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DEFORMATION METHOD AND ITS APPLICATION TO SURFACE GRID GENERATION

BY

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Abstract. We consider the problem of numerical grid generation on any parametrized surface. By a deformation method, grids are generated on the surface whose cell area elements are controllable according to a monitor function. Numerical examples are presented.

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Let S be a surface in R^3 defined by the parametrization

$$\rho : \Omega \subset R^2 \to R^3.$$

Let (u, v) be a point of Ω. Then the area element of the surface at the point $\rho(u, v) = (x(u, v), y(u, v), z(u, v)) \in S$ is given by $dA = |\rho_u \wedge \rho_v|\ du\ dv$, where $\rho_u = (x_u, y_u, z_u), \rho_v = (x_v, y_v, z_v)$. The area element controls the fineness of the mesh on the surface which is the image of a uniform mesh in Ω under ρ. Once the parametrization ρ is given, the area element is determined. In applications, one often desires to adapt the mesh according to the nature of the physical problem that is being simulated. Here, we introduce a deformation procedure by which the given area element is changed to a prescribed function defined either on the surface or on the parameter domain. We consider the former case first. Let $|\rho_u \wedge \rho_v| = f(u, v)$ in Ω.

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Given $g(u,v)$ a positive function defined on Ω, satisfying

$$\int_{\Omega} g(u,v)dudv = \text{ the area of } S = |S|, \tag{1}$$

we intend to find α, a transformation from $\Omega \to \Omega$, such that $\phi = \rho \circ \alpha$, a new parametrization of S, satisfies

$$|\phi_u \wedge \phi_v| = g(\alpha(u,v)) \quad (u,v) \in \Omega.$$

It turns out that such an α should satisfy

$$\begin{gathered}(\det \nabla\alpha(u,v))(f(\alpha(u,v))) = g(\alpha(u,v)), \quad (u,v) \in \Omega,\\ \alpha(u,v) \in \partial\Omega, \quad (u,v) \in \partial\Omega,\end{gathered} \tag{2}$$

or

$$\begin{gathered}(\det \nabla\alpha(u,v)) = g(\alpha(u,v))/f(\alpha(u,v)), \quad (u,v) \in \Omega,\\ \alpha(u,v) \in \partial\Omega, \quad (u,v) \in \partial\Omega.\end{gathered} \tag{3}$$

We will state and prove our main result on surface grid generation assuming (2), then come back to solve (2).

Theorem 1: *There exists a new parametrization of S in the form of $\phi = \rho \circ \alpha$ such that the new area element is equal to the function $g(\alpha(u,v))$.*

Proof: Assume that α is a transformation of Ω onto itself satisfying (2). Let $(u', v') = \alpha(u,v)$.

Then

$$\begin{aligned}|\phi_u \wedge \phi_v| &= |(\rho_u(u',v')\frac{\partial u'}{\partial u} + \rho_v(u',v')\frac{\partial v'}{\partial u})\wedge \\ &\wedge(\rho_u(u',v')\frac{\partial u'}{\partial v} + \rho_v(u',v')\frac{\partial v'}{\partial v})| = \\ &= |\rho_u(u',v') \wedge \rho_v((u',v')|\frac{\partial(u',v')}{\partial(u,v)} = \\ &= f(\alpha(u,v))\det\nabla\alpha(u,v) = g(\alpha(u,v)).\end{aligned}$$

□

Now we solve for α from (2). The problem can be solved by the deformation method in 2-D as discussed in [4], [5], [6]. For completeness, we discuss some relevant results here. The procedure for solving (2) is: first, find a vector field $V(u,v) \in R^2, (u,v) \in \bar{\Omega}$, such that

$$\begin{gathered} \operatorname{div} V(u,v) = \frac{g(u,v)}{f(u,v)} - 1, \quad (u,v) \in \Omega, \\ \mathbf{n} \cdot V(u,v) = 0, \quad (u,v) \in \partial\Omega, \mathbf{n} = \text{normal to } \partial\Omega. \end{gathered} \tag{4}$$

V can be found by various techniques (cf. [4], [5], [6], [9], [10]). For instance, we can solve for the potential w from the Poisson equation

$$\Delta w = \frac{g(u,v)}{f(u,v)} - 1, \quad (u,v) \in \Omega$$

with Neumann conditions on $\partial\Omega$. Then let $V = \nabla w$.

