f(X)$ and even for cones K with empty interior.

Let X be a Banach space and Z be a topological linear space ordered by the convex, pointed, closed cone Z_+. For $m \in Z_+ \setminus \{0\}$ we will denote by

$$C(m) = \{(u, a) \in X \times Z, \, a + m \|u\| \leq 0\}$$

the convex cone of $X \times Z$.

If the interior of Z_+ is nonempty and $m \in \operatorname{Int} Z_+$, we observe that $\operatorname{Int} C(m) \neq \emptyset$ and $C(m)$ is closed.

Indeed, for $a \in -\operatorname{Int} Z_+$ there exists $V \in \mathcal{V}(0)$ such that $a + V \subset -Z_+$. Since V is absorbent, there exists $\mu > 0$ such that $\mu m \in V$. If $u \in B(0, \mu)$ then $a + m \|u\| \leq 0$.

For all $b \in a + V_1$, $(V_1 + V_1 \subset V)$ the following relation does holds

$$b + V_1 \subset -Z_+.$$

Using the fact that V_1 is absorbent we obtain:

$$(u, \, b) \in C(m), \, \forall u \in B(0, \, \varepsilon), \, b \in a + V_1.$$

Thus, $(0, \, a) \in \operatorname{Int} C(m) \neq \emptyset$ and since Z_+ is closed and convex, $C(m)$ has the same properties.

Theorem 2.3. *Let X be a Banach space, Z a Hausdorff locally convex space ordered by a closed, convex, pointed cone with nonempty interior. If $f : X \to Z \cup \{+\infty\}$ is a convex, proper, lower semicontinuous function,*

$u \in \text{dom } f$, $\varepsilon \in \text{Int } Z_+$, $T \in w\partial^{\varepsilon}_{\not>} f(u)$, *then for all* $\lambda > 0$, *there exists* $u_\lambda \in X$, $T' \in \mathcal{L}(X, Z)$, $z^* \in Z_+^* \setminus \{0\}$ *such that*

$$\|u - u_\lambda\| \leq \lambda,\ z^* \circ T' \in \partial z^* \circ f(u_\lambda) \text{ and } \left\|z^*(T - T')\right\| \leq \frac{1}{\lambda} z^*(\varepsilon).$$

In addition

$$T \in w\partial_{\not>}(f + \frac{\varepsilon}{\lambda}\|\cdot\|)(u_\lambda)$$

Proof. For $u \in \text{dom } f$, $\varepsilon \in Z_+ \setminus \{0\}$ and $T \in \partial^{\varepsilon}_{\not>} f(u)$, let $g(v) = f(v) - T(v)$ for $v \in X$. We have that

$$g(u) \in w^{\,\varepsilon} MIN\ g(X).$$

Using Theorem 2.2. we obtain the existence of $u_\lambda \in X$ such that

$$\|u - u_\lambda\| \leq \lambda, \text{ epi } g \cap \{(u_\lambda, g(u_\lambda)) + C(\varepsilon/\lambda)\} = (u_\lambda,\, g(u_\lambda)).$$

Let's denote $C_\lambda = (u_\lambda,\, g(u_\lambda)) + C(\varepsilon/\lambda)$; we observe that C_λ is a convex, closed cone with nonempty interior. Since epi g is a closed, convex set we can apply the Hahn-Banach separation theorem and thus we get $(v^*,\, z^*) \in X^* \times Z^*$, $(v^*, z^*) \neq (0, 0)$ such that:

1. $v^*(v) + z^*(z) \geq 0,\ \forall\ (v,\, z) \in \text{epi } g$
2. $v^*(v) + z^*(z) \leq 0,\ \forall\ (v,\, z) \in C_\lambda$
3. $v^*(u_\lambda) + z^*(g(u_\lambda)) = 0$

