ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLVIII, s.I a, Matematică, 2002, f.2.

EXISTENCE RESULTS FOR SOME NONLINEAR IMPLICIT EVOLUTION EQUATIONS

BY

A.D. ZĂRNESCU

1. Introduction. The aim of this paper is to prove some existence results for equations of the type:

$$\begin{cases} \dfrac{d}{dt}(Bu)(t) + Au(t) \ni f(t, u(t)) \text{ a.e. } t \in (0,T) \\ \\ Bu(0) \ni Bu_0. \end{cases} \tag{1.1}$$

where A and B are (possibly nonlinear) multivalued operators of monotone type.

There exists a large variety of results for equations of the form (1.1) with f replaced by a function independent of u, i.e.:

$$\begin{cases} \dfrac{d}{dt}(Bu)(t) + Au(t) \ni y(t) \text{ a.e. } t \in (0,T) \\ \\ Bu(0) \ni Bu_0. \end{cases} \tag{1.2}$$

Equations of the form (1.1) are in fact closed loop systems associated with some control problems governed by systems of the form (1.2).

Equations of the form (1.2) are of interest on their own as extensions of evolution equations of Sobolev type. Early work on such equations is described in [5]. We mention [6] as a source of examples of initial-boundary-value problems for the pseudo-parabolic partial differential equation

$$\frac{\partial}{\partial t}(u - a\Delta u) = k\Delta u$$

with $a > 0$ and $k > 0$. Such equations arise as models for diffusion and they provide an alternative to the classical diffusion equation, where $a = 0$.

More generally, the problem (1.2) describes a large number of physical models including elastic viscoplasticity, thermodynamics and dissipation phenomena. A brief description of these kinds of models can be found in [9]. A more detailed analysis, but only for the case of Stefan problem can be found in [11]. The literature concerning doubly nonlinear equations is quite wide, and versions of problem (1.2) have been studied under a large number of different assumptions. For further examples and results concerning these types of equations we refer the reader to [2], [3], [4],[8]. Existence results for the case when A is nonlocal in time can be found in [10] .

In the present article we shall obtain some existence results for the case when B has the specific form $\Lambda + B_0$,where Λ stands for Riesz isomorphism from a Hilbert space V to its dual V' and B_0 is a subdifferential. This type of problem is called *pseudo-parabolic* and it is of interest on its own. We shall continue by using these results in an approximation procedure for fully doubly nonlinear equations. Applications are given in order to illustrate the effectiveness of the abstract results.

Acknowledgements. The author takes this opportunity to express his warmest thanks to Professor Ioan I. Vrabie for scientific advice as well as for moral support.

2. Main results. The main assumptions we shall use in the sequel are:
(H1) H and V are Hilbert spaces with

$$V \subset H \subset V'$$

algebrically and topologically, $i : V \hookrightarrow H$ is continuous and compact.

We denote by $\|.\|_H$ and $\|.\|_V$ respectively the norms on H and V and by $(.,.)$ the duality pairing between V and V' as well as the scalar product on H. Let Λ be the Riesz isomorphism between V and V'.
(H2) $B : H \to H$ is linearly bounded, i.e.

$$\|Bu\|_H \leq \ C_1\|u\|_H + C_2,$$

$C_1, C_2 > 0$, with $B = \partial\varphi$ where $\varphi : H \to (-\infty, +\infty]$ is a convex,lower semicontinuous function.
(H3) $A : V \to V'$ is maximal monotone,with $D(A) = V$ and A is bounded on bounded subsets of $L^2(0, T; V)$. Moreover, A is coercive, i.e.

$$(Au, u) \geq \omega\|u\|_H^2 + \omega_1,$$

where $\omega > 0$ and $\omega_1 \in R$.
(H4) B^{-1} is single valued and

$$\|B^{-1}x - B^{-1}y\|_H^2 \leq \alpha(B^{-1}x - B^{-1}y, x - y), \forall x, y \in H, \alpha > 0$$

First let us consider the specific form of (1.1):

$$\text{(2.1)} \qquad \begin{cases} \dfrac{d}{dt}(\Lambda + B)u(t) + Au(t) \ni f(t, u(t)) \text{ a.e. } t \in (0, T) \\ \\ \Lambda u(0) + Bu(0) \ni \Lambda u_0 + Bu_0. \end{cases}$$

Theorem 1. *Assume that (H1)-(H3) hold. Then, if $f : [0, T] \times H \to H$ is continuous and bounded, problem (2.1) has a solution $u \in W^{1,2}(0, T; V)$ and $Bu \in W^{1,2}(0, T; V')$*

Remarks.
2.1 The assumption f bounded is not restrictive. Indeed, it is a common procedure to consider f unbounded and to obtain a solution of the equation, by taking instead of f the restriction of f to a certain ball. The difference is that the solution is only local. For details see [12].
2.2 We can obtain a global existence result by assuming only that f has sublinear growth (instead of being bounded).