Second, solve for a 1-parameter family of transformations $\varphi(u,v,t)$ at fixed $(u,v) \in \Omega$, from the system of ordinary differential equations

$$\begin{gathered} \frac{\partial}{\partial t}\varphi(u,v,t) = -MV(\varphi), \quad (u,v) \in \Omega, \quad 0 \leq t \leq 1 \\ \varphi(u,v,0) = (u,v) \qquad (u,v) \in \Omega, \end{gathered} \tag{5}$$

where

$$M = \frac{1}{(1-t) + t\,e(\varphi)}, \tag{6}$$

where $e = f/|S|g$, $|S|$ is the area of S and g is a monitor function that determines the grid adaptation. Then $\alpha = \varphi(u,v,1)$ satisfies (2). Indeed, (4) and (5) imply that $H(u,v,t) = (\det\nabla\varphi(u,v,t))(1 - t + te(\varphi(u,v,t)))$ is t-invariant, and hence $H(u,v,1) = H(u,v,0)$ (cf. [6]). This is equivalent to

$$\det\nabla\varphi(u,v,1)\, e(\varphi(u,v,1)) = \det\nabla\varphi(u,v,0) e(\varphi(u,v,0)),$$

which is $\det\nabla\alpha(u,v) = g(\alpha(u,v))/f(\alpha(u,v))$ as desired.

We now present an analogous theorem with prescribed area elements evaluated on the parameter domain. Given $g(u,v)$, a real valued function defined on Ω satisfying

$$\int_\Omega g(u,v)dudv = \text{ the area of } S = |S|,$$

the equation for $\alpha(u,v)$ is

$$\begin{gathered} (\det\nabla\alpha(u,v))(f(\alpha(u,v))) = g(u,v), \quad (u,v) \in \Omega, \\ \alpha(u,v) \in \partial\Omega, \quad (u,v) \in \partial\Omega. \end{gathered} \tag{7}$$

Theorem 2: *There exists a new parametrization of S in the form of $\phi = \rho \circ \alpha$ such that the new area element is equal to the function $g(u, v)$ satisfying $\int_\Omega g(u, v) du dv = |S|$, the area of S.*
Theorem 2 can be proved as in theorem 1, assuming (7). (7) can be solved by the same techniques in [4], [5], [6]. In applications, the needed adaptation is usually specified on the surface instead of the parameter domain. Thus, theorem 1 is the relevant result for applications.

In the examples below $(u = x, v = y)$,

$$\rho : (x, y, h(x, y)), \ f(x, y) = |\rho_x \wedge \rho_y| = \sqrt{1 + h_x^2 + h_y^2}$$

where h represents the surface. Note that ρ, in general, gives rise to a non-uniform grid on the surface when (x, y) is uniform.

Example 1. Grid of uniform cell size on a Surface.
The surface is defined by

$$h(x, y) = 0.25 \sin 4\pi r \ \text{ for } (x, y) \in \Omega = [0, 1] \times [0, 1],$$

where

$$r = \sqrt{(x - 0.5)^2 + (y - 0.5)^2}.$$

In this case $g(x, y, t) = 1$ in (6). Figure 1 shows a plot of the surface. Figure 2 compares the cell surface area distribution of the resulting non-adaptive grid generated by ρ and the adapted grid of uniform cell size on the surface S. Table I compares the cell sizes of the surface grids. Figure 3 shows the grids on the surface.