Since $v^*(v) + z^*(b) \leq 0$, $\forall (v, b) \in X \times Z$ with $b + m\|v\| \in -Z_+$ it follows that the relation 2. does holds for $v = 0$, $b \in -Z_+$ which implies $z^*(b) \leq 0$, $\forall b \in -Z_+$ and thus $z^* \in Z_+^* \setminus \{0\}$.
Using 1. we derive

$$v^*(v) + z^*(g(v)) \geq 0,\ \forall v \in X$$

and just combining with 3. we obtain

$$v^*(v - u_\lambda) + z^*(g(v) - g(u_\lambda)) \geq 0,\ \forall v \in X.$$

Taking account the definition of g we have

$$z^* \circ T(v - u_\lambda) - v^*(v - u_\lambda) \le z^*(f(v) - f(u_\lambda)) \tag{1}$$

which says that

$$x^* = z^* \circ T - v^* \in \partial_{\le} z^* \circ F(u_\lambda).$$

We can find thus $T' \in \mathcal{L}(X, Z)$ such that $x^* = z^* \circ T'$, $T' \in w\partial_{\not\geq} f(u_\lambda)$.

Relation 2. implies

$$v^*(v) \le z^*(\frac{\varepsilon}{\lambda} \|v\|) = \|v\| \, z^*(\frac{\varepsilon}{\lambda})$$

and finally

$$\left\| z^*(T - T') \right\| = \|v^*\| \le \frac{1}{\lambda} z^*(\varepsilon). \tag{2}$$

Now, the relation 1. and 2. gives

$$-z^*(g(v)) \le v^*(v) \le z^*(\frac{\varepsilon}{\lambda} \|v\|), \forall v \in X.$$

Since $v^*(u_\lambda) + z^*(g(u_\lambda)) = 0$, we derive

$$z^*(g(v) - g(u_\lambda) + \frac{\varepsilon}{\lambda} \|v\| - \|u_\lambda\|) \ge 0 \tag{3}$$

Since the interior of Z_+ is nonempty, we obtain that

$$g(v) - g(u_\lambda) + \frac{\varepsilon}{\lambda} \|v\| - \|u_\lambda\| \notin -\text{Int } Z_+$$

which means that $T \in w\partial_{\not\geq}(f + \frac{\varepsilon}{\lambda} \|\cdot\|)(u_\lambda)$. $\square$

Corollary 2.1. *Let X be a Banach space, Z a Hausdorff locally convex space ordered by a convex, pointed, closed, normal cone with nonempty interior such that Z is order complete and $f : X \to Z \cup \{+\infty\}$ a proper, convex, lower semicontinuous function. Then*

$$\overline{\text{Dom } (w\partial_{\not\geq} f)} = \text{dom } f.$$

Proof. Under our hypothesis, we derive that $\partial^{\varepsilon}_{\le} f(x) \neq \emptyset$ for all $x \in \text{dom } f$ and $\varepsilon \in \text{Int } Z_+$. Applying Theorem 2.3. we get that for all $\lambda > 0$,

$\varepsilon \in \text{Int } Z_+$, $u \in \text{dom } f$, $T \in \partial^{\varepsilon}_{\leq} f(u)$ there exists $u_\lambda \in X$, $T' \in w\partial_{\not\geq} f(u_\lambda)$ such that $\|u - u_\lambda\| \leq \lambda$, $(T - T')(B) \cap (\frac{\varepsilon}{\lambda} + \text{Int } Z_+) = \emptyset$ where B is the unit sphere of X. For $\lambda_n = 1/n$ we get $u_n \in \text{Dom } (w\partial_{\not\geq} f)$ and $u_n \to u$. So, the conclusion follows. $\square$

Corollary 2.2. *If in Theorem 2.3, Z is a normed space then for all $u \in \text{dom } f$ and $\varepsilon \in \text{Int } Z_+$, there exists u_ε such that $\|u - u_\varepsilon\| \leq \sqrt{\|\varepsilon\|}$ and g_ε a convex, proper, lower semicontinuous function such that $w\partial_{\not\geq} g_\varepsilon(u_\varepsilon) \neq \emptyset$ with $\|g_\varepsilon - f\| \leq \sqrt{\|\varepsilon\|}$.*