Theorem 2. *Consider the equation (2.1) with $f = sgn_H u$, where:*

$$\text{(2.2)} \qquad sgn_H u = \begin{cases} \dfrac{u}{\|u\|_H}, \text{ if } u \neq 0 \\ \\ B(0, 1), \text{ if } u = 0 \end{cases}$$

Then, under assumptions (H1)-(H3), there exists a solution u of equation (2.1), with $u \in W^{1,2}(0, T; V)$ and $Bu \in W^{1,2}(0, T; V')$

Theorem 3. *Assume (H1)-(H4). The problem (1.1) with $f : [0, T] \times H \to H$ continuous and bounded has a solution $u \in L^2(0, T; V) \cap W^{1,2}(0, T; H)$ and $Bu \in W^{1,2}(0, T; V')$.*

Observation. The previous remarks remain valid in this case, too, but in the case of Remark 2.2 we have to assume moreover that f has a special type of growth, namely :

$$\|f(t,u)\|_H \leq L\|u\|_H + C,$$

where C is a real constant and $L < \omega$ (where ω is the constant that appears in condition (H3)).

3. The pseudo-parabolic case.

The proof of theorem 1. Consider the approximating equation:

$$\begin{cases} \dfrac{d}{dt}(\Lambda + B)_H u_\varepsilon + A_\varepsilon u_\varepsilon \ni f(t, u_\varepsilon(t)) \\ \\ \Lambda u(0) + Bu(0) \ni \Lambda u_0 + Bu_0. \end{cases} \tag{3.1}$$

where A_ε is the Yosida approximation of $A_H : H \to H$, defined by $A_H u = Au \cap H, \forall u \in D(A_H)$, $D(A_H) = \{u \in H, Au \cap H \neq \Phi\}$. We define $(\Lambda + B)_H$ similarly.

Lemma 3.1. *Under the hypotheses of Theorem 1, the problem (3.1) has a unique continuous solution* $u_\varepsilon : [0,T] \to H$ *with* $(\Lambda + B)_H u_\varepsilon \in C^1([0,T]; H)$.

Proof of the lemma. Consider the delayed integral equation:

$$(\Lambda + B)_H u_{\varepsilon,\lambda}(t) = \begin{cases} \Lambda u_0 + Bu_0, t \in [-\lambda, 0] \\ \\ \Lambda u_0 + Bu_0 + \displaystyle\int_0^t f(\tau, u_{\varepsilon,\lambda}(\tau - \lambda))d\tau - \\ \\ \qquad - \displaystyle\int_0^t A_\varepsilon u_{\varepsilon,\lambda}(\tau - \lambda)d\tau \end{cases} \tag{3.2}$$

Let us notice that for $t \in [-\lambda, 0]$ the solution is uniquely determined on $[-\lambda, 0]$ and it is Lipschitz continuous. For $\tau \in [0, \lambda]$ we have that $\tau - \lambda \in [-\lambda, 0]$, hence $(\Lambda + B)_H u_{\varepsilon,\lambda}(\tau - \lambda) = \Lambda u_0 + Bu_0$, and $u_{\varepsilon,\lambda}(\tau - \lambda)$ is uniquely determined on $[-\lambda, 0]$, and Lipschitz continuous.

Analogously, $u_{\varepsilon,\lambda}$ is determined on $[\lambda, 2\lambda]$ and it can be noticed that it is Lipschitz continuous because A_ε is continuous and therefore bounded on compact sets.Obviously, after a finite number of steps, in this way we can determine the solution on the entire $[-\lambda, T]$.

We can assume with no loss of generality that $[0,0] \in G(A)$ and then $[0,0] \in G(A_\varepsilon)$, thus there exists $L > 0$ such that:

$$\|A_\varepsilon u\|_H \leq L\|u\|_H, \forall u \in H \tag{3.3}$$

Taking the inner product of (3.2) with u_ε, using the monotonicity of B, the continuity of $i : V \hookrightarrow H$, (3.3) and Gronwall's Lemma we have that $\{u_{\varepsilon,\lambda}\}_\lambda$ is bounded in H,uniformly in t and λ.Since A_ε is Lipschitz continuous and f is bounded, we have that $\{(\Lambda+B)_H u_{\epsilon,\lambda}\}_\lambda$ is equiuniformly continuous. On the other hand $(\Lambda + B)_H$ is strongly monotone in H, and therefore$(\Lambda + B)_H^{-1}$ is Lipschitz continuous.Hence $\{u_{\varepsilon,\lambda}\}_\lambda$ is equiuniformly continuous in H.