Table I. Comparison of Cell Sizes for Example 1

	Non-adaptive	Adaptive grid with different SOR Tolerance ϵ		
cell area		$\epsilon = 5 \times 10^{-7}$	$\epsilon = 1 \times 10^{-7}$	$\epsilon = 5 \times 10^{-8}$
maximum	2.1789×10^{-4}	1.6627×10^{-4}	1.6564×10^{-4}	1.6571×10^{-4}
minimum	0.6619×10^{-4}	1.3554×10^{-4}	1.4049×10^{-4}	1.4060×10^{-4}
mean	1.5423×10^{-4}	1.5423×10^{-4}	1.5423×10^{-4}	1.5423×10^{-4}
$\frac{\text{max}-\text{min}}{\text{mean}}$	98.3%	19.9%	16.3%	16.2%
standard dev.	5.2204×10^{-5}	3.9383×10^{-6}	2.5532×10^{-6}	2.4939×10^{-6}

Example 2. Grid Adapted to a Circle.
Set $\Omega = [0, 1] \times [0, 1]$ and the surface

$$h(x, y) = 4(x - \frac{1}{2})^2 - 4(y - \frac{1}{2})^2 \ \text{ for } (x, y) \in \Omega.$$

Let $g(x,y,t)$ in (6) be associated with a circle with center $(x_c, y_c) = (0.5, 0.8)$ and radius $r = 0.15$ as follows. Let

$$d = 2((x - x_c)^2 + (y - y_c)^2 - r^2).$$

Then we define g in (6) as

$$g(x,y,t) = \begin{cases} 1 & \text{if } d \leq -0.1 \\ (1-t) + t(0.2 - 8d) & \text{if } -0.1 < d \leq 0.0 \\ (1-t) + t(0.2 + 8d) & \text{if } 0.0 < d \leq 0.1 \\ 1 & \text{if } d > 0.1 \end{cases} \tag{8}$$

Figure 4 shows the grid generated by α in Ω. Figure 5 shows the adapted grid on the surface S.

Example 3. Unstructured Grid on a surface. Set $\Omega = [0,1] \times [0,1]$ and the surfaces as in example 1 and 2. Figures 6 and 8 show unstructured grids on the parameter domain and figures 7 and 9 show the resulting (unstructured) grids on the respective surfaces.

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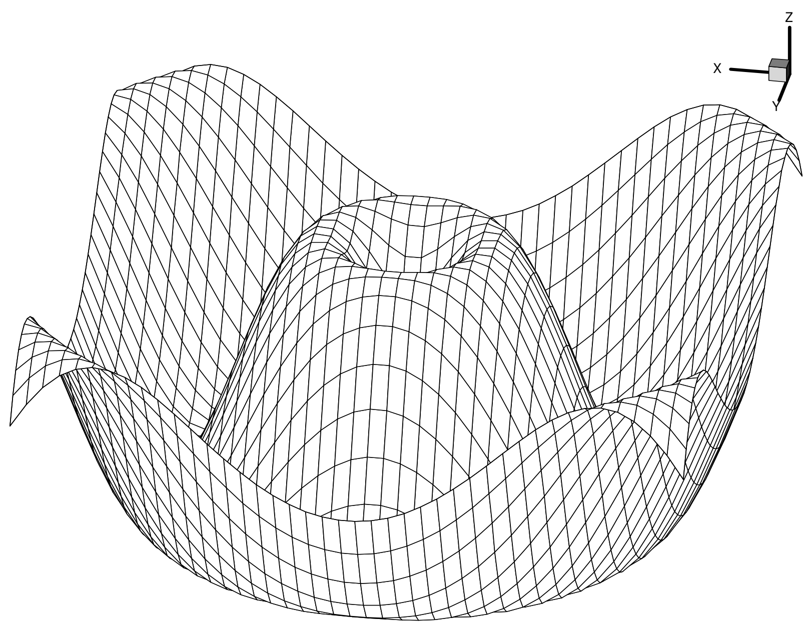

Figure 1: Surface Plot for example 1.