Proof. The corollary follows from Theorem 2.3 if we take $\lambda = \sqrt{\|\varepsilon\|}$ and $g_\varepsilon = f + \frac{\varepsilon}{\sqrt{\|\varepsilon\|}} \|.\|$. $\square$

If we apply the calculus rules for the subdifferentials in relation (3)

$$z^* \circ T \in \partial_{\leq}(z^* \circ f + z^*(\frac{\varepsilon}{\lambda} \|.\|))(u_\lambda),$$

we get

$$z^* \circ T \in \bigcap_{\alpha > 0} (\bigcup_{\alpha_1 + \alpha_2 = \alpha,\, \alpha_1,\, \alpha_2 > 0} \partial^{\alpha_1}_{\leq} z^* \circ f(u_\lambda) + \partial^{\alpha_2}_{\leq} z^*(\frac{\varepsilon}{\lambda} \|.\|))(u_\lambda).$$

This means that $\forall \alpha > 0$, $\exists \alpha_1, \alpha_2 > 0$ with $\alpha_1 + \alpha_2 = \alpha$ such that

$$z^* \circ T = U_1 + U_2,\ U_1 \in \partial^{\alpha_1}_{\leq} z^* \circ f(u_\lambda),\ U_2 \in \partial^{\alpha_2}_{\leq} z^*(\frac{\varepsilon}{\lambda} \|.\|)(u_\lambda).$$

The relation $U_2 \in \partial^{\alpha_2}_{\leq} z^*(\frac{\varepsilon}{\lambda} \|.\|)(u_\lambda)$ implies

$$U_2(u - u_\lambda) \leq z^*(\frac{\varepsilon}{\lambda} \|u\|) - z^*(\frac{\varepsilon}{\lambda} \|u_\lambda\|) + \alpha_2,\ \forall u \in X$$

and thus, $\frac{\lambda}{z^*(\varepsilon)} U_2 \in \partial^{\frac{\lambda \alpha_2}{z^*(\varepsilon)}}_{\leq} \|.\| (u_\lambda)$. Using the special form of the elements belonging to $\partial^{\frac{\lambda \alpha_2}{z^*(\varepsilon)}}_{\leq} \|.\| (u_\lambda)$, we obtain

$$\frac{\lambda}{z^*(\varepsilon)} U_2 = M,\ \|M\| \leq 1,\ M u_\lambda \geq \|u_\lambda\| - \frac{\alpha_2 \lambda}{z^*(\varepsilon)}.$$

By this reason, for all $T \in w\partial^{\varepsilon}_{\not\geq} f(u_0)$ and $\lambda > 0$ there exists u_λ, $z^* \in Z^*_+ \setminus \{0\}$, such that $\|u_0 - u_\lambda\| \leq \lambda$, and for all $\alpha > 0$, there exists $M \in X^*$, $\|M\| \leq 1$, $M u_\lambda \geq \|u_\lambda\| - \frac{\alpha_2 \lambda}{z^*(\varepsilon)}$ and

$$z^* \circ T - M \in \partial^{\alpha}_{\leq} z^* \circ f(u_\lambda).$$

If we replace α by $\alpha z^*(\varepsilon)$ we get that for all $T \in w\partial^{\varepsilon}_{\not>} f(u_0)$ and $\lambda > 0$ there exists u_λ, and $z^*_\lambda \in Z^*_+ \setminus \{0\}$ such that $\|u_0 - u_\lambda\| \leq \lambda$ and for all $\alpha > 0$ there exists $M_\alpha \in X^*$ with $\|M_\alpha\| \leq 1$, $M_\alpha u_\lambda \geq \|u_\lambda\| - \alpha\lambda$, $z^*_\lambda \circ T - M_\alpha \in \partial^{\alpha}_{\leq} z^*_\lambda \circ f(u_\lambda)$.