Inasmuch as $(\Lambda + B)_H$ is coercive in V and $(\Lambda + B)_H u_{\varepsilon,\lambda}$ is bounded in H, it follows that $\{u_{\varepsilon,\lambda}\}_\lambda$ is bounded in V and therefore $\{u_{\varepsilon,\lambda}\}_\lambda$ is compact in H. Since the latter is also equicontinuous,from

Arzelá-Ascoli Theorem, letting λ to tend to zero, we have (using the demiclosedness of $(\Lambda + B)_H$):

$$(\Lambda + B)_H u_\varepsilon(t) = \Lambda u_0 + Bu_0 + \int_0^t f(\tau, u_\varepsilon(\tau))d\tau - \int_0^t A_\varepsilon u_\varepsilon(\tau)d\tau$$

We continue with the proof of the theorem noticing that the solution of (3.1) also solves the equation:

$$\begin{cases} \dfrac{d}{dt}(\Lambda + B)u_\varepsilon + A_\varepsilon u_\varepsilon \ni f(t, u_\varepsilon) \\ \\ \Lambda u(0) + Bu(0) \ni \Lambda u_0 + Bu_0. \end{cases} \tag{3.1$'$}$$

Taking the inner product of (3.1') with u_ε and integrating on $[0,t]$ we have:

$$\begin{aligned} &\frac{\|u_\varepsilon(t)\|_V^2}{2} - \frac{\|u_\varepsilon(0)\|_V^2}{2} + \varphi^*(Bu_\varepsilon(t)) - \varphi^*(Bu_0) + \\ &+ \int_0^t (A_\varepsilon u_\varepsilon(s), u_\varepsilon(s))ds \leq \int_0^t (f(s, u_\varepsilon(s)), u_\varepsilon(s))ds \end{aligned} \tag{3.4}$$

Since B is linearly bounded, its domain is all of H, and therefore $\varphi(0) < \infty$.Thus, with no loss of generality, we may take $\varphi(0) = 0$ and $\varphi^*(x) \geq 0, \forall x \in H$. Taking into account this, as well as f boundedness and the coercivity of A_H, and hence that of A_ε, we have $\{u_\varepsilon\}_\varepsilon$ bounded in $L^2(0,T;H)$, uniformly in ε.

Knowing that $u_\varepsilon = (I+\varepsilon A_H)^{-1}u_\varepsilon + \varepsilon A_\varepsilon u_\varepsilon$ and $A_\varepsilon u_\varepsilon \in A(I+\varepsilon A_H)^{-1}u_\varepsilon$, we have:

$$(A_\varepsilon u_\varepsilon, u_\varepsilon) \geq \omega \|(I+\varepsilon A_H)^{-1}u_\varepsilon\|_V^2 \tag{3.5}$$

Taking into account the boundedness of $\{u_\varepsilon\}_\varepsilon$ in $L^2(0,T;H)$, from (3.4) and (3.5) it results that $\{(I+\varepsilon A_H)^{-1}u_\varepsilon\}_\varepsilon$ is bounded in $L^2(0,T;V)$, uniformly in ε. From (3.1') and the boundedness of A on bounded sets we obtain the boundedness of $\{\frac{d}{dt}(\Lambda + B)u_\varepsilon\}_\varepsilon$ in $L^2(0,T;V')$, and then we have that $\{(\Lambda + B)u_\varepsilon\}_\varepsilon$ is equiuniformly continuous in V', hence $\{u_\varepsilon\}_\varepsilon$ is equiuniformly continuous in V.

Let us observe that $\{(\Lambda + B)u_\varepsilon\}_\varepsilon$ is bounded, uniformly in t and ε in V' (from boundedness of the derivatives in $L^2(0,T;V')$ and the coincidence of initial data). From coercivity we have $\{u_\varepsilon\}_\varepsilon$ bounded in V, hence compact in H. The conditions of Arzelá-Ascoli Theorem are fulfilled, and thus there exists $u \in W^{1,2}(0,T;V)$ such that $u_\varepsilon \to u$ in C([0,T];H) and $u_\varepsilon \rightharpoonup u$ in $W^{1,2}(0,T;V)$.