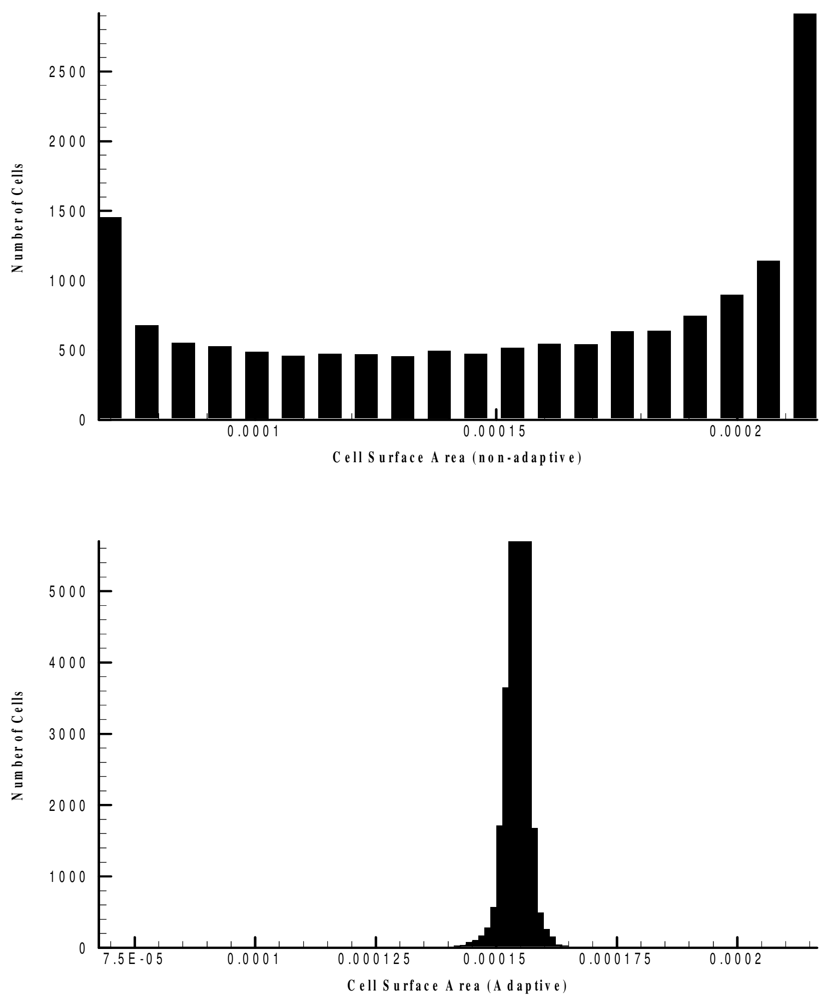

Figure 2: Cell surface area distribution for non-adaptive and adpative grids.