Since $\bigcap\limits_{\alpha>0} \partial^{\alpha}_{\leq} z^*_\lambda \circ f(u_\lambda) = \bigcap\limits_{1>\alpha>0} \partial^{\alpha}_{\leq} z^*_\lambda \circ f(u_\lambda) = \partial_{\leq} z^*_\lambda \circ f(u_\lambda)$ we derive for all $T \in w\partial^{\varepsilon}_{\not>} f(u_0)$ and $\lambda > 0$ there exists u_λ, $z^*_\lambda \in Z^*_+ \setminus \{0\}$ and $M \in X^*$ such that $\|u_0 - u_\lambda\| \leq \lambda$, $\|M\| \leq 1$, $M u_\lambda \geq \|u_\lambda\| - \lambda$ and $z^* \circ T - M \in \partial_{\leq} z^* \circ f(u_\lambda)$.

If in addition f is regular sudifferentiable, i.e. $\partial z^* \circ f(x) = z^* \circ \partial_{\leq} f(x)$ for all $x \in \operatorname{dom} f$ and $z^* \in Z^*_+$, we obtain that for all $T \in w\partial^{\varepsilon}_{\not>} f(u_0)$ and $\lambda > 0$ there exists u_λ, $z^*_\lambda \in Z^*_+ \setminus \{0\}$, $\exists\, M \in X^*$ such that $\|u_0 - u_\lambda\| \leq \lambda$, $\|M\| \leq 1$, $M u_\lambda \geq \|u_\lambda\| - \lambda$, $U \in \partial_{\leq} f(u_\lambda)$ and $z^*(T - U) = M$.

Using this observations, we get the following corollary:

Corollary 2.3. *If in the previous theorem Z is a normed space and f is regular subdifferentiable, then for all $u_0 \in X$ and $\lambda > 0$, $\varepsilon \in \operatorname{Int} Z_+$ there exists u_λ, z^*_ε such that $\|u_0 - u_\lambda\| < \lambda$ and*

$$w\partial^{\varepsilon}_{\not>} f(u_0) \subset \partial_{\leq} f(u_\lambda) + B_{z^*_\varepsilon}(0,1)$$

where we denote $B_{z^*}(0,1) = \{T \in \mathcal{L}(X,Z),\ \|z^*(T)\| \leq 1\}$.

It is known that for real maps the multifunction $(x,\lambda) \rightrightarrows \partial^{\lambda} f(x)$ is w^*-continuous on $\operatorname{Int} \operatorname{dom} f \times \mathbb{R}^*$ endowed with the product topology [1].

A similar result does holds in the vector case.

Proposition 2.7. *Let (X,τ), (Z,μ) be two Hausdorff locally convex spaces, Z ordered by the closed cone Z_+ which has nonempty interior, Z is order complete and the intervals are weakly compact. Let $f : X \to \overline{Z}$ be a convex, proper function upper bounded on a neighborhood of a point in* $\operatorname{dom} f$.

Then, the multifunction $F : X \times Z \to \mathcal{L}_p(X, Z_w)$,

$$F(x,\lambda) = \begin{cases} \partial^{\lambda}_{\leq} f(x), & x \in \operatorname{dom} f \\ \emptyset, & x \notin \operatorname{dom} f \end{cases}$$

is continuous on $\operatorname{Int} \operatorname{dom} f \times \operatorname{Int} Z_+$ *endowed with the product topology $\tau \times \mu$.*

Proof. Our hypothesis ensure that $\partial^{\lambda}_{\leq} f(x) \neq \emptyset$, $\forall x \in \operatorname{dom} f$, $\forall \lambda \in \operatorname{Int} Z_+$ and thus

$$F(x,\lambda) \neq \emptyset,\ \forall (x,\lambda) \in \operatorname{dom} f \times \operatorname{Int} Z_+.$$

We know from Proposition 2.5 that $\partial^{\lambda}_{\leq} f(x)$ is compact in $\mathcal{L}_p(X, Z_w)$, $\forall \lambda \in$ Int Z_+ and $x \in$ dom f.