From $u_\varepsilon \rightharpoonup u$ in $W^{1,2}(0,T;V)$ we have $\Lambda u_\varepsilon \rightharpoonup \Lambda u$ in $W^{1,2}(0,T;V')$. From $u_\varepsilon \to u$ in C(0,T;H), (H2)and the demiclosedness of B, we have $Bu_\varepsilon \rightharpoonup Bu$ in $L^2([0,T];H)$. Inasmuch as $\{u_\varepsilon\}_\varepsilon$ is equilipschitz in V, then $\{\frac{d}{dt}\Lambda u_\varepsilon\}_\varepsilon$ is bounded in $L^2(0,T;V')$, and taking into account the boundedness of $\{\frac{d}{dt}(\Lambda + B)u_\varepsilon\}_\varepsilon$ in $L^2(0,T;V')$ we have the boundedness of $\{\frac{d}{dt}Bu_\varepsilon\}_\varepsilon$ in $L^2(0,T;V')$. Therefore we have $Bu_\varepsilon \rightharpoonup Bu$ in $W^{1,2}(0,T;V')$.

Since $A_\varepsilon u_\varepsilon$ is bounded in $L^2(0,T;V')$, there is an $\eta \in L^2(0,T;V')$ such that $A_\varepsilon u_\varepsilon \rightharpoonup \eta$ in $L^2(0,T;V')$. To prove that $\eta \in Au$ is enough to show:

$$\limsup_{\varepsilon \to 0} \int_0^t (A_\varepsilon u_\varepsilon(s), u_\varepsilon(s))ds \leq \int_0^t (\eta(s), u(s))ds$$

But

$$\int_0^T (A_\varepsilon u_\varepsilon(s), u_\varepsilon(s))ds = -\varphi^*(Bu_\varepsilon(T)) + \varphi^*(Bu_0) -$$

$$-(\Lambda u_\varepsilon(T), u_\varepsilon(T)) + (\Lambda u_0, u_0) + \int_0^T (f(s, u_\varepsilon(s)), u_\varepsilon(s))ds$$

and recalling that $(\Lambda u(T), u(T)) \leq \liminf_{\varepsilon\to 0}(\Lambda u_\varepsilon(T), u_\varepsilon(T)), u_\varepsilon \to u$ in $C([0,T];H)$ and the lower semicontinuity of φ^*, by passing to upper limit we obtain exactly what we were looking for.

Remark. Let us observe that from (3.4) we can obtain the following estimate:

$$\frac{1}{2}\|u_\varepsilon(t)\|_V^2 \leq C + \int_0^t (f(s,u_\varepsilon(s)), u_\varepsilon(s))ds$$

Let us suppose now that f has underlinear growth, instead of being bounded. Using the fact that the imbedding of V in H is continuous and Gronwall's Lemma, we obtain the boundedness of $\{u_\varepsilon\}_\varepsilon$ in V uniformly in ε. In order to prove Remark 2.2 one can continue reasoning as in the previous proof.

Proof of theorem 2. Let us observe that $sgn_H u = \partial\|u\|_H$. We consider the approximating equation:

$$\begin{cases} \dfrac{d}{dt}(\Lambda + B)u_\lambda + Au_\lambda \ni \partial_\lambda\|u_\lambda\|_H^2 \\ \\ \Lambda u_\lambda(0) + Bu_\lambda(0) \ni \Lambda u_0 + Bu_0. \end{cases} \tag{3.6}$$

where $\partial_\lambda\|u\|_H$ is the Yosida approximation of $\partial\|u\|_H$.

According to theorem 1, equation (3.6) has a solution $u_\lambda \in W^{1,2}(0,T;V)$. Taking the inner product of (3.6) with u_λ and integrating on $[0,t]$, where we took into account the coercivity of A and the boundedness of $\partial\|u\|_H$, we have $\{u_\lambda\}_\lambda$ bounded in $L^2(0,T;V)$, and therefore $\{Au_\lambda\}_\lambda$ is equibounded in $L^2(0,T;V')$. Hence $\{\frac{d}{dt}(\Lambda + B)u_\lambda\}_\lambda$ is equibounded in $L^2(0,T;V')$, $\{(\Lambda + B)u_\lambda\}_\lambda$ equibounded and equicontinous in V', and this also happens for $\{u_\lambda\}$ in V and therefore from Arzelá-Ascoli Theorem we have $u_\lambda \to u$ in $C([0,T];H)$ and $u_\lambda \rightharpoonup u$ in $W^{1,2}([0,T];V)$.