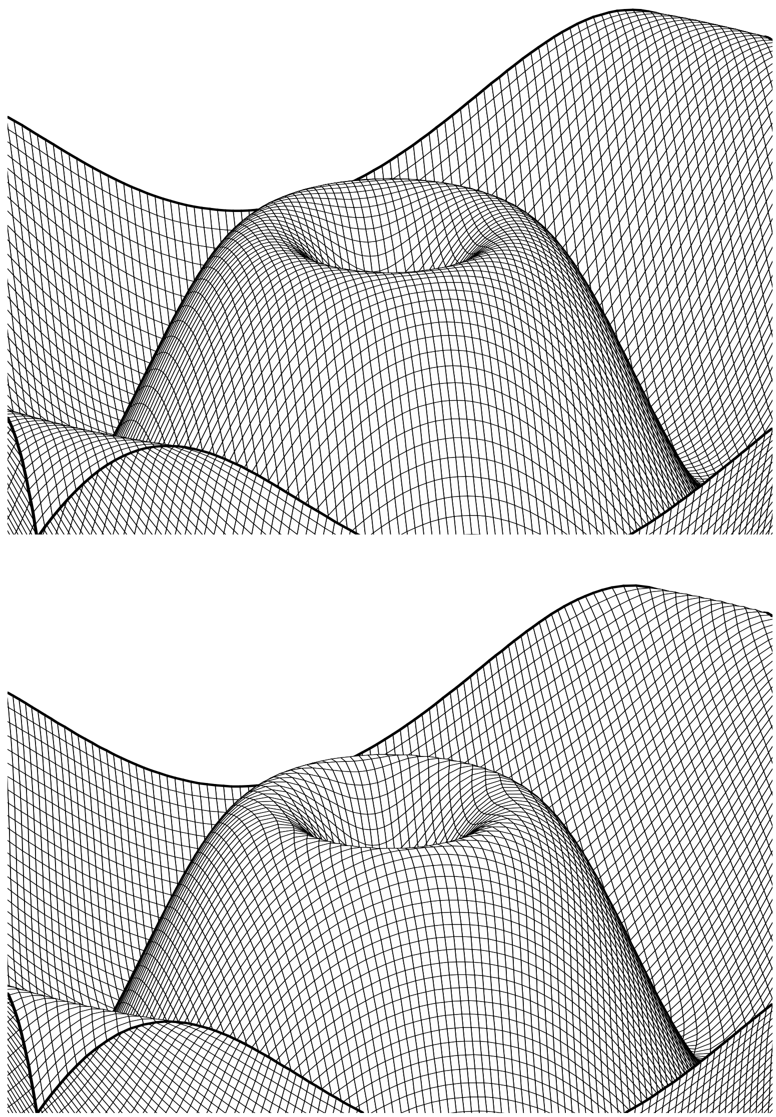

Figure 3: Non-adaptive (top) and adaptive (bottom) surface grids.

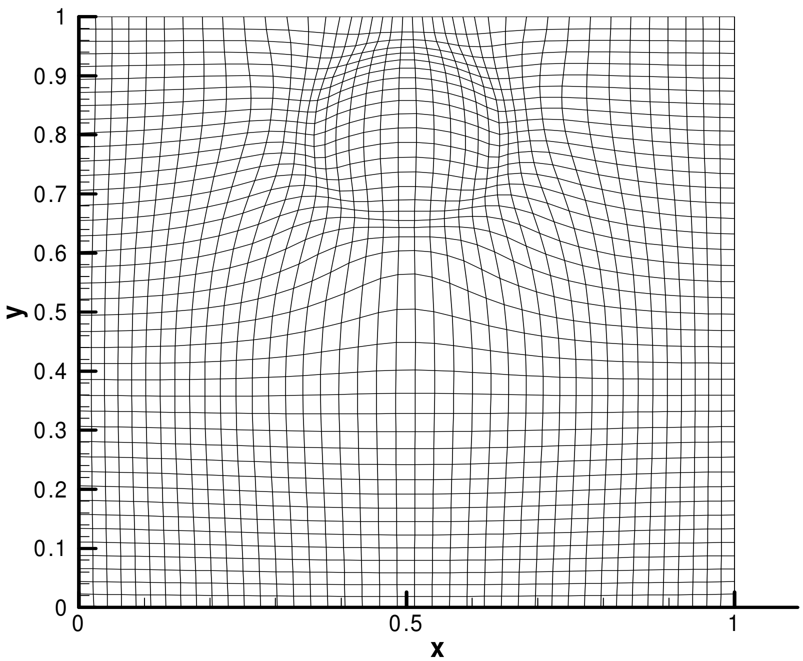

Figure 4: Grid adapted to a circle on parameter domain.