Let $(x, \delta) \in$ Int dom $f \times$ Int Z_+. We shall prove that for all $V \in \mathcal{V}_{\mathcal{L}_p(X,Z_w)}(0)$ there exists $U \in \mathcal{V}_X(0)$ and $\varepsilon \in$ Int Z_+ such that

$$\partial^{\lambda}_{\leq} f(x+u) \subset \partial^{\delta}_{\leq} f(x) + V;$$

$$\partial^{\delta}_{\leq} f(x) \subset \partial^{\lambda}_{\leq} f(x+u) + V$$

$\forall\, u \in U$, $\lambda \in (\delta - \varepsilon, \delta + \varepsilon)$. The first relation implies that F is upper semicontinuous at (x, δ) and the second implies that F is lower semicontinuous at (x, δ).

Let observe that our hypothesis ensures that f is continuous on Int Dom f and $\{(-\varepsilon, \varepsilon),\ \varepsilon \in \text{Int } Z_+\}$ forms a fundamental system of neighborhoods in (Z, μ).

Let V be a neighborhood of 0 in $\mathcal{L}_p(X, Z_w)$ and for $\mu = 2\delta$, let U be a neighborhood of 0 in (X, τ) for which the set $M = \{\partial^{\mu}_{\leq} f(x+u),\ u \in U\}$ is equicontinuous (the existence of this set is ensured by Proposition 2.4). Thus M is bounded in $\mathcal{L}_p(X, Z_w)$ and there exists $\rho > 0$ such that $M - M \subset \rho V$.
Consider $\varepsilon \in$ Int Z_+ such that $2\varepsilon < \frac{1}{\rho+1}\delta$ (it exists since $\delta \in$ Int Z_+).
By the hypothesis we have $W \subset U$ such that

$$-\frac{\varepsilon}{2} < f(x+u_1) - f(x+u_2) < \frac{\varepsilon}{2},\ \forall\, u_1, u_2 \in W$$

and

$$-\frac{\varepsilon}{4} < T(w) < \frac{\varepsilon}{4},\ \forall\, w \in W,\ T \in M.$$

Let $\alpha < \beta,\ u_1,\ u_2 \in \text{dom } f - x$.

$$\begin{aligned}\partial^{\alpha}_{\leq} f(x+u_1) = \{&T \in \mathcal{L}(X, Z) \mid\ \forall z \in X, f(z) \geq \\ &\geq f(x+u_1) + T(z - x - u_1) - \alpha\} = \\ =\{&T \in \mathcal{L}(X, Z) \mid\ \forall z \in X,\ f(z) \geq \\ &\geq f(x+u_2) + T(z - x - u_2) - \\ &\quad -(\alpha + f(x+u_2) - f(x+u_1) + T(u_2 - u_1)\}.\end{aligned}$$

For $u_1,\ u_2 \in W,\ \alpha < 2\delta,\ T \in \partial^{\alpha} f(x+u_1) \subset M$ it follows

$$\alpha + f(x+u_2) - f(x+u_1) + T(u_2 - u_1) \leq \alpha + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq \beta$$

if $\beta - \alpha \geq \varepsilon$.
Thus,

$$\partial_{\leq}^{\alpha} f(x+u_1) \subset \partial_{\leq}^{\beta} f(x+u_2),\ \forall\, u_1,\, u_2 \in W,\ \alpha < 2\delta,\ \beta - \alpha \geq \varepsilon$$

which implies

$$\partial_{\leq}^{\lambda-\varepsilon} f(x) \subset \partial_{\leq}^{\lambda} f(x+u) \subset \partial_{\leq}^{\lambda+\varepsilon} f(x),\ \forall\, u \in W,\ \forall\, \lambda \leq 2\delta.$$

In the same time for all $\theta > 1$ we have $\gamma = \beta + \theta(\alpha - \beta) < \alpha$ and

$$\frac{1}{\theta} \partial_{\leq}^{\gamma} f(x) + (1 - \frac{1}{\theta}) \partial_{\leq}^{\beta} f(x) \subseteq \partial_{\leq}^{\alpha} f(x).$$