From $u_\lambda \to u$ in $C([0,T];H)$, (H2) and the demiclosedness of B we have $Bu_\lambda \rightharpoonup Bu$ in $L^2([0,T];H)$ and similarly $\partial_\lambda\|\lambda\|_H \rightharpoonup \partial\|u\|_H$ in $L^2([0,T];H)$. Inasmuch as $\{\frac{d}{dt}(\Lambda+B)u_\lambda\}_\lambda$ is bounded in $L^2(0,T;V')$ and $\{\Lambda u_\lambda\}_\lambda$ is equilipschitz in V' we get the boundedness of $\{\frac{d}{dt}Bu_\lambda\}_\lambda$ in $L^2(0,T;V')$, and then we have $Bu_\lambda \rightharpoonup Bu$ in $W^{1,2}([0,T];V')$.

Since Au_λ is bounded in $L^2(0,T;V')$, there is an $\eta \in L^2(0,T;V')$ such that $Au_\lambda \rightharpoonup \eta$ in $L^2(0,T;V')$. To prove that $\eta \in Au$ is enough to show:

$$\limsup_{\lambda\to 0}\int_0^t (Au_\lambda(s),u_\lambda(s))ds \le \int_0^t (\eta(s),u(s))ds$$

But

$$\int_0^T (Au_\lambda(s),u_\lambda(s))ds = -\varphi^*(Bu_\lambda(T)) + \varphi^*(Bu_0) - (\Lambda u_\lambda(T),u_\lambda(T)) +$$

$$+(\Lambda u_0,u_0) + \int_0^T (\partial_\lambda(s,u_\lambda(s)),u_\lambda(s))ds$$

and recalling that $(\Lambda u(T),u(T)) \le \liminf_{\lambda\to 0}(\Lambda u_\lambda(T),u_\lambda(T)), u_\lambda \to u$ in $C([0,T];H)$, the lower semicontinuity of φ^* and the fact that $\partial_\lambda\|u_\lambda\|_H \rightharpoonup \partial\|u\|_H$ in $C([0,T];H)$, by passing to upper limit one obtains the desired conclusion.

Remark. Following mainly the same proof,one can obtain existence results for equations of the form:

$$\begin{cases} \dfrac{d}{dt}(\Lambda+B)u(t) + Au(t) \ni Cu(t) \text{ a.e. } t\in(0,T) \\ \\ \Lambda u(0) + Bu(0) \ni \Lambda u_0 + Bu_0, \end{cases} \tag{3.7}$$

where C is a maximal monotone operator such that there is a L with $\|Cu\|_H \le L\|u\|_H + C$ where C is a real constant and $L < \omega$(where ω is the constant that appears in condition (H3)). Condition $L<\omega$ is necessary in order to obtain the boundedness of solutions of the approximating equations.

4. The doubly nonlinear case.

Proof of the Theorem 3. Consider the approximating equation:

$$\begin{cases} \dfrac{d}{dt}(\lambda\Lambda+B)u_\lambda(t) + Au_\lambda(t) \ni f(t,u_\lambda(t)) \text{ a.e. } t\in(0,T) \\ \\ Bu_\lambda(0) \ni Bu_0 \end{cases} \tag{4.1}$$

It is immediate from Theorem 1 that problem (4.1) has a solution $u_\lambda \in W^{1,2}(0,T;V)$ and $Bu_\lambda \in W^{1,2}(0,T;V')$. Taking the inner product in (4.1) with u_λ in H, and integrating on [0,t], we obtain:

$$\begin{aligned}&\frac{\|\lambda u_\lambda(t)\|_V^2}{2} - \frac{\|\lambda u_\lambda(0)\|_V^2}{2} + \varphi^*(Bu_\lambda(t)) - \varphi^*(Bu_0) + \\ &+ \int_0^t (Au_\lambda(s), u_\lambda(s))ds \leq \int_0^t (f(s,u_\lambda(s)), u_\lambda(s))ds\end{aligned} \tag{4.2}$$

As before we may assume that $\varphi^*(x) \geq 0, \forall x$ in H. From Schwarz inequality and the coercivity of A, using (4.2), we obtain the uniform boundedness of $\{u_\lambda\}_\lambda$ in $L^2(0,T;V)$. The equicontinuity of $\{u_\lambda\}_\lambda$ comes from the boundedness of $\{\frac{d}{dt}(\lambda\Lambda + Bu_\lambda)\}_\lambda$ in $L^2(0,T;V')$ and (H4). Thus we obtain the equicontinuity of $\{u_\lambda\}_\lambda$ in H and then, using the coincidence of initial data as well as the fact that $\{u_\lambda\}_\lambda$ are equihölder we have the boundedness of $\{u_\lambda\}_\lambda$ in H. Hence we have the strong compactness of $\{u_\lambda\}_\lambda$ in $C([0,T];H)$. The proof follows as in Theorem 1.