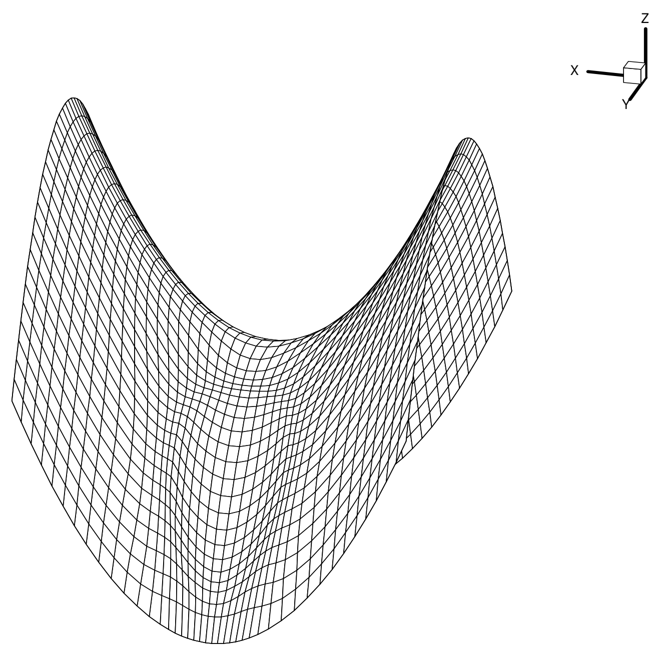

Figure 5: Grid adapted to a circle on surface.

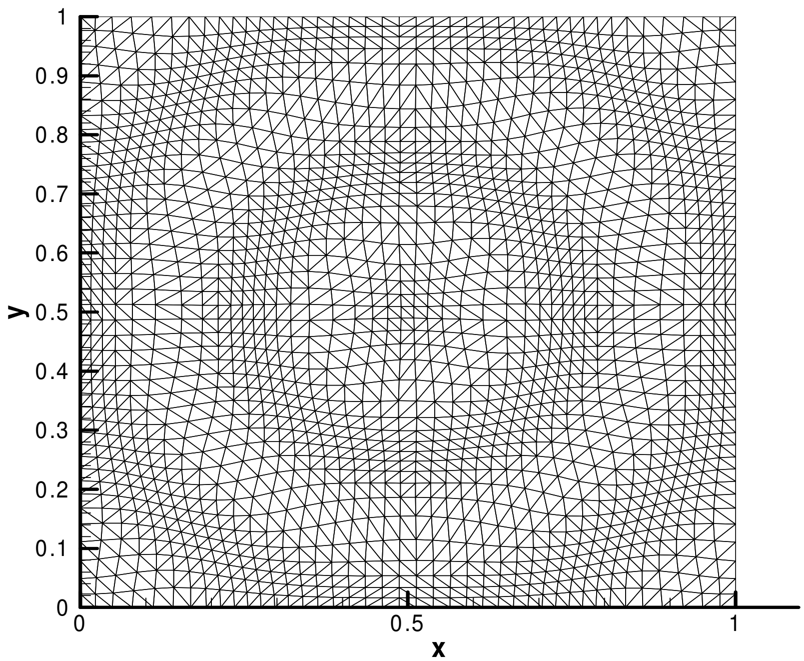

Figure 6: Unstructured grid on parameter domain.