Let $\alpha = \delta - 2\varepsilon,\ \beta = \delta,\ \theta = \rho + 1$. We have $\alpha \in \operatorname{Int} Z_+,\ \alpha < \beta < 2\delta,\ \theta > 1$,

$$\gamma = \delta + (\rho+1)(\delta - 2\varepsilon - \delta) = \delta - 2\varepsilon(\rho+1) \in Z_+ \backslash \{0\},\ \gamma < \delta - 2\varepsilon < 2\delta.$$

Thus,

$$\begin{array}{rcl} \partial_{\leq}^{\delta} f(x) & \subseteq & (1 - \frac{1}{\rho+1})^{-1} (\partial_{\leq}^{\delta-2\varepsilon} f(x) - \frac{1}{\rho+1} \partial_{\leq}^{\gamma} f(x)) \\ & \subseteq & \partial_{\leq}^{\delta-2\varepsilon} f(x) + \frac{1}{\rho} (\partial_{\leq}^{\delta-2\varepsilon} f(x) - \partial^{\gamma} f(x)) \\ & \subseteq & \partial_{\leq}^{\delta-2\varepsilon} f(x) + \frac{1}{\rho} (M - M) \subseteq \partial_{\leq}^{\delta-2\varepsilon} f(x) + V. \end{array}$$

If we take $\alpha = \delta - 2\varepsilon,\ \beta = \delta - \varepsilon,\ u_1 = 0,\ u_2 \in W$ we obtain

$$\partial_{\leq}^{\delta-2\varepsilon} f(x) \subseteq \partial_{\leq}^{\delta-\varepsilon} f(x+u) \subseteq \partial_{\leq}^{\lambda} f(x+u), \forall\, \lambda \in (\delta - \varepsilon, \delta + \varepsilon)$$

and finally

$$\partial_{\leq}^{\delta} f(x) \subseteq \partial_{\leq}^{\lambda} f(x+u) + V, \forall\, u \in W,\ \lambda \in (\delta - \varepsilon, \delta + \varepsilon).$$

Analogously, for $\alpha = \delta,\ \beta = \delta + 2\varepsilon,\ \theta = \rho + 1,\ \gamma = \delta + 2\varepsilon + (\rho+1)(-2\varepsilon) = \delta - 2\rho\varepsilon \in Z_+ \backslash \{0\}$ we obtain

$$\partial_{\leq}^{\delta+2\varepsilon} f(x) \subset \partial_{\leq}^{\delta} f(x) + V$$

and

$$\partial_{\leq}^{\lambda} f(x+u) \subseteq \partial_{\leq}^{\delta+\varepsilon} f(x+u) \subseteq \partial_{\leq}^{\delta+2\varepsilon} f(x), \forall u \in W, \lambda \in (\delta - \varepsilon, \delta + \varepsilon).$$

Thus, $\partial_{\leq}^{\lambda} f(x+u) \subseteq \partial_{\leq}^{\lambda} f(x) + V$ and the conclusion follows. $\square$

3. ε-subdifferentials via efficient points. Another tool for study the ε-subdifferential of a vector map is, similarly to the real case, the directional derivative.

Definition 3.1. Let $f : X \to \overline{Z}$ be a convex, proper function, where Z is ordered by a closed cone Z_+ and $x_0 \in \operatorname{dom} f,\ h \in X$. The set given by

$$f'_\varepsilon(x_0, h) = INF_1 \left\{ \frac{f(x_0 + th) - f(x_0) + \varepsilon}{t}, t > 0 \right\}$$

will be called the ε-directional derivative of f at (x_0, h).

We denote $\tilde{f}'_\varepsilon(x_0, h) = INF \left\{ \frac{f(x_0+th)-f(x_0)+\varepsilon}{t}, t > 0 \right\}$; if the interior of the cone is nonempty, we denote $wf'_\varepsilon(x_0, h)$ and $w\tilde{f}'_\varepsilon(x_0, h)$ the sets obtained with the aid of $wINF_1$, respectively $wINF$.