Remark. Using mainly the same arguments as in the previous proof, one can prove the existence for the problem (1.1) when $f = sgn_H u$, or $f = C$,where C is maximal monotone. In the former case we have to assume an extra condition as in Remark 2.2

5. Applications. Let Ω be a bounded domain in $R^N (N \geq 1)$ with sufficiently smooth boundary $\partial\Omega$. We denote $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$.

5.1 Let β be a maximal monotone graph in $R \times R$, with:

$$| \beta(z) | \leq Q(1 + | z |), z \in R, Q > 0,$$

and a_i,i=1,..,N are continuous increasing functions such that:

$$0 \leq a_i(r) \leq C_1 | r | + C_2, \forall r \in R$$

$$a_i(r) \cdot r \geq \omega | r |^2 + C_3, \forall r \in R, i = 1, .., N, \omega > 0$$

A is defined by:

$$(Au, v) = \sum_{i=1}^N \int_\Omega a_i(\frac{\partial u}{\partial x_i}) \cdot \frac{\partial v}{\partial x_i} dx, \forall u, u \in V$$

Λ is defined as usual:

$$(\Lambda u, v) = \sum_{i=1}^{N} \int_{\Omega} \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{x_i} dx, \forall u, v \in V$$

and $Bu = \{v \in V; v(x) \in \beta(u(x)), a.e.x \in \Omega\}$

Applying Theorem 1 or Theorem 2 we obtain the existence for the problem:

$$\frac{\partial}{\partial t}(\beta(u) - \Delta u) - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(a_i(\frac{\partial u}{\partial x_i})) \ni f(t,u), \text{ in } \Omega \times (0,T)$$

$$u(x,0) = u_0(x), \text{ in } \Omega$$

$$u(x,t) = 0, \text{ in } \partial\Omega \times (0,T)$$

This problem appears in [4].Other problems of this kind appear in [5],[7] and [10]. We mention that in the last one A is nonlocal.

5.2 Under the same assumptions as above,assuming in addition that there exists a ϖ such that

$$(*)(\beta(r_1) - \beta(r_2)) \cdot (r_1 - r_2) \geq \varpi \mid r_1 - r_2 \mid_2, \forall r_1, r_2 \in R, \varpi > 0$$

Theorem 3 proves the existence for problems of the form:

$$\frac{\partial}{\partial t}(\beta(u) - \Delta u) - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(a_i(\frac{\partial u}{\partial x_i})) \ni f(t,u), \text{ in } \Omega \times (0,T)$$

$$u(x,0) = u_0(x), \text{ in } \Omega$$

$$u(x,t) = 0, \text{ in } \partial\Omega \times (0,T)$$

Such a problem appears for instance in [2].

5.3 Theorem 3 also applies to the problem:

$$\frac{\partial}{\partial t}\beta_1(u) + \gamma(u-v) \ni f(t,u), \text{ in } \Omega \times (0,T)$$

$$\frac{\partial}{\partial t}\beta_2(u) - \gamma(u-v) - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(a_i(\frac{\partial v}{\partial x_i})) \ni g(t,u), \text{ in } \Omega \times (0,T)$$

$$u(x,0) = u_0(x), v(x,0) = v_0(x), \text{ in } \Omega$$

$$u(x,t) = 0, v(x,t) = 0, \text{ in } \partial\Omega \times (0,T),$$

where $\beta_i, i = 1,2$ are maximal monotone graphs in $R \times R$, linearly bounded which satisfy (*),$\gamma : R \to R$ is a monotone continuous function such that $\gamma(0) = 0$ and $\mid \gamma(u) \mid \leq C \mid u \mid^{\frac{Np}{2(N-p)}}, \forall u \in R$ and a_i satisfy the above assumptions. Such a problem has been studied in [2] and [8].

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Received: 09.X.2001

Faculty of Mathematics,
University "AL.I.Cuza" Iaşi
Bd. Copou Nr. 11
6600 Iaşi
ROMANIA
zarnescu@math.uchicago.edu