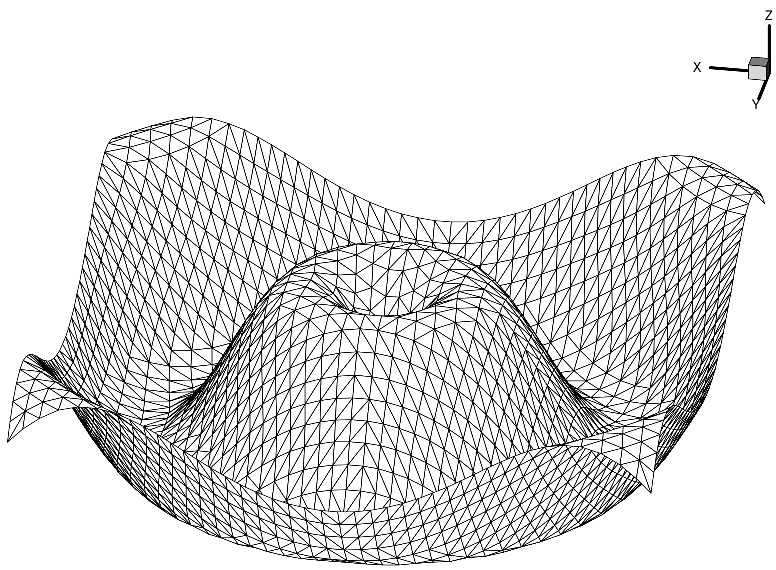

Figure 7: Unstructed grid on the surface $h(x,y) = \sin\sqrt{x^2+y^2}$.

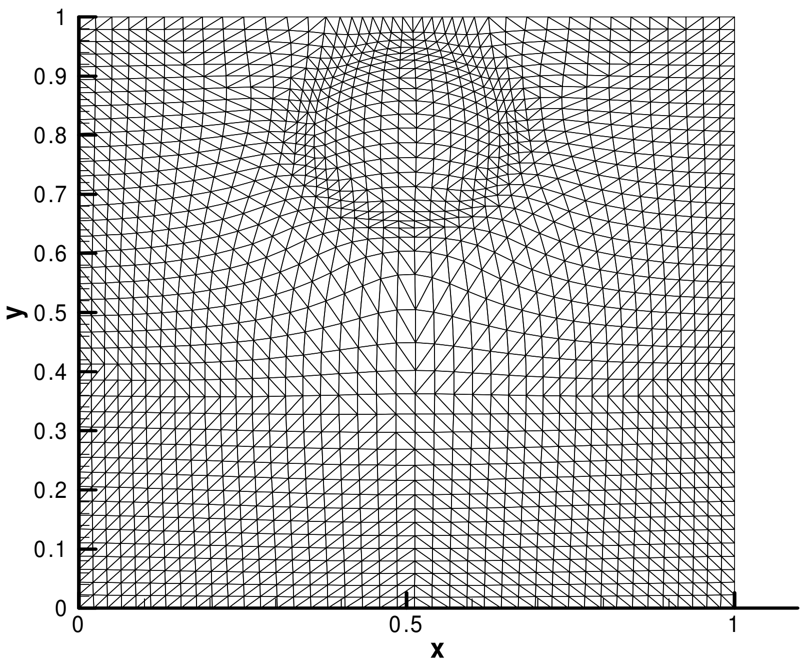

Figure 8: Unstructured grid adapted to a circle on parameter domain.

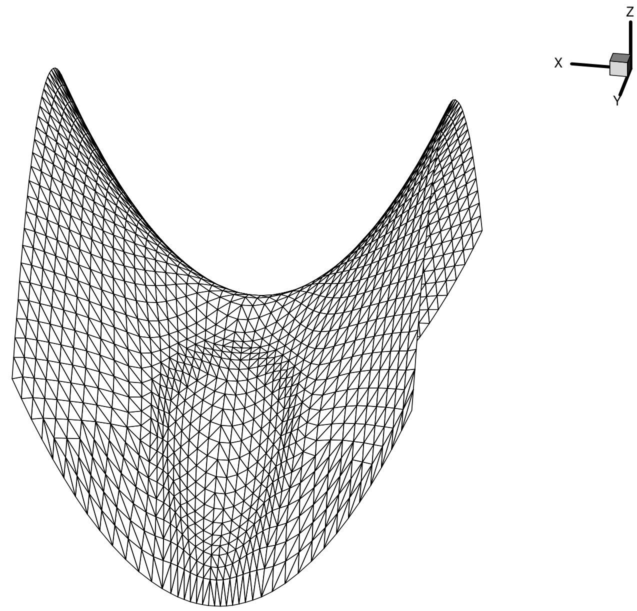

Figure 9: Unstructured grid adapted to a circle on the surface $h(x, y) = 4(x - \frac{1}{2})^2 - 4(y - \frac{1}{2})^2$.

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