Proposition 3.1. *Let Z ordered by the normal, closed cone Z_+ with nonempty interior and which is Daniell. If $f : X \to \overline{Z}$ is a convex, proper function, then for all $h_1, h_2 \in X$, $\varepsilon \in Z_+$, $x_0 \in \operatorname{dom} f$ we have*

$$\begin{cases} wf'_\varepsilon(x_0, h_1 + h_2) \subseteq wINF_1(wf'_\varepsilon(x_0, h_1) + wf'_\varepsilon(x_0, h_2)) - Z_+ \\ wINF_1(wf'_\varepsilon(x_0, h_1) + wf'_\varepsilon(x_0, h_2)) \subseteq wf'_\varepsilon(x_0, h_1 + h_2) + Z_+ \\ wf'_\varepsilon(x_0, \lambda h) = \lambda wf'_\varepsilon(x_0, h),\ \forall \lambda \in R_+,\ h \in X. \end{cases}$$

Proof. Let $t_2 > t_1 > 0$, $x_0 \in \operatorname{dom} f$. Since f is convex we have

$$\frac{f(x_0 + t_1 h) - f(x_0)}{t_1} + Z_+ \supseteq \frac{f(x_0 + t_2 h) - f(x_0)}{t_2}.$$

Let $h_1, h_2 \in X$ and $\mu \in \operatorname{Int} Z_+$. For all $\alpha \in f'_\varepsilon(x_0, h_1) + \mu$ there exists $t'_\varepsilon > 0$ such that $\alpha - \frac{f(x_0+2t'_\varepsilon h_1)-f(x_0)+\varepsilon}{2t'_\varepsilon} \in \operatorname{Int} Z_+$ which implies that for all $t < t'_\varepsilon$,

$$\alpha - \frac{f(x_0 + 2th_1) - f(x_0) + \varepsilon}{2t} \in \operatorname{Int} Z_+.$$

For all $\beta \in wf'_\varepsilon(x_0, h_2) + \mu$, $\exists t''_\varepsilon > 0$ such that $\beta - \frac{f(x_0+2t''_\varepsilon h_2)-f(x_0)+\varepsilon}{2t''_\varepsilon} \in \operatorname{Int} Z_+$ which implies that for all $t < t''_\varepsilon$,

$$\beta - \frac{f(x_0 + 2th_2) - f(x_0) + \varepsilon}{2t} \in \operatorname{Int} Z_+.$$

Thus, $\forall \gamma \in wf'_\varepsilon(x_0, h_1) + wf'_\varepsilon(x_0, h_2) + 2\mu, \exists t > 0$ such that

$$\gamma - \frac{f(x_0 + 2th_1) - f(x_0) + \varepsilon}{2t} + \frac{f(x_0 + 2th_2) - f(x_0) + \varepsilon}{2t} \in \operatorname{Int} Z_+.$$

It follows

$$\begin{aligned} &wf'_\varepsilon(x_0,h_1)+wf'_\varepsilon(x_0,h_2)+2\mu\subseteq\\ &\quad\subseteq\left\{\tfrac{f(x_0+2th_1)-f(x_0)+\varepsilon}{2t}+\tfrac{f(x_0+2th_2)-f(x_0)+\varepsilon}{2t},t>0\right\}+\text{Int } Z_+\\ &\quad\subseteq\left\{\tfrac{f(x_0+t(h_1+h_2))-f(x_0)+\varepsilon}{t},t>0\right\}+\text{Int } Z_+. \end{aligned}$$

Taking infimum in the previous relation we derive

$$wf'_\varepsilon(x_0,h_1+h_2)\subseteq wINF\,(wf'_\varepsilon(x_0,h_1)+wf'_\varepsilon(x_0,h_2))+2\mu.$$

If $wf'_\varepsilon(x_0,h_1)=\{-\infty\}$ or $wf'_\varepsilon(x_0,h_2)=\{-\infty\}$ then $wf'_\varepsilon(x_0,h_1+h_2)=\{-\infty\}$.
If $wf'_\varepsilon(x_0,h_1)\neq\{-\infty\}$ and $f'_\varepsilon(x_0,h_2)\neq\{-\infty\}$ it follows $\widetilde{f}'_\varepsilon(x_0,h_i)\neq\{-\infty\}$ and $wf'_\varepsilon(x_0,h_i)\neq\emptyset$, $i=1,2$,

$$wf'_\varepsilon(x_0,h_1)+wf'_\varepsilon(x_0,h_2)\subset Z.$$

If

$$wINF\,(wf'_\varepsilon(x_0,h_1)+wf'_\varepsilon(x_0,h_2))=\{-\infty\}$$

we derive $wf'_\varepsilon(x_0,h_1+h_2)=\{-\infty\}$ and $wINF_1(wf'_\varepsilon(x_0,h_1)+wf'_\varepsilon(x_0,h_2))=\{-\infty\}$, so the conclusion follows.
If

$$wINF\,(wf'_\varepsilon(x_0,h_1)+wf'_\varepsilon(x_0,h_2))\neq\{-\infty\}$$

we obtain by using Theorem 1.3 [16] that

$$wf'_\varepsilon(x_0,h_1+h_2)\subseteq wINF_1(wf'_\varepsilon(x_0,h_1)+wf'_\varepsilon(x_0,h_2))-\text{Int } Z_++2\mu.$$

Since the sets "$wINF_1A$" has the property (see Proposition 1.13 [16])

$$\bigcap_{\mu>0}(wINF_1A-\text{Int } Z_+\cup\{0\}+\mu)=wINF_1A-\text{Int } Z_+\cup\{0\},$$

we derive

$$wf'_\varepsilon(x_0,h_1+h_2)\subseteq wINF_1(wf'_\varepsilon(x_0,h_1)+wf'_\varepsilon(x_0,h_2))-\text{Int } Z_+\cup\{0\}.$$

Therefore $wINF\,(wf'_\varepsilon(x_0,h_1)+wf'_\varepsilon(x_0,h_2))\neq\{-\infty\}$, then

$$wf'_\varepsilon(x_0,h_1+h_2)\neq\{-\infty\}\text{ and }\widetilde{wf'_\varepsilon}(x_0,h_1+h_2)\neq\{-\infty\}.$$

Thus, $wf'_\varepsilon(x_0, h_1 + h_2) \neq \emptyset$ and

$$\begin{aligned} &wINF_1(f'_\varepsilon(x_0, h_1) + f'_\varepsilon(x_0, h_2)) + 3\mu \subseteq \\ &\subseteq wf'_\varepsilon(x_0, h_1) + wf'_\varepsilon(x_0, h_2) + 2\mu + \text{Int } Z_+ \subseteq \\ &\subseteq wf'_\varepsilon(x_0, h_1 + h_2) + \text{Int } Z_+, \forall \mu \in Z_+ \backslash \{0\}. \end{aligned}$$

But the set "$wINF_1A$" has property

$$\bigcap_{\mu > 0} (wINF_1A + \text{Int } Z_+ \cup \{0\} - \mu) = wINF_1A + \text{Int } Z_+ \cup \{0\},$$

and finally

$$wINF_1(wf'_\varepsilon(x_0, h_1) + wf'_\varepsilon(x_0, h_2)) \subseteq wf'_\varepsilon(x_0, h_1 + h_2) + \text{Int } Z_+ \cup \{0\}.$$

The homogeneity follows trivially by the definition. $\square$

Corollary 3.1. *Under the previous hypothesis, for $x_0 \in \text{dom } f$,*

$$T \in w\partial^\varepsilon_{\not\geq} f(x_0) \Longleftrightarrow \forall h \in X,\ Th \in wf'_\varepsilon(x_0, h) - \text{Int } Z_+ \cup \{0\}.$$

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Received: 27.XI.2001

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